ON THE RANK OF EDMONDS POLYTOPES

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ABSTRACT

Let \( P \) be a bounded polyhedron, given by the linear system \( \{Ax \leq b\} \). The convex hull of the integral points of \( P \), denoted by \( E(P) \), has been called by Chvátal: the Edmonds Polytope of \( P \). He also defined the term rank of an integer linear programming (with a given linear system \( S \)). The rank of Edmonds polytope, \( E(P) \), where \( P \) is given by a linear system \( S \), is defined analogously.

In general, it seems that the rank of \( E(P) \) gives an indication on the difficulty of describing \( E(P) \) by a linear system, and (as was pointed out by Chvátal) of solving the associated combinatorial problem.

We discuss two theorems concerning the rank of \( E(P) \) and their usefulness for estimating the difficulty of combinatorial optimization problems. We briefly discuss a combinatorial optimization problem with rank of \( E(P) > 1 \).

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Chvátal in [2] introduced the notion of an Edmonds polytope and formulated in [3] the following consequence of Gomory’s cutting planes method [9] (see also [14] for a geometric approach). For graph-theoretical terminology we refer the reader to [1] and for the theory of polyhedra to [15]. Let $S$ be a set of linear inequalities that determines a bounded polyhedron $P$. The closure of $S$ is the smallest set of inequalities that contain $S$ and is closed under two operations:

(i) taking linear combinations of inequalities,

(ii) adjoining the inequality $\sum a_jx_j \leq a$ where $a_1, a_2, \ldots, a_n$ are integers if the inequality $\sum a_jx_j \leq a_0$ with $\left\lfloor a_0 \right\rfloor \leq a$ is present.

The Edmonds polytope, $E(P)$, is the convex hull of all the integral points in $P$ and Chvátal has shown that the closure of $S$ determines it. The elementary closure of $S$ is defined as the set that contains every inequality which is the result of an application of (i) on $S$ and then an application of (ii). $e'(S)$ denotes the elementary closure of $S$. For any $k > 1$, $e^k(S)$ is defined recursively by $e^k(S) = e'(S \cup e^{k-1}(S))$. The rank of $E(P)$ is the minimum $k$ for which $e^k(S)$ determines $E(P)$.

Let $P$ be a class of bounded polyhedra, defined by a class of systems of linear inequalities $S$. The rank of $E(P)$ is defined by: $\operatorname{rank} E(P) = \max \{ \operatorname{rank} E(P) : P \in P \}$.

The matching polytope is a classical example of an Edmonds polytope: Edmonds [7] proved that if we take the class of polytopes $P$, defined by the class of linear systems $S$: $A x \leq 1, x \geq 0$, where $A$ is the node-edge incidence matrix of a graph $G$, then $E(P)$ has rank $= 1$.

For many combinatorial problems, it is not difficult to define a class of systems of linear inequalities $S$ that defines a class of polytopes $P$ such that for every instance, the convex hull of all the solutions to the problem is exactly $E(P)$. However, in many cases $E(P) \neq P$ (i.e. rank $E(P) > 0$) and a linear description of $E(P)$ for every $P \in P$ might lead to a good characterization or even to a good algorithm for that problem.

Karp and Papadimitriou [10] proved that unless $NP = co-NP$, there is no $NP$ description of the facets of $E(P)$ for every instance of an associated $NP$-complete problem. Chvátal [3] showed that there is no upper bound on the rank of $E(P)$ for (all the instances of) the problem of maximum independent set of a graph (i.e. rank $E(P) = \infty$).
In general, it seems that the rank of $E(P)$ gives an indication on the difficulty of describing $E(P)$ by a class of linear systems, and (as was pointed out by Chvátal) of solving the associated combinatorial problem. In fact, if the rank of $E(P)$ is greater than 1, it probably indicates that the description of $E(P)$ by a class of systems of linear inequalities is extremely difficult. However, the associated combinatorial problem might still be solved by a polynomial algorithm as is the case in the problem of finding maximum weight independent set in claw-free graphs. Giles and Trotter [8] proved that the rank of $E(P)$ is greater than 1 where $P$ is given by the clique inequalities, and as far as we know, no description of all the facets of $E(P)$ is known. On the other hand, there is a polynomial algorithm for solving this problem [13]. Another problem with rank $> 1$, for which the complexity is still open, is mentioned in the sequel. Several other works on the relation between cutting planes, and combinatorial proofs and complexity, appeared in [4,5,6].

The following theorem is a characterization of full dimensional Edmonds polytopes of rank $> 1$.

**Theorem 1.** Let $P$ be a full dimensional bounded polyhedron defined by $\{x: Ax \leq b\}$ and assume that $E(P)$ is full dimensional. The following are equivalent:

(i) $\text{rank}(E(P)) > 1$

(ii) $\exists w, c_0$ integral such that $wx \leq c_0$ is a facet inequality of $E(P)$ and $\max \{wx: x \in P\} \geq c_0 + 1$.

Before proving this theorem, we would like to state a theorem by Chvátal, that can be used to prove lower bounds on the rank of $E(P)$ in some cases. He used it in [3] to show that there is no upper bound on the rank of the problem of finding maximum independent set in a graph.

Unfortunately, this method cannot be used in general; we provide here an example for which this method cannot be applied.

An inequality $\sum a_j x_j \leq b$ is called positive regular if $a_j \geq 0 \ (j = 1,2,\ldots,n)$ and $b \geq \max(a_1,a_2,\ldots,a_n) > 0$. The strength of such an inequality is the ratio $\frac{\sum a_j}{b}$. A linear inequality is called negative regular if it reads $-x_j \leq 0$. 


Theorem 2. (Chvátal [3]). Let $S$ be a set of regular inequalities; let $k$ be a positive integer. If all positive regular inequalities in $S$ have strength $\leq s$ then all positive regular inequalities in $e^k(S)$ have strength $< 2^k s$. □

Given a set $S$ of regular inequalities that defines $P$ such that the strength of every positive regular inequality is $\leq s$, we can show that \( \text{rank}(P) > k \) if we could find a positive regular inequality which is valid for $E(P)$ and has strength $\geq 2^k s$.

Observation 1. Let $P$ be a full dimensional bounded polyhedron, given by a regular system of inequalities, such that $\text{rank} E(P) > 1$. It is not always possible to use the method of Theorem 2 for proving that the rank of $E(P)$ is greater than 1.

Proof of observation 1. The following example shows that a positive regular inequality, with an adequate strength, does not always exist.

Example 1. Consider the following regular system:

\[
\begin{align*}
3x + y &\leq 6 \\
x + 3y &\leq 6 \\
x, y &\geq 0
\end{align*}
\]

First let us use Theorem 1 to show that the rank of $P$ is $> 1$. The set of integral points in $P$ is \{(0,0), (0,1), (1,0), (1,1), (2,0), (0,2)\}. Clearly, \( x + y = 2 = c_0 \) is a facet of $E(P)$.

\[\max(x + y: (x,y)\in P) = 3 \geq c_0 + 1 = 2 + 1,\]

hence by Theorem 1, \( \text{rank} E(P) > 1. \)

The strength of any positive regular inequality here is \( \frac{3+1}{6} = \frac{2}{3} \). Any positive regular inequality in the closure of $P$ has the form:

\[ax + by \leq c \quad \text{where} \quad a, b \geq 0 \quad \text{and} \quad c \geq \max\{a, b\}\]

and

\[
\frac{c}{a} \geq 2; \quad \frac{c}{b} \geq 2 \Rightarrow a \leq \frac{c}{2}; \quad b \leq \frac{c}{2}.
\]

Hence, the strength of such an inequality is \( \frac{a+b}{c} \leq \frac{12+c/2}{c} = 1 \). That is to say: there is no positive regular inequality in the closure of $P$ that has strength $\geq 2 \cdot \frac{2}{3} = \frac{4}{3}$ and, therefore, we cannot apply Theorem 2 to conclude that \( \text{rank} E(P) > 1. \) □
Remark. The method of Theorem 2 cannot be applied to linear systems where there are inequalities which are not regular. However, the method of Theorem 1 is applicable also in that case.

Example 2. Consider the following linear system of inequalities (which is a non-regular system) from the example in [3] (which is attributed to J.A. Bondy):

\[
P \begin{cases} 
0 \leq x \leq 1 \\
y \geq 0 \\
-tx + y \leq 1 \\
tx + y \leq t+1
\end{cases}
\]

where \( t \) is an arbitrary positive number.

Clearly \( y \leq 1 \) is a facet inequality of \( E(P) \) (since \( E(P) = \text{convex hull of} \{ (0,0),(0,1),(1,0),(1,1) \} \) and \( y \leq 1 \) is satisfied as equality by \((0,1)\) and \((1,1)\)). However, \( \max \{ y : (x,y) \in P \} = \frac{t}{2} + 1 \). So for \( t \geq 2 \) we have \( \max \{ y : (x,y) \in P \} \geq c_0 + 1 \) where \( c_0 = 1 \). Hence, by using Theorem 1, we have proved that for \( t \geq 2 \), \( \text{rank} \ E(P) > 1 \). □

Example 3. (Integral packing of T-cuts)

Let \( G = (V,E) \) be a graph and a weight function \( w : E \to \mathbb{Z}^+ \) (where \( \mathbb{Z}^+ \) is the set of the non-negative integers). Let \( T \subseteq V \) be an even subset of the vertices of \( G \). A \( T \)-cut is an edge-cutset of the graph which divides \( T \) into two odd sets. Let \( A \) be the incidence matrix of \( T \)-cuts in \( G \). Then the maximum weighted integral packing of \( T \)-cuts problem is the following integer linear programming problem:

\[
\max \{ yc : y \in P, y \text{ integral} \}
\]

where \( P \) is given by the system: \( \{ yA \leq w, y \geq 0 \} \) and \( c \) is the vector of all ones.

The complexity of this problem is not known (i.e. polynomial? \( NP \)-hard? or other?) and some partial results have been proved (e.g. [11,12,16]). The rank of the \( E(P) \) of this problem is greater than one as can be seen from the following lemma. This gives some indication that this problem is not easy to solve.

Lemma 1. Let \( A \) be the matrix of \( T \)-cuts in \( K_4 \) (the complete graph on 4 vertices) with \( T = V(K_4) \).

Then \( \text{rank} \ E(P) > 1 \) where \( P \) is given by: \( \{ yA \leq 1, y \geq 0 \} \), or:

\[
\begin{align*}
y_1 + y_2 & \leq 1 \\
y_1 + y_3 & \leq 1 \\
y_1 + y_4 & \leq 1 \\
y_2 + y_3 & \leq 1 \\
y_2 + y_4 & \leq 1 \\
y_3 + y_4 & \leq 1 \\
y_i & \geq 0 \ (i = 1,2,3,4)
\end{align*}
\]
Proof. Clearly, \( y_1 + y_2 + y_3 + y_4 \leq 1 = c_0 \) is a valid inequality for \( E(P) \) and since \((1,0,0,0), (0,1,0,0), (0,0,1,0)\)
and \((0,0,0,1)\) satisfy it as an equation, it is a facet inequality of \( E(P) \).

\[
\max \{ y_1 + y_2 + y_3 + y_4: y \in P \} = 2 = v^*
\]

and is achieved by \( y = \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] \) hence, \( v^* \geq c_0 + 1 \), and by Theorem 1 \( \text{rank } E(P) > 1 \).

It follows that the problem of \textit{maximum packing of T-cuts} in a graph has \textit{rank} \( > 1 \). \( \square \)

Let us now proceed by the

**Proof of Theorem 1:** (ii) \( \Rightarrow \) (i) (Giles and Trotter [8]).

Since \( wx \leq c_0 \) is a facet inequality of \( E(P) \), it cannot be obtained as a non-negative linear combination of any other valid inequalities of \( E(P) \) (except for the trivial ones); clearly, it does not belong to \( P \) since \( \max\{wx: x \in P\} \geq c_0 + 1 \).

If it is in \( e'(P) \) then \( \exists \lambda \geq 0 \) such that \( \lambda A = w \) and \( \lfloor \lambda b \rfloor = c_0 \) or \( \exists \lambda \geq 0 \) such that \( \lambda A = w \) and \( \lambda b < c_0 + 1 \leq \max\{wx: x \in P\} \) which is impossible since such a \( \lambda \) is feasible for the dual l.p. of \( \max\{wx: x \in P\} \) but the dual objective at this point is strictly smaller than the primal one.

Hence, we have \( wx \leq c_0 \notin e'(P) \).

(i) \( \Rightarrow \) (ii)

Since we assume that \( E(P) \) is full-dimensional, there exists a facet of \( E(P): wx \leq c_0 \notin e'(P) \) where \( w' \) and \( c'_0 \) are integral.

We have to consider two cases: (1) \( \max\{w'x: x \in P\} = v^* \) is an integer, (2) \( v^* \) is not an integer.

(1) \( v^* \) is an integer. By duality theory \( \exists \lambda \geq 0 \) such that \( \lambda A = w' \) and \( \lambda b = v^* \). \( v^* \neq c'_0 \) since \( wx \leq c'_0 \notin e'(P) \).

Hence \( v^* \geq c'_0 + 1 \).

(2) \( v^* \) is not an integer \( \Rightarrow \min\{\lambda b: hA = w' \lambda \geq 0\} = v^* \) and since \( wx \leq c'_0 \notin e'(P) \) we have \( v^* > c'_0 \) and since \( c'_0 \) and \( v^* \) are integral we have \( v^* \geq c'_0 + 1 \) or \( v^* \geq c'_0 + 1 \). \( \square \)
Open problem:

*Theorem* 1 characterizes Edmonds polytopes with \( \text{rank} > 1 \). For each \( k \geq 2 \), find a similar characterization of full dimensional Edmonds polytopes with \( \text{rank} > k \).
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REFERENCES