NONMONOTONIC DEFAULT MODAL LOGICS

by

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Abstract. Conclusions by the failure to prove the opposite are frequently used in reasoning about incompletely specified world. This naturally leads to logics for default reasoning which, in general, are nonmonotonic, i.e., introducing new axioms can invalidate old theorems. Accordingly, a set of axioms of a nonmonotonic theory is called (nonmonotonically) degenerating, if adding new axioms does not invalidate already proved theorems. We study nonmonotonic modal logics based on various sets of defaults and present a necessary and sufficient condition for a set of axioms of modal nonmonotonic theory to be degenerating. Also we establish some closure properties of sets of defaults defining nonmonotonic modal logics.

1. Introduction

Nonmonotonic reasoning is very natural in human reasoning and AI.

For example, in science the different theories explaining the world are discovered. Then, sooner or later, these theories are failed when a contradiction to some consequence of a theory is found on a lower level (cf. [Po]).

When an expert system derives a conclusion based on incomplete knowledge, this conclusion may be invalidated in the future by the new facts about the external world. Also, while dealing with probabilistic reasoning, the derived probabilities of different events can change if new facts are added to the knowledge base. Therefore, when we use threshold probabilities for making conclusions, the accepted truths can also change completely. In Prolog, with its negation by failure semantics, the proved goals can become invalid after adding new facts to the data base.
Logics which formalize nonmonotonic reasoning have been introduced in [MCDD], [Re], [MCC] and [Ga]. In particular, a detailed example for nonmonotonic commonsense reasoning can be found in [Ga]. Most of these logics are based on the "fixed point" semantics or proof theory. The "default" logic of Reiter ([Re]) is based on the theories which are the fixed points of some operator. The logic of McDermott and Doyle ([MCDD]) is based on the intersection of all the fixed points of a similar operator. The "circumscription" of McCarthy ([MCC]) is the definition of a predicate as the minimal relation satisfying some property.

Later McDermott in [MCD] introduced nonmonotonic modal logics which are more suitable for describing the dynamic world. These logics are based on the modal systems $T$, $S4$, and $S5$. Unfortunately, the above logics are somewhat problematic in view of the following (cf. [MCDD]). First, it is (yet) unknown whether McDermott's logics are consistent in the first order $T$ or $S4$. Second, the logic based on $S5$ degenerates, i.e., it is equivalent to the monotonic one.

We shall study here the nonmonotonic modal logic, containing the operators $M$ and $L$ - "eventually" and "necessarily", more precisely, extensions of the well known modal system $T$ (cf. [Fe], [HC]). Our definition of the nonmonotonic logic is the relativization of that appearing in [MCD]. It is based on the intersection of all the extensions of certain default theory (cf. [Re]). The main difficulty of dealing with a nonmonotonic modal logic is that the underlying modal logic lacks the deduction theorem ($A, \varphi \vdash \psi$ implies $A \vdash \varphi \Rightarrow \psi$). This is the reason why we cannot prove that every consistent$^*$ theory has a consistent nonmonotonic extension etc., as it was done in [Re]. Despite of this, in modal logics which contain $T$ we have a weak deduction theorem stating that $A, \varphi \vdash \psi$ implies $A \vdash (L^k \varphi) \Rightarrow \psi$ for some $k$, where $L^0 \varphi$ is $\varphi$, and $L^{k+1} \varphi$ is $LL^k \varphi$. Using this deduction theorem we can give a condition when modal default logic degenerates, i.e., becomes monotonic. This condition states that a modal default logic degenerates if and only if the set of defaults is, in some sense, closed under negation.

The paper is organized as follows. In the next section we give necessary definitions and derive some simple properties of nonmonotonic default logics. We also prove that in modal logic with the "strong" equality (cf. [HC]) McDermott's nonmonotonic logic is consistent in first order $T$ and $S4$. Unfortunately,

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$^*$ In this paper the term "consistent" means "noncontradictory in the usual (monotonic) logic".
the "strong" equality seems to be inconvenient for describing incompletely known world by means of modal logic, because objects known as equal may become different when we give more information, and vice versa. In Section 3 we prove that a set of defaults can always be taken closed under the operators $\land$, $\lor$, and $L$. Section 4 contains the main result of this paper, i.e. a condition for degeneracy of nonmonotonic modal logics. Finally, in Section 5, we present a slightly different version of McDermott's nonmonotonic logic, which does not suffer the lacks of logics mentioned above.

2. Monotonic and nonmonotonic modal logics

We consider here the language of the predicate calculus containing the propositional constants $true$ and $false$, and operators $\supset$ and $\exists x$. Other operators $\land$, $\lor$, $\neg$, and $\forall x$ are defined as usual. The language $\text{Lang}$ of modal logic is obtained from the language of the predicate calculus by extending it with the modal connective $L$ (necessarily). As usual, the connective $M$ (possibly) is defined by $-L-$. A formula without free variables is called a sentence. $Fm$ denotes the set of the formulas, and $St$ denotes the set of the sentences of $\text{Lang}$.

In this paper we shall deal with modal logics which result from classic predicate calculus by adding the rule of inference

Necessitation (NEC): \[
\frac{\Phi}{L\Phi}.
\]

and the axiom schemata below.

M1. $L\phi \supset \phi$
M2. $L(\phi \supset \psi) \supset (L\phi \supset L\psi)$
BF. $\forall x L\phi \supset L\forall x \phi$ - this is the so called Barcan formula.

The above system is called $T+BF$. Sometimes we add the axiom scheme

M4. $L\phi \supset LL\phi$
resulting $S4+BF$, or the scheme

M5. $M\phi \supset LM\phi$
resulting $S5$. One can easily prove M4 from S5, so S5 is the extension of S4 (cf. [Fe] or [HC]). Below we omit the suffix $+BF$ and simply call the above systems $T$, $S4$ and $S5$. 

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In this paper we deal with modal systems that are extensions of \( T \) with additional axioms, e.g., \( T \) itself, \( S4, S5 \), etc.

For a set of axioms \( A \subseteq \text{Fm} \) we define the theory of \( A \), denoted by \( \text{Th}(A) \), as

\[
\text{Th}(A) = \{ \varphi \in \text{Fm} : A \vdash \varphi \},
\]

where \( A \vdash \varphi \) means "\( \varphi \) is provable from \( A \)."

Next we recall the definition of Kripke semantics for modal logics.

A Kripke model is a triple \( M = \langle W, R, D \rangle \), where \( W \) is a set of models for the nonmodal part of \( \text{Llg} \) over the same domain \( D \) and \( R \) is a relation on \( W \). Notice that the equality in these models is interpreted by a binary relation that does not have to be the real equality, but satisfies its first order axioms. Alternatively, we can define the strong equality Kripke model as a model where the equality is interpreted by the real equality in the domain \( D \). \( W \) is thought as a set of possible worlds, and \( w_1 R w_2 \) means that \( w_2 \) is possible with respect to \( w_1 \).

Let \( \varphi \) be a formula from \( \text{Fm} \) with elements of \( D \) assigned to the free variables, and \( w \) be a world from \( W \). We define the validity of \( \varphi \) in \( w \), denoted by \( w \models \varphi \), by induction on \( \varphi \) as follows.

If \( \varphi \) is an atomic formula, then \( w \models \varphi \) if \( \varphi \) is valid in \( w \) considered as a first order model.

\[
w \models \varphi \to \psi \text{ if } w \not\models \varphi \text{ or } w \models \psi.
\]

\[
w \models \exists x \varphi(x) \text{ if there exists an } a \in D \text{ such that } w \models \varphi(a).
\]

\[
w \models L \varphi \text{ if for every } v \in W \text{ such that } w R v, v \models \varphi.
\]

Finally, \( M \models \varphi \) if and only if \( \varphi \) is valid in all the \( w \in W \).

The Kripke models with reflexive \( R \) are sound and complete for \( T \). The Kripke models with reflexive and transitive \( R \) are sound and complete for \( S4 \), and the Kripke models, where \( R \) is an equivalence relation, are sound and complete for \( S5 \), cf. [HC, p. 169].

Definition 1. Let \( D \) be a set of sentences from \( \text{St} \) called the set of "defaults". Following Reiter [Re] and McDermott [MF0], we define the nonmonotonic theory \( \text{NM}_D \) by adding the following rule of nonmonotonic inference:

\[
\frac{\varphi, \psi \in D}{\psi}
\]
which is formalized as follows. For a fixed theory (set of formulas) \( A \), we say that \( C \) is an extension of \( A \) in \( NM_D \), if it satisfies the conditions 1) and 2) below.

1) \( C = \text{Th}(A \cup (C \cap D)) \);
2) For every \( \psi \in D \setminus C \), \( C \not\vdash \neg \psi \).

Condition 1) says that an extension is generated by the formulas added by the rule of nonmonotonic inference, i.e. that this rule is the only one used. Condition 2) says that the rule of nonmonotonic inference is satisfied. This definition is equivalent to Definition 1 in [Re] for a closed normal default theory. However, in modal logic the results of Reiter for closed normal default theories do not hold.

Now, similarly to that in [MFD], we define the set of formulas derivable \( D \)-nonmonotonically from \( A \) as

\[
\text{TH}_D(A) = \bigcap \{ C : C \text{ is an extension of } A \text{ in } NM_D \}. 
\]

Remark 1. The conditions 1) and 2) of Definition 1 are equivalent to the following:

1') \( C = \text{Th}(A \cup D') \) for some \( D' \subseteq D \) - "\( C \) is generated in \( A \) by a part of \( D \)";
2') For every \( \psi \in D \setminus C \), \( C \not\vdash \neg \psi \) or \( C \not\vdash \psi \) - "\( C \) is complete in \( D \)".

Remark 2. One can show that the fixed points of McDermott (cf. [MFD]) are equivalent to our extensions with \( D = \{ M \psi : \psi \in S \} \). Of course, the nonmonotonic theory \( \text{TH}(A) \) of McDermott is our \( \text{TH}_D(A) \).

Remark 3. Condition 1') of Remark 1 implies that if the set of defaults \( D \) is of finite cardinality \( n \), then any set of axioms has at most \( 2^n \) extensions. Therefore, in the propositional nonmonotonic logic \( T \), \( S4 \) or \( S5 \) based on a finite set of defaults \( D \), if the set of axioms \( A \) is also finite, then the nonmonotonic theory \( NM_D(A) \) is decidable. The decision procedure is as follows. By Definition 1, and the decidability of the above propositional modal logics, it is possible to construct all the finite sets of axioms \( A \cup D', D' \subseteq D \) such that \( \text{Th}(A \cup D') \) is complete in \( D \). Then, for a formula \( \varphi \), one can easily decide whether for every such \( D' \subseteq D \cup D \) \( \neg \varphi \), i.e., whether \( \varphi \) belongs to all the extensions of \( A \).

It is known from [MFD] that McDermott’s nonmonotonic \( S5 \) is equivalent to the monotonic \( S5 \), and thus, trivially, it is consistent with the empty set of axioms (cf. [MFD]). Also, yet McDermott’s nonmonotonic propositional \( T \) and \( S4 \) are consistent, cf. [MFD], nothing is known about the consistence of the first
order nonmonotonic $T$ and $S4$. However it is not hard to show that the first order nonmonotonic $T$ and $S4$
with the strong equality are consistent.

A modal logic with strong equality is obtained from modal logic with the ordinary equality by adding
the axioms $(x=y) \supset L(x=y)$ and $(x \neq y) \supset L(x=y)$, cf. [HC, p. 190]. In $T$ the above two axioms are
equivalent to the strong equality axiom $M(x=y) \supset L(x=y)$.

For the consistency proof of the first order nonmonotonic $T$ and $S4$ with the strong equality we con­
struct a consistent extension as follows. First, we add the formula $M \forall x \forall y (x=y)$. This formula together
with the strong equality axiom imply $L \forall x \forall y (x=y)$. Therefore, for each formula $\phi(x) \in Fm$, the formulas
$\phi(x)$, $\phi(y)$, $\forall x \phi(x)$, and $\exists x \phi(x)$ are equivalent. Hence in modal logic with the strong equality the formula
$M \forall x \forall y (x=y)$ implies that each formula is equivalent to a formula with only one free variable and without
quantifiers, and the result follows from the consistency of nonmonotonic propositional logics $T$ and $S4$.

We must note, however, that the strong equality leads to some philosophical inconvenience, since it
imposes very strict constraints on the logic and its Kripke interpretations. For example, in Kripke models
for the logic with the strong equality, two elements which are equal in some world must be equal in all the
worlds, cf. [HC, pp. 190-192] for a general discussion. Then, using modalities in order to reflect our
(incomplete) knowledge of the external world, we must assume complete knowledge of the equality predi­
cate. Therefore there seems to be only a few possible applications for nonmonotonic modal logics with the
strong equality.

In Section 5 we present a slightly modified version of McDermott's logic, called nonmonotonic
ground logic. This logic is consistent in the first order $T$, $S4$ and $S5$ with the empty set of axioms and
possesses many other desirable properties. In particular, it does not degenerate when the underlying modal
logic is $S5$. So, it seems to be an appropriate candidate for AI systems based on $S5$.

3. Closure properties of sets of defaults

In this section we establish a closure property of set of defaults under connectives $\wedge$, $\vee$, and $L$. This
closure property is given by Lemma 1 below and can be considered as a motivation for Theorem 2 in the
next section.
Lemma 1. Let $\overline{D}$ denote the closure of $D \cup S$ under the operators $\land, \lor$ and $L$. Then, for every set of axioms $A$, $TH_D(A) = TH_{\overline{D}}(A)$. Moreover, the corresponding nonmonotonic theories have the same extensions.

Proof.

1) If $C$ is an extension of $A$ in $NM_D$, then $C = Th(A \cup D')$, for some $D' \subseteq D$, and $C$ is complete in $D$, cf. Remark 1. This $D'$ is also a subset of $\overline{D}$. We prove that $C$ is complete in $\overline{D}$ by induction on the formula $\varphi \in D$.

- If $\varphi \in D$. Then by the completeness of $C$ in $D$, $C \models \varphi$ or $C \models \neg \varphi$.
- If $\varphi \equiv \psi \lor \xi$. We have either $C \models \neg \psi$ and $C \models \xi$, or $C \models \psi$ and $C \models \xi$. In the former case $C \models \neg \varphi$.
- In the latter one $C \models \varphi$.
- If $\varphi \equiv \psi \land \xi$. We have either $C \models \psi$ and $C \models \xi$, or $C \models \neg \psi$ and $C \models \neg \xi$. In the first case $C \models \varphi$. In the second and third one $C \models \neg \varphi$.
- If $\varphi \equiv L\psi$. We have either $C \models \psi$, or $C \models \neg \psi$. In the former case $C \models \varphi$ by NEC. In the latter one $C \models \neg \varphi$ by M1.

2) If $C$ is an extension of $A$ in $NM_{\overline{D}}$, then $C = Th(A \cup (C \cap \overline{D}))$, and $C$ is complete in $\overline{D}$. Then $C$ is also complete in $D$. Define $C' = Th(A \cup (C \cap D))$. Clearly, $C' \subseteq C$. We intend to prove that $C \subseteq C'$. I.e., for a formula $\psi \in Lax$, if $A, \varphi_1, \ldots, \varphi_n \models \psi$, where $\varphi_1, \ldots, \varphi_n \in C \cap \overline{D}$, then there exist formulas $\xi_1, \ldots, \xi_m \in C \cap D$ such that $A, \varphi_1, \ldots, \varphi_n \models \psi$. For the proof we show that we can always decrease the positive number of additional logical operators $\land, \lor$, and $L$ in the formulas $\varphi_1, \ldots, \varphi_n \in C \cap \overline{D}$ over the formulas from $D$. Suppose that $A, \varphi_1, \ldots, \varphi_n \models \psi$, where $\varphi_1, \ldots, \varphi_n \in C \cap D$. Then one of the following holds.

- a) $\varphi_1 = \xi_1 \land \xi_2$. Then, by completeness of $C$ in $\overline{D}$, $\xi_1, \xi_2 \in C \cap \overline{D}$. Thus $A, \varphi_1, \ldots, \varphi_n, \xi_1, \xi_2, \varphi_{n+1}, \ldots, \varphi_m \models \psi$.
- b) $\varphi_1 = \xi_1 \lor \xi_2$. Then either $\xi_1 \in C \cap \overline{D}$, or $\xi_2 \in C \cap \overline{D}$. Clearly, $A, \varphi_1, \ldots, \varphi_n, \xi_1, \varphi_{n+1}, \ldots, \varphi_m \models \psi$.
- c) $\varphi_1 = L\xi$. Then $\xi \in C \cap \overline{D}$ and $A, \varphi_1, \ldots, \varphi_n, \xi, \varphi_{n+1}, \ldots, \varphi_m \models \psi$.

We proved that $C = C'$. Then $C = Th(A \cup (C \cap D))$, and thus $C$ is an extension of $A$ in $NM_{\overline{D}}$. \(\square\)
Remark 4. We can always assume that the set of defaults $D$ is closed under conjunction, disjunction, and $L$. By Lemma 1, the extensions and the nonmonotonic theories remain the same. We can also assume that $D$ contains the constants $\text{true}$ (conjunction of the empty set) and $\text{false}$ (disjunction of the empty set). This does not change the nonmonotonic theory, and would add at most one, inconsistent extension.

4. Degeneration of nonmonotonic theories

After proving different closure properties for the set of defaults in Lemma 1, the natural question is what about negation. The answer we give here is that if the set of defaults $D$ is closed under negation, then the corresponding nonmonotonic logic becomes monotonic. Moreover, in $S4$ for certain $D$'s the nonmonotonic default logic is monotonic if and only if $D$ is closed under negation.

By the definition of the nonmonotonic default logic, for a formula $\psi \in D$ we always have either $(L \psi) \in \text{TH}_D(A)$, or $(L \neg \psi) \in \text{TH}_D(A)$. So the formula $(L \psi) \lor (L \neg \psi)$ must belong to $\text{TH}_D(A)$. We formulate a somewhat stronger fact in the following definition and lemma.

Definition & Lemma 2. For the set of defaults $D$ we define the set $Ax_D$ of inherent axioms of $D$ by $Ax_D = \{(L \psi) \lor (L \neg \psi) : \psi \in D\}$. The following holds for this set of axioms.

(i) $\text{TH}_D(A) \supseteq Ax_D$,

(ii) If the underlying modal logic contains $T$, then $(L \psi) \lor (L \neg \psi) \models (L^k \psi) \lor (L^m \neg \psi)$ for $k, m = 1, 2, \ldots$.

That $Ax_D \models \{(L^k \psi) \lor (L^m \neg \psi) : \psi \in D, k, m = 1, 2, \ldots\}$.

Proof.

(i) Follows easily from the definition of $\text{TH}_D(A)$ by the NEC rule.

(ii) The proof uses Kripke models for $T$. Let $M = \langle W, R, D \rangle$ be a model of $T$ (i.e. $R$ is reflexive). Suppose that $M$ contains a state $w$ such that $w \models (L^k \psi) \lor (L^m \neg \psi)$ and that $w \models \psi$ (the case where $w \models \neg \psi$ is treated similarly). Then there exists a path $w = w_0, w_1, \ldots, w_n$, such that $w_n \models \neg \psi$. Thus for some $i < m$ we have $w_i \models \psi$ and $w_{i+1} \models \neg \psi$. Then $w_i \models L \psi \lor L \neg \psi$, and so $M \models L \psi \lor L \neg \psi$.

Definition 3. In this section we consider the following properties of nonmonotonic default modal logic.

1) For every formula $\psi \in D$ there exists a formula $\xi \in D$ such that $A, \xi \models \neg \psi$, and $A, Ax_D, \neg \psi \models \xi$, i.e. $D$ is "closed" under $\neg$ in $A$. 

2) $TH_D(A) = \text{Th}(A \cup Ax_D)$, i.e. the nonmonotonic inference on $A$ is equivalent to the usual one with the necessary additional axioms.

2') For every $A' \supseteq A$ holds $TH_D(A') = \text{Th}(A' \cup Ax_D)$, i.e. the nonmonotonic inference on the theories based on $A$ is equivalent to the usual one with the necessary additional axioms.

3) For every $A' \supseteq A$ holds $TH_D(A') \subseteq TH_D(A)$, i.e. adding new axioms to $A$ does not invalidate the assumptions made from $A$.

3') For every $A'$ and $A''$ such that $A'' \supseteq A$ holds $TH_D(A'' \cup Ax_D) \subseteq TH_D(A')$, i.e. the nonmonotonic inference is monotonic on the theories based on $A$.

The following two theorems show that the above properties of nonmonotonic theories are tightly connected.

**Theorem 1.** For any set of defaults $D$ and for any set of axioms $A$ we have

1 $\iff$ 2 $\iff$ 2'.

where $\iff$ denotes "equivalence", and $\implies$ denotes "implication".

**Proof.**

We shall prove the theorem in the following order: $1 \implies 2'$, $2' \implies 2$, $2 \implies 3$, $3' \implies 3$, $3 \implies 2'$.

1 $\implies$ 2'. Assume 1, and prove 2', i.e., that for every $A' \supseteq A$, $TH_D(A') \subseteq \text{Th}(A' \cup Ax_D)$. By Lemma 2 it is sufficient to prove $TH_D(A') \subseteq \text{Th}(A' \cup Ax_D)$. For a formula $\psi$, such that $A', Ax_D \vdash \psi$, we construct an extension of $A$ in $NM_D$ without $\psi$. Assume that $D = \{ \varphi_i \}_{i \in I}$, where $I$ is an ordinal, and construct inductively the sequence of theories

$A' = A \cup \ldots \cup A_\infty \cup \ldots$

where for every $n$, $A_n, Ax_D \vdash \psi$. Let $\{ A_i \}_{i \in \omega}$ be already constructed. Define $A_\infty = \bigcup A_i$. Trivially, $A_\infty, Ax_D \vdash \psi$ by the compactness of $\vdash$.

If $A_\infty, Ax_D, \varphi_i \vdash \psi$, define $A_n = A_\infty \cup \{ \varphi_i \}$. Clearly, $A_n \subseteq A_\infty, A_n \vdash \varphi_i$, and $A_n, Ax_D \vdash \psi$.

Otherwise, if $A_\infty, Ax_D, \varphi_i \vdash \psi$, we take a formula $\xi \in D$ such that $A_\infty \vdash \neg \varphi_i$ and $A, Ax_D, \neg \varphi_i \vdash \xi$, and define $A_n = A_\infty \cup \{ \xi \}$. Then $A_n \subseteq A_\infty$ and $A_n \vdash \neg \varphi_i$.

Assume that $A_\infty, Ax_D, \xi \vdash \psi$, then $A_\infty, Ax_D, \neg \varphi_i \vdash \psi$. By the deduction theorem (cf. Lemma A.1 in
We define the set of nonlogical axioms $A = \{Lq_i \lor L\neg q_i \} = Ax_w$, making the modal logic equivalent to the propositional one, and the set of defaults $D$ consisting of all the finite positive boolean expressions on the formulas from $(q_n)$. Let $\models_{PC}$ denote provability in the propositional calculus. The formulas $q_n$ are pairwise disjoint (inconsistent), and thus each consistent extension is generated by some $q_n$. Hence, a formula $\psi$ belongs to $TH_D(A)$ if and only if for every $n$, $q_n \models_{PC} \psi$. Because $\psi$ contains only finitely many variables, let $q_n$ be a propositional variable not appearing in $\psi$. Then $q_n \models_{PC} \psi \lor q_n, q_n \models_{PC} \psi$, and thus $\models_{PC} \psi$.

We proved that $TH_D(A) = Th(A \cup Ax_D)$, the condition 2 from Definition 3. But the condition 2' (and then, by Theorem 1, also the condition 1) does not hold, because the set of formulas $C = \{q_i \supset q_0\}_{i>0}$ is consistent, but for every $n>0$, $C \cup \{q_0\}$ is not. Hence, $q_0 = q_0 \notin TH_D(C)$ (recall that every consistent extension is generated by some $q_n$), but $C, Ax_D \models q_0$. Thus $Th(C \cup Ax_D) \neq TH_D(C)$.

Remark 5. The same result can be shown for the first order Modal Logic with finite signature. In Arithmetic with one additional monadic predicate $P(x)$, we define, for every $n \geq 0$, $q_n = P(1 + \cdots + 1)$. These formulas satisfy the same properties of independency that the propositional variables in the example above.

5. Nonmonotonic ground logic

One of the inconvenient properties of the nonmonotonic modal logic of McDermott is that a consistent theory may have no consistent extensions, i.e., be inconsistent in the nonmonotonic logic. This is true for many other nonmonotonic modal logics (cf. Proposition 1 below). A possible reason for this may be the lack of clear separation between the ground (first order) formulas about the real world, and the meta-formulas about knowledge, necessity, contingency etc. In this section we propose a nonmonotonic logic which seems to be much more comfortable to deal with.

The "nonmonotonic" rule looks as follows.

$$\vdash \neg q, q \text{ does not contain modalities}$$

$Mq$

More precisely, this logic is a $NM_{DC}$ nonmonotonic one with the set of defaults

$$DG = \{Mq_0 \supset q : q \text{ does not contain modalities} \}.$$ 

We shall call this logic $NMG$ (nonmonotonic ground logic). As in McDermott's logic, in $T$ or $S4$ a con-
sistent theory $[ML \varphi]$ is inconsistent in $NMG$, cf. Proposition 1 below. But, fortunately, this logic is free from other defects of McDermott's logic. First, it can be easily proved for $T$, $S4$ and $S5$ that a consistent theory without modalities always has a consistent extension, and in $S5$ this is true for any theory, even if it contains modal formulas. This implies, for example, that the empty 1-order nonmonotonic theory is consistent in $NMG$, even if the underlying logic is $T$ or $S4$. Second, $NMG$ does not degenerate in $S5$, it remains nonmonotonic. The precise statements of the above results are given below.

**Proposition 1.** Let the underlying modal logic be $T$ or $S4$, and let $\psi$ be a formula without modalities such that $\models \neg \psi$ and $\models \neg \neg \psi$. If $(M \models \psi) \in D \subseteq \{ M \models \varphi : \varphi \in S \}$, then the consistent set of axioms $[ML \psi]$ is nonmonotonically inconsistent.

**Proof.** Let $C$ be an extension of $[ML \psi]$. We contend that $C$ is inconsistent.

Assume that $C$ has a model $M = <W, R, D>$ with sufficiently large domain $D$. Recall that we deal here with the logic without the strong equality, and we can always add new elements indistinguishable from an existing one. Let $w$ be a (first order) model for $\neg \psi$ on a domain $D$ (because $\psi$ is a formula without modalities such that $\models \neg \psi$ and $\models \neg \neg \psi$). Consider a model $M' = <W', R', D>$, where $W' = W \cup \{ w \}$, and $R' = R \cup (\{ w \} \times W')$. If $M$ is a $T$ ($S4$) model, then $M'$ is also a $T$ ($S4$) one. Also, $M'$ satisfies $ML \psi$ and all the formulas from $C \cap D$. However, $M'$ doesn't satisfy $L \psi$. Hence $C \vdash L \psi$, and, by the "completeness" of $C$ in $D$, $C \vdash \neg M \psi$. This formula contradicts $ML \psi$ in $T$. $\Box$

**Proposition 2.** If the underlying logic is somewhere between $T$ and $S5$, and a consistent theory $A$ does not contain modalities, then $A$ has in $NMG$ a unique consistent extension $EX_A = Th(A \cup \{ M \models \varphi : \varphi \in DG, A \models \neg \varphi \})$.

**Proof.** First, notice that $EX_A$ is indeed an extension. We added only formulas from $DG$, and for every formula $M \models \varphi$ either $M \models EX_A$, or $L \models \varphi EX_A$.

Second, $EX_A$ is consistent. To prove this we use the corollary of the well known Löwenheim-Skolem theorem, i.e., that in the first order logic every consistent theory in a signature $\sigma$ has a model of the power $\max(\aleph_0, |\sigma|)$. The equality in this model does not have to be the real equality. It is an equivalence relation satisfying the standard properties of the equality. There may be different elements in the domain which are equal in the model. So, for every consistent first order theory containing $A$ we construct its
model on the same domain of the power max(κ₁, σ₁). Then we marry all these models into an S5 model making all the states connected one with another. This model satisfies EX₄. Notice that here we use the fact that the underlying logic is the sublogic of S5, i.e. allows S3 models.

Third, no consistent extension C contains a formula MₜₑₑDG such that A ⊨ ¬φ. Thus, such a C must be a subset of EX₄. Also, for any MₜₑₑEX₄ ∩ DG, EX₄ ⊨ ¬φ because EX₄ is consistent. Then C ⊨ ¬φ because C ⊆ EX₄, and MₜₑₑC because C is an extension. Hence, C = EX₄.

Proposition 3. Let the underlying logic be S5. If the theory A is consistent, then TH₀D(A) is also consistent.

Proof. Assume that DG = (Mₜₑₑ)ₗ∈I, where I is an ordinal, and construct inductively the sequence of consistent theories

A = A₀ ⊆ ... ⊆ Aₙ ⊆ ... .

Let (Aₙ)₁ⁿ be already constructed. Define aₙ = ∪ Aₙ. aₙ is consistent by the compactness of ⊨.

If aₙ, (Mₜₑₑ) is consistent, define Aₙ+₁ = aₙ ∪ (Mₜₑₑ). Clearly, aₙ ⊆ Aₙ, Aₙ ⊨ Mₜₑₑ, and Aₙ is consistent.

Otherwise, if aₙ, (Mₜₑₑ) is inconsistent, then, by the deduction theorem for extensions of S₄ (cf. Corollary to Lemma A.1 in Appendix), aₙ ⊨ (Mₜₑₑ) ⊨ false. In S₅ this formula is equivalent to L¬φ. Then we take Aₙ+₁ = aₙ. Clearly, aₙ ⊆ Aₙ, Aₙ ⊨ ¬φₙ, and Aₙ is consistent.

Thus, C = Th (∪ₙ Aₙ) is a consistent extension of A (we added only elements of DG).

Now we shall study when the nonmonotonic logic NMG degenerates. Clearly, if a theory A is complete in the first order logic, then TH₀D(A) = Th(A) (property 2 from Definition 2) holds, and then, by Theorem 1, we prove that the nonmonotonic logic NMG is monotonic on extensions of A (property 3 from Definition 2). We show that NMG is nonmonotonic on extensions of every incomplete theory without modalities.

Proposition 4. If the underlying logic is somewhere between T and S5, and a consistent theory A without modalities is not complete in the first order logic, then there exists a theory A' ⊆ A without modalities, such that TH₀D(A') ⊊ TH₀D(A).

Proof. If a consistent theory A without modalities is not complete, then there is a formula φ without
modalities, such that \( A \vdash \neg \varphi \) and \( A \vdash \varphi \). Then, by Proposition 2, \( M \models TH_{DG}(A) \). Also, the theory \( A \cup \{ \neg \varphi \} \) is consistent. Hence, by the same Proposition 2 it has a consistent extension, and then \( M \models TH_{DG}(A \cup \{ \neg \varphi \}) \). Thus, \( TH_{DG}(A \cup \{ \neg \varphi \}) \models TH_{DG}(A) \).

Appendix

We prove here some modal logic metatheorems used in the paper.

First, we present the proof of deduction theorem for modal logic. The proof is well-known from the folklore, but, unfortunately, no reference is known to us.

**Lemma A.1.** Let \( \varphi \) be a sentence, and let the modal logic be an extension of \( T \). If \( A, \varphi \models \psi \), then for some \( k \geq 0 \) we have \( A \vdash L^k \varphi \supset \psi \).

**Proof.** The theorem is proved by induction on the length of deduction of \( \psi \) from \( A \cup \{ \varphi \} \). Recall that a deduction of \( \psi \) from \( A \cup \{ \varphi \} \) is a sequence of formulas \( \varphi_1, \varphi_2, \ldots, \varphi_n = \psi \) such that each \( \varphi_i \) is a logical axiom, or belongs to \( A \cup \{ \varphi \} \), or is obtained from the formulas \( \varphi_1, \varphi_2, \ldots, \varphi_{i-1} \) by one of the rules of inference.

If \( \varphi_n \) is a logical axiom, or \( \varphi_n \in A \), then \( A \vdash \varphi_n \), which implies \( A \vdash \varphi \supset \varphi_n \).

If \( \varphi_n \) is \( \varphi \), then, trivially, \( A \vdash \varphi \supset \varphi_n \).

Assume that \( \varphi_n \) is obtained from \( \varphi_i \) and \( \varphi_i \supset \varphi_n \) by modus ponens. Then, by induction hypothesis we have \( A \vdash L^k \varphi \supset \varphi_i \), and \( A \vdash L^k \varphi \supset (\varphi_i \supset \varphi_n) \). Applying M1 sufficiently many times, if necessary, we may assume that \( k = k_2 \). Now the situation is similar to that occurring in the case of predicate calculus, cf. [Me, Proposition 2.4, p. 61].

Assume that \( \varphi_n \) is obtained from \( \varphi_i \) by generalization, i.e., \( \varphi_n = \forall x \varphi_i \). Then, by induction hypothesis, for some \( k \geq 0 \) we have \( A \vdash L^k \varphi \supset \varphi_i \). Again this case can be treated similarly to that of predicate calculus, cf. [Me, Proposition 2.4, p. 61].

Assume that \( \varphi_n \) is obtained from \( \varphi_i \) by NEC, i.e., \( \varphi_n = \exists x \varphi_i \). By the induction hypothesis, for some \( k \geq 0 \), we have \( A \vdash L^k \varphi \supset \varphi_i \). Applying NEC and M2 we obtain \( A \vdash L^{k+1} \varphi \supset \exists x \varphi_i \), i.e., \( A \vdash L^{k+1} \varphi \supset \varphi_n \), which completes the proof of the theorem. \( \square \)
Now we turn to prove the mutual inconsistency of $\mathcal{A}^* \cup \mathcal{A}^*$ in propositional $\mathcal{S}_5$. For $i \neq j$, the theory $A \cup A_i \cup A_j$ is inconsistent, i.e., $A, \xi_1, \ldots, \xi_n \vdash \text{false}$ in $\mathcal{S}_4$ for some formulas $\xi_1, \ldots, \xi_n \in A \cup A_i \cup A_j$. Then, by Corollary to Lemma A.1, $A \vdash \bigwedge_{\xi \in \mathcal{A}} \xi \vdash \text{false}$, i.e. $A \vdash \bigwedge_{\xi \in \mathcal{A}} \neg L \xi$. That is the formula $\bigwedge_{\xi \in \mathcal{A}} \neg L \xi$ belongs to $A^*$, therefore its translation $\bigwedge_{\xi \in \mathcal{A}} \neg L \xi$ belongs to $A^*$. But, the theory $\langle \bigwedge_{\xi \in \mathcal{A}} \neg L \xi, \xi_A^*, \ldots, \xi_A^* \rangle$ is inconsistent even in propositional $\mathcal{T}$, and the formulas $\xi_A^*, \ldots, \xi_A^*$ belong to $A^* \cup A^*$. Hence $A^* \cup A^* \cup A^*$ is also inconsistent in propositional $\mathcal{S}_5$.

In $\mathcal{S}_5$ every formula is equivalent to a formula of modal degree less or equal to 1 (cf. [HC, p. 55]). Hence, propositional $\mathcal{S}_5$ with finite signature has only finite number of nonequivalent formulas. Thus there may be only finitely many different theories in such $\mathcal{S}_5$. □

References


