THE DIOPHANTINE PROBLEM OF FROBENIUS:
A CLOSE BOUND

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ABSTRACT

The conductor of \( n \) positive integer numbers \( a_1, a_2, \ldots, a_n \), whose greatest common divisor is equal to 1, is defined as the minimal \( K \), such that for every \( m \geq K \), the equation \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = m \), has a solution over the nonnegative integers. In this note we give a polynomial algorithm computing a close bound \( B \) for the conductor \( K \) of \( n \) given positive integers, when \( n \) is fixed. The bound \( B \) satisfies \( B/\sqrt{n} \leq K \leq B \).
INTRODUCTION

Definition: The conductor of \( n \) positive integer numbers \( a_1, a_2, \ldots, a_n \), whose greatest common divisor is equal to 1, is the minimal \( K \), such that for every \( m \geq K \), the equation \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = m \) has a solution over the nonnegative integers. We denote this minimum by \( K = \mathcal{K}(a_1, a_2, \ldots, a_n) \).

The Diophantine Problem of Frobenius is to determine the conductor of \( n \) given positive integers \([\text{Fro}, \text{Bra}]\).

In this paper we present an algorithm for the computation of a close bound \( B \) for the conductor \( K \). This bound has the property that \( B/n \leq K \leq B \). For fixed \( n \), the time complexity of the algorithm is bounded by a polynomial in the length of the input integers \( a_1, a_2, \ldots, a_n \).

Up to the present no polynomial algorithm for the computation of the conductor for \( n > 3 \) has been found, not even for the case where the number \( n \) of integers is not part of the input. For the case of \( n = 2 \), a very simple solution due to Sylvester is known \([\text{Syl}]\).

While no such solution is known for \( n = 3 \), it is easy to show that a polynomial algorithm can be derived from the work of Brauer and Shockley \([\text{BS}]\) combined with Lenstra's polynomial algorithm for solving integer linear programs with a fixed number of variables \([\text{Len}]\). We are not aware of any polynomial algorithm existing or implied in the literature for the \( n \geq 4 \) case. A non-polynomial algorithm for the computation of the conductor can be found in \([\text{Nij}]\).

Several authors have tried to find a good upper bound for the conductor (see \([\text{Sel}, \text{Sch}]\) for an extensive bibliography). Many such bounds have been found but all those bounds are of the order of magnitude of the square of the minimal \( a_i \) or higher. The bound we give in this note is, as far as we know, the first bound which is, for fixed \( n \), of the same order of magnitude as the conductor and computable in polynomial time.

THE MAIN RESULT

Let us denote by \( \alpha_i, 1 \leq i \leq n \), the minimal integer \( \alpha \) such that there exists a solution over the nonnegative integers to the equation

\[
a_1 x_1 + \cdots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \cdots + a_n x_n = \alpha a_i - 1
\]

Denote

\[
B = (\alpha_i - 1) a_1 + (\alpha_{i-1} - 1) a_2 + \cdots + (\alpha_n - 1) a_n
\]

Theorem 1: Let \( K \) be the conductor of \( n \) positive integers \( a_1, a_2, \ldots, a_n \), whose greatest common divisor is equal to 1, and let \( B \) be defined as above then

\[
\frac{B}{n} \leq K \leq B
\]

Proof: First we prove the lower bound. We must show that \( B/n \leq K \). We do this by showing for every \( i, 1 \leq i \leq n \), that \( (\alpha_i - 1) a_i \leq K \). If this is not true, then it follows from the definition of \( K \) that the equation \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = (\alpha_i - 1) a_i - 1 \), has a
solution over the nonnegative integers, implying that the equation

\[ a_1 x_1 + \cdots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \cdots + a_n x_n = (a_{i-1} - x_i) a_i - 1 \]

has such a solution, contradicting the minimality of \( \alpha_i \).

To prove the upper bound we shall show that for any \( m \geq B \), there exists a solution to the equation \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = m \). We prove this by showing that if for any \( m > B \) a solution exists, then a solution exists also for \( m-1 \). Let \( \beta_1, \beta_2, \ldots, \beta_n \), be a solution for such an \( m \). As \( m > B \), there exists some index \( i, 1 \leq i \leq n \), such that \( \beta_i > \alpha_i - 1 \). On the other hand, by the definition of \( \alpha_j \), there exist nonnegative integers \( \alpha'_1, \ldots, \alpha'_{i-1}, \alpha'_{i+1}, \ldots, \alpha'_n \) such that

\[ \alpha'_1 a_1 + \cdots + \alpha'_{i-1} a_{i-1} - \alpha_i a_i + \alpha'_{i+1} a_{i+1} + \cdots + \alpha'_n a_n = -1 \]

Combining the two equations we get that

\[ \beta_1 + \alpha'_1, \ldots, \beta_{i-1} + \alpha'_{i-1}, \beta_i - \alpha_i, \beta_{i+1} + \alpha'_{i+1}, \ldots, \beta_n + \alpha'_n \]

is a (nonnegative) solution for \( m-1 \). \( \square \)

**Theorem 2:** The bound \( B \) can be computed in polynomial time for every fixed value \( n \).

**Proof:** The equation (*) can be solved for any value of \( \alpha \) as a linear integer program, and therefore every \( \alpha_i \) can be found by binary-search of the minimal value of \( \alpha \) for which a solution to (*) exists. Using Lenstra’s polynomial algorithm for the Integer Linear Programming [Len], we can compute the bound \( B \) in polynomial-time when \( n \) is fixed. \( \square \)

**Remark 1:** In the proof of Theorem 1 we have shown that \( (\alpha_i - 1)a_i \leq K \). Combining this with the well-known bound \( K \leq (a_{\text{min}} - 1)(a_{\text{max}} - 1) \) [Bra] (where \( a_{\text{min}}, a_{\text{max}} \) are the minimal and maximal elements, respectively, among \( a_1, a_2, \ldots, a_n \)), we get that \( \alpha_i < a_{\text{max}} \), a bound that can be used for initialization in the above binary search for \( \alpha_i \).

**Remark 2:** The above bound also induces a bound for the number of nonnegative integers \( m \) for which no solution exists to the equation \( a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = m \). This number, denoted by \( N \) was also investigated in the literature [NW] and it is easy to prove that \( K/2 \leq N \leq K \). Thus, our bound provides also a close bound for \( N \). Namely, \( B/2n \leq N \leq B \).
REFERENCES


