HAMILTONIAN AND DEGREE RESTRICTED DFS TREES
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ABSTRACT

The Depth First Search (DFS) algorithm is one of the basic techniques which is used in a very large variety of graph algorithms. Every application of the DFS involves, beside traversing the graph, constructing a special structured tree, called a "DFS tree" that may be used subsequently.

In some problems the degree of some vertices in the DFS tree obtained in a certain run are crucial and therefore we consider the following problem: Let $G=(V,E)$ be a connected undirected graph where $|V|=n$ and let $d \in \mathbb{N}^n$ be a degree sequence upper bound vector. Is there any DFS tree $T$ with degree sequence $d_T$ that violates $d$, (i.e. $d_T \notin d$ which means: $\exists j$ such that $d_T(j) > d(j)$)? We show that this problem is NP-Complete even for the case where we restrict the degree of only one specific vertex to be less than or equal to $k$ for a fixed $k \geq 2$ (i.e. $d=(n-1,\ldots,n-1,k,n-1,\ldots,n-1)$). However, for the special interesting case where $d=(2,2,\ldots,2)$, which means that the underline graph of every DFS tree is a Hamiltonian path, a complete characterization of all graphs that support this restriction is given. The main theorem which implies this characterization is the following: $G$ is an undirected graph in which every DFS tree is a directed Hamiltonian path if and only if $G$ is one of the following graphs: a simple circuit, a $K_n$ or a $K_{n,n}$. This characterization enables us to recognize such graphs by efficient and simple sequential and parallel algorithms.

Key words: DFS trees, complete characterization, degree restricted, graph algorithms, Hamiltonian path, NP-Complete, parallel algorithms.

AMS(MOS) subject classification: 05C05, 05C38, 05C45, 05C75, 68Q25, 68R10
1. INTRODUCTION

The Depth First Search (DFS) algorithm is one of the basic techniques which is used in a very large variety of graph algorithms. The history of this algorithm (in a different form) goes back to 1882 when Tremaux' algorithm for the maze problem was first published (see [BLW, page 18]). The impact of DFS grew rapidly since the Hopcroft and Tarjan version of it was published (see [Ta], [HT a], [HT b] and [HT c]). This algorithm is used in many areas of computer science, and recently it also has penetrated the field of parallel and distributed algorithms (e.g. [AA], [Aw], [HY], [KO b], [LMT], [Re], [Sm] and [Ti]).

Every use of the DFS, beside traversing the graph, constructs a special structured directed rooted tree, called a DFS tree, that may be used subsequently. However, in some application the degree of some vertices in a DFS tree obtained in a certain run might be too high. For example, in distributed and parallel algorithms the degree in the tree might be restricted because of technical limitations (e.g. if the DFS tree is going to be used as a massive communication tree then each vertex may have a limited number of ports to be used in the tree).

This leads us to the following problem: Let $G=(V,E)$ be a connected undirected graph where $|V|=n$ and let $d \in \mathbb{N}^n$ be a degree sequence upper bound vector. Is there any DFS tree $T$ with degree sequence $d_T$ that violates $d$, (i.e. $d_T(\cdot) \notin d$ which means: $\exists j$ such that $d_T(j) > d(j)$)?

In section 3 we show that this problem is NP-Complete even for the case where we restrict the degree of only one specific vertex to be less than or equal to $k$ for a fixed $k \geq 2$ (i.e. $d=(n-1,\ldots,n-1,k,n-1,\ldots,n-1)$). However, for the special interesting case where $d=(2,2,\ldots,2)$ (which means that the underline graph of every DFS tree is an Hamiltonian path) we give in section 4 a complete characterization of all graphs that support this restriction. This characterization enables us to solve the problem for this special case by efficient and simple sequential and parallel algorithms.

Other characterizations and algorithms that concern spanning trees that can be obtained by the DFS algorithm appear in [HN], [KO a], [KO b] and [Sy].

2. SOME DEFINITIONS and CONVENTIONS

Let $T$ be an undirected spanning tree in an undirected graph $G=(V,E)$ and let $s \in V$. $T_s$ is the tree $T$...
with an orientation that makes \( s \) the root of \( T_s \). \( T \) is called a DFS tree in \( G \) if there exists a vertex \( s \in V \) such that \( T_s \) is a DFS tree in \( G \) (i.e. \( T_s \) can be constructed by a DFS run in \( G \)).

To simplify the discussion we assume that all graphs in this paper are without loops and parallel edges. This assumption does not affect the characterizations and the complexity of the sequential algorithm presented here. As for the parallel algorithm, by using the sorting algorithm in [Co], which has \( O(\log |E|) \) time complexity and uses \( O(|E|) \) processors on an exclusive read exclusive write parallel random access machine (EREW PRAM), we can eliminate loops and parallel edges.

The symbol: " \( \square \) " stands for "end of the proof" or "end of the statement and a proof is not provided".

3. DEGREE RESTRICTED DFS TREES

The Degree Restricted DFS Trees (DRDT) problem: Let \( G=(V, E) \) be a connected undirected graph where \( |V|=n \) and let \( \bar{d} \in N^n \) be a degree sequence upper bound vector. Is there any DFS tree \( T \) with degree sequence \( d_T \) that violates \( \bar{d} \), (i.e. \( d_T \not\leq \bar{d} \) which means: \( \exists j \) such that \( d_T(j) > \bar{d}(j) \))?

This question is important in problems where the degree of some vertices in a DFS tree obtained in a certain run might be too high. For example, in distributed and parallel algorithms the degree in the tree might be restricted because of technical limitations (e.g. if the DFS tree is going to be used as a massive communication tree, then each vertex may have a limited number of ports to be used in the tree).

Theorem 3.1: The DRDT problem is \( NP-Complete \).

Proof: It is easy to see that \( DRDT \in NP \). The completeness follows from Lemma 3.2 where we show that this problem is \( NP-Complete \) even for the case where we restrict the degree of only one specific vertex to be less than or equal to \( k \) for a fixed \( k \geq 2 \) (i.e. \( \bar{d} =(n-1, \ldots, n-1, k, n-1, \ldots, n-1) \)). \( \square \)

Lower Bound Degree (LBD) problem: Given a graph \( G=(V, E) \), a vertex \( v \in V \) and a constant \( q \geq 3 \). Is there a DFS tree where the degree of \( v \) is greater than or equal to \( q \)?

Lemma 3.2: The LBD problem is \( NP-Complete \).

Proof: It is obvious that \( LBD \in NP \). To show completeness we transform Hamiltonian path to \( LBD \). Let \( G'=(V', E') \) be any graph and let \( q \) be any integer such that \( |V'| \geq q \geq 3 \). We construct a graph \( G=(V, E) \), with a specific vertex \( v \in V \) such that there is a DFS tree in \( G \) in which the degree of \( v \) is greater than or equal to \( q \) if and only if there is an Hamiltonian path in \( G \).
The construction of $G=(V,E)$ is as follows: $V=V' \cup V''$ where $V''=\{v,v_1,v_2,\ldots,v_q\}$ and $V' \cap V'' = \emptyset$. $E=E' \cup \{(v,v_i)\text{ for all }1 \leq i \leq q\} \cup \{(v',v_i)\text{ for all }1 \leq i \leq q \text{ and } v' \in V'\}$. Note that the construction of $G$ is polynomial in the size of $G'$.

Assume $G'$ has an Hamiltonian path. Consider a DFS run in $G$ that first progress along an Hamiltonian path in $G'$. It is easy to see that the degree of $v$ in the DFS tree obtained by this run is $q$ (note that the vertices of $V'$ are discovered before the $q+1$ vertices of $V''$).

On the other hand if $G'$ has no Hamiltonian path then in every DFS run in $G$ starting at a vertex $v' \in V'$, at least two vertices in $V''$ are discovered by vertices in $V'$. Therefore, the degree of $v$ is less than $q$ in any DFS tree in $G$ rooted at any vertex in $V'$. Now consider a DFS run in $G$ rooted at a vertex in $V''$. Let $v_1 \in V'$ be the first vertex that was discovered in $V'$, and assume it was discovered by $v_i$. Since $q \geq 3$, it is easy to see that when $v_1$ was discovered at least one vertex in $\{v_1,\ldots,v_q\}$ had not yet been discovered. Therefore, there is a path $\pi=(v_i,v_1',v_2',\ldots,v_p',v_j)$ in the DFS tree, where $\{v_i,v_j\} \subset V''$ and $\{v_1',v_2',\ldots,v_p'\} \subset V'$. This implies that $(v,v_i)$ and $(v,v_j)$ can not be together in that DFS tree since they would create a circuit with $\pi$. Hence the degree of $v$ in any DFS tree rooted at $V''$ is also less than $q$.

Corollary 3.3: The following decision problem is NP-Complete: Given a weighted graph $G=(V,E)$ and an integer $W$, is there a DFS tree having total weight $W$ or less?

Proof: We leave it as an easy exercise to see how the corollary follows from Lemma 3.2. Also, one can easily see that Hamiltonian path polynomially transforms into that problem.

4. HAMILTONIAN DFS GRAPHS

Definition 4.1: An undirected connected simple graph is called Hamiltonian-DFS-Graph ($H$-DFS-$G$) if every DFS tree in it is an Hamiltonian directed path.

Definition 4.2: An undirected connected simple graph is called Underline Hamiltonian-DFS-Graph ($UH$-DFS-$G$) if the underline graph of every DFS tree in it is an Hamiltonian path.

By Theorem 3.1 the DRDT problem is NP-Complete. However, for the special interesting case where $d=(2,2,\ldots,2)$ (which means that the underline graph of every DFS tree is an Hamiltonian path) we give a complete characterization of all graphs that support this restriction. This characterization enables us to decide whether or not a given graph is an $UH$-DFS-$G$ by efficient and simple sequential and paral-
be a simple circuit of length $k$ in $G$. Since $G$ is an $H$-$DFS$-G it is possible to extend the path $\pi_{v_1,v_k} = \pi_{v_1,v_4} + \pi_{v_4,v_{k-1}} + (v_{k-1} - v_k)$ to an Hamiltonian path in $G$ by a DFS run in $G$ starting at $v_1$, then moving along $\pi_{v_1,v_k}$ and then moving along the path $v_k-u_1-u_2-\ldots-u_{n-k-1}-u_{n-k} = (v_k-u_1) + \pi_{u_1,u_{n-k}}$, where $\{u_1, \ldots, u_{n-k}\} = V-V'$.

By Proposition 4.4, $(v_1,u_{n-k})\in E$ and therefore the following path exists in $G$: $v_3-v_4 + \pi_{v_4,v_{k-1}} + (v_{k-1} - v_k - v_1 - u_{n-k}) + \pi_{u_n-k,u_1}$ (where the path $\pi_{u_n-k,u_1}$ is the path $\pi_{u_1,u_{n-k}}$ in the reverse order). If $(v_2,u_1)\not\in E$ it is impossible to extend this path to an Hamiltonian path in contradiction to the definition of $G$. Hence $(v_2,u_1)\in E$.

By analogue arguments $(v_{k-1},u_{n-k})\in E$. It follows that $\pi_{v_4,v_3} = \pi_{v_4,v_{k-1}} + (v_{k-1} - u_{n-k}) + \pi_{u_{n-k},u_1} + (u_1 - v_2) + (v_2 - v_3)$ is a path in $G$ and therefore there is a DFS run in $G$, the first part of which is this path. $\pi_{v_4,v_3}$ includes all the vertices in $G$ excluding $v_1$ and $v_k$. Since $G$ is an $H$-$DFS$-G one of the following must hold:

(i) $(v_3,v_1)\in E$ which implies the existence of the following triangle (i.e. a $K_3$) in $G$: $(v_1,v_2,v_3)$ and by Lemma 4.6 - $G$ is a complete graph.

(ii) $(v_3,v_k)\in E$ which implies the existence of the following circuit in $G$: $(v_3,v_k) + \pi_{v_4,v_{k-1}} + (v_{k-1} - v_k - v_3)$.

Since $k > 4$ it follows that in both of these cases $G$ has a circuit of length $k-2$. □

**Corollary 4.8:** Let $G=(V,E)$ be an $H$-$DFS$-G which is neither a simple circuit nor a bipartite graph, then it is a complete graph.

**Proof:** $G$ contains a triangle: This is easily checked for $|V|\leq 5$ and for $|V|\geq 6$ it follows from Lemma 4.7. Hence, by Lemma 4.6 - $G$ must be a complete graph. □

**Lemma 4.9:** Let $G=(V,E)$, with $|V|\geq 3$ be an $H$-$DFS$-G which is bipartite but not a simple circuit. Then $G$ is the complete bipartite $K_{n,n}$.

**Proof:** By Corollary 4.5, $G$ has an Hamiltonian circuit and therefore any 2-coloring of $G$ induces two color classes $V_1$ and $V_2$ of the same cardinality. Since we assume that $G$ is not a simple circuit it follows that $|V|\geq 6$ and hence, by Lemma 4.7 it follows that $G$ contains a circuit of size 4 (called a square). Let
$G'=(V',E')$ be a maximal complete bipartite subgraph of $G$ that contains a square. Let $V'_1 = V_1 \cap V'$ and $V'_2 = V_2 \cap V'$. Clearly, both $|V'_1|$ and $|V'_2|$ are at least two. If $G = G'$ we are done. Otherwise let $V'' = V - V'$.

We may assume w.l.o.g. that there are five vertices $\{u,v_1,v_2,v_3,v_4\} \subseteq V$ such that: $u \in V''$, $\{v_1,v_3\} \subseteq V'_1$, $\{v_2,v_4\} \subseteq V'_2$, $(u,v_1) \in E$ and $(u,v_3) \notin E$. Since $G$ is an $H-DFS-G$ the path $u-v_1-v_2-v_3-v_4$ can be extended to an Hamiltonian circuit in $G$ by the path $v_4-x-\cdots-y-u = \pi_{v_4,u}$. Consider a DFS run that starts at $v_1$ and moves along the path $(v_1-v_4) + \pi_{v_4,u}$. Since $G$ is an $H-DFS-G$ and since $v_2$ and $v_3$ are the only vertices that have not yet been discovered, we have either $(u,v_2) \in E$ or $(u,v_3) \in E$. Since $u$ and $v_2$ have the same color (both are connected to $v_1$) it follows that $(u,v_2) \notin E$ and we must have $(u,v_3) \in E$, a contradiction. 

Corollary 4.11 below is an extension of Theorem 4.3. In order to prove it we need the following proposition:

**Proposition 4.10:** If $G=(V,E)$ has a separating vertex $v \in V$ with degree greater than or equal to 3 then there is a DFS tree $T$ in $G$ where the degree of $v$ is greater than or equal to 3 (which means that $G$ is not an $UH-DFS-G$).

**Proof:** Let $\{x,y,z\} \subseteq V$ be such that $\{(x,v),(y,v),(z,v)\} \subseteq E$ where $x$ and $y$ are in different 2-connected components. The DFS run beginning with the path $z-v-x$ will create a tree where the degree of $v$ is at least 3.

**Corollary 4.11:** Let $G=(V,E)$ be an undirected graph. $G$ is an $UH-DFS-G$ if and only if $G$ is one of the following four possibilities: a simple circuit, a $K_n$, a $K_{n,n}$ or a simple path.

**Proof:** One can easily check the correctness of the if part. To show the only if part let us first assume that $G$ has no separating vertex. In this case the root of every DFS tree in $G$ has degree one. Therefore every DFS tree in $G$ is an Hamiltonian directed path and by Theorem 4.3 the proof of this case is completed. The correctness of the corollary for graphs which have separating vertices is an immediate consequence of Proposition 4.10.

**Corollary 4.12:** The characterization given in Corollary 4.11 enables us to solve the DRDT problem for a graph $G=(V,E)$ where $d=(2,2,\ldots,2)$ by a simple sequential algorithm with $O(|E|)$ time complexity and
by a simple parallel algorithm with $O(\log |V|)$ time complexity and $O(|E|)$ processors on an exclusive read exclusive write parallel random access machine (EREW PRAM).

**Proof:** As we assume throughout the paper, $G$ has no loops nor parallel edges. Hence, it is easy to check whether $G$ is a simple circuit, a $K_n$ or a simple path by observing the degrees of the vertices (recall that $G$ is connected). As for the recognition of a $K_{n,n}$ we have three steps:

(i) Choosing a vertex $s \in V$.

(ii) Finding the set of vertices $V' \subset V$ which are neighbors of $s$ in $G$.

(iii) Checking that the two sets of vertices $V'$ and $V'' = V - V'$ form the $K_{n,n}$. □

5. CONCLUSION

We have shown that on one hand the DRDT problem is \textit{NP-Complete} even for the case where \(d = (n-1, \ldots, n-1, k, n-1, \ldots, n-1)\) for a fixed $k \geq 2$. On the other hand we have completely characterized the set of \textit{Underline Hamiltonian-DFS-Graphs}. This characterization implies that the DRDT problem can be solved by efficient and simple sequential and parallel algorithms in the case where $d = (2, 2, \ldots, 2)$. It is left open to characterize the graphs that support the restriction or to classify the complexity of recognition of such graphs, for all vectors between those two extreme cases.
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