TOTAL DUAL INTEGRALITY OF COVERS OF ODD CIRCUITS
AND NP-HARDNESS OF THE NEGATIVE CUT PROBLEM

by

E. Korach and M. Penn

Technical Report #440

November 1986

* Faculty of Industrial and Management Engineering, Technion
TOTAL DUAL INTEGRALITY OF COVERS OF ODD CIRCUITS AND NP-HARDNESS OF THE NEGATIVE CUT PROBLEM

by

Ephraim Korach* and Michal Penn**

Technion - Israel Institute of Technology
Haifa 32000, Israel

ABSTRACT

Consider the clutter \( C_o \) of odd circuits in an undirected graph, with incidence matrix \( C \). The integral solutions of the system \( CC' = \{ x \in R^E : Cx \geq 1, x \geq 0 \} \) are transversals of \( C_o \). \( D_o \) is the blocker of \( C_o \) with the analogous system \( CD = \{ x \in R^E : Dx \geq 1, x \geq 0 \} \). Barahona gave excluded minors characterizations of graphs for which \( CC \) is TDI and graphs for which \( CD \) is TDI. We derive from Barahona's results, excluded subgraph characterizations of these systems.

Given \( w : E \rightarrow R^+ \), the problem : \( MCC = \min \{ wx : x \in CC ; x \text{ integral} \} \) is known to be NP-complete (equivalent to the max-cut problem). Let \( f : E \rightarrow \{ +1, -1 \} \), we say that a cut \( C \) is negative if \( \sum w(e) \cdot f(e) : e \in C \) < 0: The negative cut problem is to find a negative cut in a given graph \( G \) or, to decide that there is no negative cut in \( G \). We show that the NP-completeness of MCC implies NP-hardness of the negative cut problem.

---

* Computer Science Dept., Technion-IIT, Haifa, Israel.
** Faculty of Industrial and Management Engineering, Technion-IIT, Haifa, Israel.
1. INTRODUCTION

If \( G = (V, E) \) is an undirected graph, and \( S \subseteq V \), \( \delta(S) \) denotes the set of edges with exactly one extremity in \( S \), \( \delta(S) \) is called a cut. If \( C_o \) is the clutter of odd circuits in \( G \) with an incidence matrix \( C \), the integral solutions of the system \( CC = \{ x \in R^E : Cx \geq 1, x \geq 0 \} \) are the covers of \( C_o \).

Given \( w: E \rightarrow R_+ \), the problem \( MCC = \min \{ wx : x \in CC, x \text{ integral} \} \) is known to be NP-complete (equivalent to the max-cut problem). Let \( f: E \rightarrow \{+1,-1\} \). We say that a cut \( \delta(S) \) is negative if \( \sum_{e \in \delta(S)} w(e) f(e) < 0 \). The negative cut problem is to find a negative cut in a given graph \( G \) or to decide that there is no negative cut in \( G \). In Chapter 2 we show that the NP-completeness of the MCC implies the NP-hardness of the negative cut problem.

Let \( T \subseteq V \), \( |T| \) even, \( a T \)-cut is a cut which divides \( T \) into two odd sets. In Chapter 3, we present some similarities and some important differences between the following two problems: finding minimum cover of odd circuits and finding minimum cover of \( T \)-cuts.

Consider the clutter \( C_o \), and the system \( CC = \{ x \in R^E : Cx \geq 1, x \geq 0 \} \). Let \( D \) be the blocker of \( C_o \) with the analogous system \( CD = \{ x \in R^E : Dx \geq 1, x \geq 0 \} \). Barahona gave excluded minors characterizations of graphs for which \( CC \) is TDI and graphs for which \( CD \) is TDI. In Chapter 4, we derive from Barahona's results, excluded subgraphs characterizations of these systems.

2. ON THE MAX-CUT AND NEGATIVE-CUT PROBLEMS

The max-cut problem can be stated as follows: Given a graph \( G = (V,E) \) with edge weights \( w(e) > 0 \) for all \( e \in E \), find a cut \( \delta(S) \) such that \( w(\delta(S)) = \sum_{e \in \delta(S)} w(e) \) is as large as possible. Let \( C_o \) be the collection of all odd circuits in \( G \). \( X \subseteq E \) is a cover of odd circuits if for each \( C_o \in C_o \), \( |X \cap C_o| \geq 1 \). The minimum cover problem is to find a cover \( X \) such that \( w(X) = \sum_{e \in X} w(e) \) is as small as possible. It is easy to see that if \( X \) is a minimum cover then \( E - X \) is a maximum cut. From now on we shall consider the problem of minimum covering odd circuits
instead of considering the problem of max-cut.

Definitions: Let \( G = (V,E) \) be a graph, \( C \) the collection of circuits, \( C_e \) the collection of even circuits, \( B \subseteq C \) a base of the cycle space of \( G \). Let \( |C| := |E(C)| \).

Theorem 1: \( X \subseteq E \) is a minimal cover of odd circuits if and only if the following conditions hold:

1. For each \( C_o \in C_o \cap B, |C_o \cap X| \) is odd.
2. For each \( C_e \in C_e \cap B, |C_e \cap X| \) is even.
3. \( X \) is a minimal set for which the above conditions hold.

Proof:

(i) \( X \) is a minimal cover \( \Rightarrow \) the conditions hold

1. Assume that (1) does not hold, then there exists a circuit \( C_o \in C_o \) such that \( |C_o \cap X| = 2n \). Since \( X \) is a minimal cover then for each \( x \in X \cap C_o \) there exists an odd circuit \( C_o^i \in C_o, i = 1,\ldots,2n \), such that \( |C_o^i \cap X| = 1 \). Let \( C = C_o \Delta C_o^1 \Delta \cdots \Delta C_o^{2n} \). \( C \) is a collection of edge-disjoint circuits and \( |C| \) is odd, so there exists \( C_o' \in C \cap C_o \) such that \( |C_o' \cap X| = \Phi \), a contradiction.

2. Assume that (2) does not hold, then there exists an even circuit \( C_e \) such that \( |C_e \cap X| = 2n+1 \). Let \( C_e^i \in C_o \) be such that \( |C_e^i \cap C_e \cap X| = 1, i = 1,\ldots,2n+1 \). Let \( C = C_e \Delta C_o^1 \Delta \cdots \Delta C_o^{2n+1} \). Since \( |C| \) is odd there must be \( C_o \in C \cap C_o \) such that \( |C_o \cap X| = \Phi \), a contradiction. So, (1) and (2) hold for all circuits (bases or not).

3. Assume that (3) does not hold, then there exists a set \( Y \subset X \) such that conditions (1) and (2) hold for \( X-Y \) and \( X \) is a minimal cover. For each \( y \in X \cap Y \) there exists an odd circuit \( C_y \in C_o \) such that \( |C_y \cap X| = \{y\} \). Therefore \( C_y \cap (X-Y) = \Phi \). Hence condition (1) is not satisfied.
(ii) The conditions hold \( \Rightarrow \) \( X \) is a minimal cover

We shall show briefly that if conditions (1) and (2) hold for each circuit \( b \in B \) then they hold for each circuit \( C \in C \).

Let \( b_o \in B \cap C_o \) and \( b_e \in B \cap C_e \). If \( C_o \in C_o \) then \( C_o = b_o \Delta \cdots \Delta b_{ok} \Delta b_{e1} \cdots \Delta b ej \)

where \( k \) is an odd number. Since (1) and (2) hold for each \( b \in B \) we have that 

\(|b_o \Delta \cdots \Delta b_{ok} \cap X| \) is odd and \(|b_{e1} \Delta \cdots \Delta b ej \cap X| \) is even. It follows that \(|C_o \cap X| \)

is odd. So, (1) holds for each \( C_o \in C_o \). In a similar way (2) holds for each \( C_e \in C_e \).

Since (1) holds for each \( C_o \in C_o \) we have that \(|C_o \cap X| \geq 1 \) which implies that \( X \) is a cover.

It remains to show the minimality of \( X \). Assume \( X \) is not a minimal cover. Then there is a set \( \emptyset \neq Y \subset X \) such that \( (X-Y) \) is a minimal cover.

So, from (i) we have that conditions (1), (2) and (3) hold for it, contradicting the minimality of \( X \).

Q.E.D.

Remark: We cannot omit any one of these three conditions. Clearly (1) and (3) are necessary. We shall show the necessity of (2) with an example.

Let \( G \) be the following graph:

![Graph](image)

Let \( T \) be a spanning tree of \( G \):

![Spanning Tree](image)

Then \( B = \{(1,5,2),(1,5,4,3)\} \)

Let \( X = \{1\} \). Then (1) and (3) hold for it while \( X \) is not a cover.

Definition: \( \overline{X} = E - X \).

Lemma 2: Let \( X \) be a minimal cover of odd circuits and \( \delta(S) \) be a cut, then \( X' = X \Delta \delta(S) \) is a cover, not necessarily minimal.

Proof: From Theorem 1 we have that \(|C_o \cap X| \) is odd for each \( C_o \in C_o \). For each \( \delta(S) \) we
have that $|C_o \cap \delta(S)|$ is even. Let $C_o \in C_o$. If $|C_o \delta(S) \cap \bar{X}| \geq 1$ then $|C_o \cap X'| \geq 1$. If $|C_o \delta(S) \cap \bar{X}| = 0$ then $|C_o \delta(S) \cap X| = |C_o \delta(S)|$ and, even, therefore $|C_o \delta(S) \cap X| \geq 1$ and again $|C_o \cap X'| \geq 1$. Q.E.D.

Lemma 3: Let $X$ and $Y$ be two minimal covers such that $X \neq Y$. Then there is a cut, $\delta(S)$, such that $\delta(S) \subseteq X \Delta Y$.

Proof: Assume that $X \Delta Y$ does not contain a cut, then $G \setminus (X \Delta Y)$ is connected. Let $T$ be a spanning tree. Let $e \in X \Delta Y$, assume that $e \in X$ and $e \notin Y$, then $T \cup \{e\}$ contains a single circuit $C$ and $|C \cap X| \equiv (mod 2) \neq |C \cap Y| \equiv (mod 2)$. From Theorem 1 we have that $|C \cap X| \equiv (mod 2) = |C \cap Y| \equiv (mod 2)$.

Remark: $X \Delta Y$ contains a cut and is not necessarily a cut.

Definition: Let $X \subseteq E$ and $\delta(S)$ be a cut. We say that $\delta(S)$ is a negative cut relative to $X$ if $w(\delta(S) \setminus X) < w(\delta(S) \cap X)$. We shall omit relative to $X$ when no confusion can be arisen.

Theorem 4: A minimal cover of odd circuits $X$ is an optimal one if and only if there is no negative cut relative to $X$.

Proof.

(i) $X$ is a minimal cover which is optimal $\Rightarrow$ there is no negative cut.

Assume that there is a negative cut $\delta(S)$. So, $w(\delta(S) \setminus X) < w(\delta(S) \cap X)$. Let $X' = X \Delta \delta(S)$ be a cover. Then

$$w(X) = w(X \cap \delta(S)) + w(X \setminus \delta(S)) > w(X \cap \delta(S)) + w(\bar{X} \cap \delta(S)) = w(X').$$

A contradiction to the optimality of $X$.

(ii) There is no negative cut relative to $X \Rightarrow X$ is an optimal cover.

Let $X$ be a minimal cover with no negative cut relative to $X$ and assume that $X$ is not optimal. Let $Y$ be an optimal cover. From (i) there is no negative cut relative to $Y$. Since $X$ is not optimal, $X \neq Y$.

Let $Y = \{Y_o, Y_1, \ldots, Y_i, \ldots\}$ be a sequence of optimal covers obtained in the following way:
\(Y_0 = Y.\)

\(Y_{i+1} = Y_i \Delta \delta(S),\) where \(\delta(S)\) is a cut contained in \(X \Delta Y_i\) (from Lemma 3 there is such a cut).

Since \(\delta(S) \subseteq X \Delta Y_i\), we have:

\[
w(\delta(S) \cap X) = w(\delta(S) \cap Y_i),
\]

\[
w(\delta(S) \cap X) = w(\delta(S) \cap Y_i).
\]

There is no negative cut relative to \(X\) so

\[
w(\delta(S) \cap Y_i) = w(\delta(S) \cap Y_i) = w(\delta(S) \cap Y_i).
\]

Since there is no negative cut relative to \(Y_i\) equality must hold and \(w(Y_i) = w(Y_{i+1})\). This with Lemma 2 implies that \(Y_{i+1}\) is an optimal cover. \(Y_{i+1} \neq X,\) because otherwise it will contradict the nonoptimality of \(X\). From the definition of \(Y_{i+1}\) we have that \(|Y_{i+1} \Delta X| < |Y_i \Delta X|\) which implies that there must be \(j\) such that \(Y_j = X\), a contradiction.

Q.E.D.

**Definition:** Let \(G = (V,E), w: E \rightarrow R\) be a cost function and \(X \subseteq E\). The **negative cut problem** is the following: Either find negative cut relative to \(X\) or claim that there is no negative cut relative to \(X\).

**Theorem 5:** The negative cut problem is NP-hard.

**Proof:** We shall show that if we can solve the negative cut problem in polynomial time then we can solve the max-cut problem in a graph \(G\) with \(w(e) = 1, \forall e \in E\) in polynomial time, but this problem is known to be NP-complete (cf. Garey and Johnson [4]).

Let \(X\) be a minimal cover. There is a polynomial time algorithm for finding minimal cover based on the algorithm for computing shortest paths on even lengths ('Waterloo-Folklore', Grotschel and Pulleyblank [5]). Then, we have an algorithm for the max-cut problem in \(G\) with \(w(e) = 1, \forall e \in E\), based on the negative cut problem. We can state the algorithm as follows:

1. Let \(X'\) be a cover (clearly \(E\) is a cover).

   1. (i) Let \(X \subseteq X'\) be a minimal cover, obtained by the above method.
(ii) Use the algorithm for the negative cut problem relative to \( X \) in \( G \) with \( w \) and either claim that there is no negative cut or find a negative cut.

2. (i) If the result of the algorithm is no then \( X \) is an optimal cover of odd circuits and \( E - X \) is a max-cut. Otherwise:

(ii) We get a negative cut \( \delta(S) \).

Let \( X' = X \Delta \delta(S) \). Go to (ii).

Clearly, every time we come to 2(ii) the cardinality of \( X \) is decreasing. So, after at most \( O(E) \) times, we must reach step (i) and find a max-cut.

Q.E.D.

3. ON COVERING ODD-CIRCUITS AND COVERING ODD-CUTS

There is an interesting correspondence between the covering odd-circuits problem and the covering odd-cuts problem.

Let \( G = (V, E) \) be a graph with edge weights \( w(e), \forall e \in E \) and \( d(v) \) be the degree of \( v \), the number of edges in \( G \) incidence to \( v \). Let \( T \subseteq V \) be an even subset of the vertices of \( G \). A \( T \)-cut is an edge cut set of the graph which divides \( T \) into two odd sets. A special case is when \( T = \{v \in V \mid d(v) \text{ is odd}\} \), then a \( T \)-cut is an odd-cut. A \( T \)-join is a minimal subset of edges that meets every \( T \)-cut. A \( T \)-join is a minimal cover of \( T \)-cut.

3.1 The Planar Case

If \( G = (V, E) \) is a plane graph and \( G^* = (V^*, E^*) \) is its planar dual, the following relations are well known (e.g. Bondy and Murty [2]).

(i) Faces of \( G \) correspond to vertices of \( G^* \).

(ii) Edges of \( G \) correspond to edges of \( G^* \).

(iii) Circuits of \( G \) correspond to co-boundaries of \( G^* \).
Using this relation it is easy to see that if \( T = \{ v \in V: d(v) \text{ is odd} \} \) and \( F \) is a T-join in \( G \) then \( F^* \) is a minimal cover of odd-circuits in \( G^* \) and vice versa. So, in the planar case, one can solve the covering odd circuits problem in \( G \) by finding minimum T-join in \( G^* \). This can be done polynomially using for example Edmonds and Johnson [3] algorithm.

3.2 The General Case

In fact, in general graphs finding minimum cover of odd-circuits is known to be NP-complete. So, for general graphs it is not possible to solve one problem by solving the other. But there are still some similar properties.

It is known that \( F \subseteq E \) is a T-join if and only if the following conditions hold:

(i) \(|F \cap \delta(v)| \) is odd, \( \forall v \in T \).
(ii) \(|F \cap \delta(v)| \) is even, \( \forall v \in T \).
(iii) \( F \) is a forest in \( G \).

Theorem 1 in this paper states that \( X \) is a minimal cover of odd circuits in \( G \) if and only if the following conditions hold:

(i) \(|C_o \cap X| \) is odd, \( \forall C_o \in C_o \cap B \).
(ii) \(|C_e \cap X| \) is even, \( \forall C_e \in C_e \cap B \).
(iii) \( X \) is a minimal set for which the above conditions hold.

So, one can see the similarity by using the known fact that the stars are the base for the bond space.

G. Meigu [10] characterized the optimal covers of odd-cuts by proving that a minimal cover of odd-cuts is optimal if and only if there is no circuit \( C \) such that \( w(C \cap F) > w(C \setminus F) \). (We call such a circuit a "negative" circuit.) A similar result hold for the other problem. We proved in Theorem 4 that a minimal cover of odd-circuits \( X \) is optimal if and only if there is no negative-cut, i.e. a cut for which \( w(\delta(S) \cap X) > w(\delta(S) \setminus X) \). Aside from these similarities, there are two important differences between these two problems that we are going to discuss below.
Let $A$ be the incidence matrix of all $T$-cuts in $G$, and define the problem $P_1$ to be:
$$P_1: \min\{wx: Ax \geq 1, x \geq 0\}.$$ 

Let $C$ be the incidence matrix of all odd circuits in $G$ and define the problem $P_2$ to be:
$$P_2: \min\{wx: Cx \geq 1, x \geq 0\}.$$ 

Edmond and Johnson [3] proved that if $w \in \mathbb{R}^+$, then there always exists an integral optimal solution for $P_1$.

There are cases for which there does not exist an integral solution for $P_2$. For example if $G=K_5$ and $w(e) = 1$, $\forall e \in E$ (see Grotschel and Pulleyblank [5]).

Grotschel and Pulleyblank [5] defined the graph $G$ to be weakly bipartite if the convex-hull of all the incidence vectors of the bipartite subgraphs in $G$ is defined by the trivial inequalities and $x(C) \leq |C| - 1$ for odd circuit $C$ (the odd circuit inequalities). Therefore if $G$ is weakly bipartite, then there always exists an integral solution for $P_2$. Let $P_2' = \max\{wx: Cx \leq |C| - 1, x \leq 0\}$, if $x^*$ is an optimal solution for $P_2'$, then $1-x^*$ is an optimal solution for $P_2$, e.g. Barahona [1]).

Let $V \subseteq E$ then the negative circuit problem is the following: Either find negative circuit relative to $F$ or claim that there is no negative circuit relative to $F$.

One can solve that problem in polynomial time by using the following arguments: Let $T = \{v \in V: \delta(v) \cap F$ is odd $\}$ so $F$ is a $T$-join. Using Edmonds and Johnson algorithm, find $F^*$, an optimal $T$-join. If $w(F) = w(F^*)$ then $F$ is an optimal $T$-join and there is no negative circuit relative to $F$. Otherwise, $F \delta F^*$ contains a negative circuit. In contrast, we showed in Theorem 5 that the negative cut problem is NP-hard.

4. ON THE CLUTTER OF ODD CIRCUITS AND ITS BLOCKER

Let $C$ be a clutter on a set $E$, and let $A$ be the incidence matrix of $C$. If the pair of programs
$$\max\{y: yA \leq w, y \geq 0\} \text{ and } \min\{wx: Ax \geq 1, x \geq 0\}$$
both have integer optimization vectors, for every integer vector $w \geq 0$ such that the solution to both problems exist, then the clutter is called mengerian, (see Seymour [11]). Let $G$ be a graph, let $C_o(G)$ be the clutter of odd circuits and $D(G)$ be the blocker of $C_o(G)$. 
In both cases we shall show that there exists a circuit $C$ such that $|C \cap F|$ is odd which implies that $\overline{G}$ is not a parity subdivision.

1. $G_o$ is not a bipartite, so there exists an odd circuit $C_o = (f_1, ..., f_{2t+1})$. Let $C$ be the expansion of $C_o$ in $G$, i.e. we extend $C_o$ to a circuit $C$ using the contracted vertices. Since each component $G_i$ in $G$ is connected we have that for each pair $(f_j, f_{j+1})$, where $f_j, f_{j+1} \in C_o$, there exists a path $P_j$ and a component $G_i$ such that $P_j \subseteq G_i$ and $|P_j \cap F| = 0$. So $|C \cap F|$ is odd. A contradiction.

2. $G_o$ is a bipartite. $F \neq F_o$ since $F$ is not a cut. So, there exists $f \in F$ such that the number of components in $(G - F) \cup f$ remains the same as the number of components in $G - F$. Assume without loss of generality that $f \in G_1$ where $G'_1 = G_1 \cup \{f\}$, and $G'_1$ is a connected component in $(G - F) \cup f$. Since $G_1$ is a connected component there exists a path $P \in E(G_1)$ which connects the ends of $f$. Therefore, $P \cup \{f\}$ is a circuit, $C$, such that $|C \cap F| = 1$. A contradiction.

Q.E.D.

We present some definition from Barahona [1]. A signed graph is a pair $(G, s)$ where $G=(V,E)$ is a graph and $s: E \rightarrow \{-1, 1\}$ is called a sign function. Let us denote by $\overline{s}$ the function $\overline{s} = -1$. A negative circuit $C$ is a circuit such that $\prod_{e \in C} s(e) = -1$. For instance, negative circuits of $(G, \overline{s})$ correspond to odd circuits in $G$.

Let us denote by $\mathcal{C}_o(G)$ the clutter of negative circuits of $(G, s)$ and let $\mathcal{B}(G)$ be its blocker. $\mathcal{C}_o(G)$ and $\mathcal{B}(G)$ are binary clutters.

A sign graph $(G, s)$ is said to be reducible to $(G', s')$ if this last graph can be obtained from the former one by a sequence of the following operations:

(i) deletion of an edge.

(ii) contraction of an edge $e$ with positive sign.

(iii) Changing the signs of the edges of a star $\delta(v)$, $v \in V$.

$\mathcal{C}_o(G')$ and $\mathcal{B}(G')$ are minors of $\mathcal{C}(G)$ and $\mathcal{B}(G)$ respectively.

Seymour [11] showed that the clutter $Q_6$ defined by
\[ Q_6 = \{\{1,3,5\}, \{1,4,6\}, \{2,3,5\}, \{2,4,5\}\} \]

is a minimal nonmengerian clutter. Moreover, he proved that a binary clutter is mengerian if and only if it has no \( Q_6 \) minor.

**Theorem 7 (Barahona):** Given a graph \( G \), \( C_o(G) \) is mengerian if and only if \( (G,s) \) is not reducible to \( (K_4, \mathcal{S}) \).

\[
\begin{array}{c}
\text{And } D(G) \text{ is mengerian if and only if } (G,s) \text{ is not reducible to the following sign graph}
\end{array}
\]

We shall show a slightly different necessary and sufficient conditions for \( C_o(G) \) and \( D(G) \) to be mengerian.

**Lemma 8:** Let \( G=(V,E) \) be a graph, there exists \( G \subseteq G \) which is \( \delta(S) \)-even subdivision of \( K_4 \) if and only if \( (G,s) \) is reducible to \( (K_4, \mathcal{S}) \).

**Proof:** Since the reduction operations keep the sign of the circuits unchanged and cannot create \( K_4 \) if there is not a homeomorph of \( K_4 \), it is easy to see that there exists a minor \( (K_4, \mathcal{S}) \) \( \iff \) there exists a subgraph \( G \subseteq G \) which is a subdivision of \( K_4 \) such that the sign of the circuits in \( (K_4, \mathcal{S}) \) is equal to the sign of their subdivision in \( (G,s) \).

Clearly the collection of negative circuits in \( (G,s) \) is the same as the collection of odd circuits in \( G \).

So, there exists a subgraph \( G \subseteq G \) which is a parity subdivision of \( K_4 \) \( \iff \) there exists a minor \( (K_4, \mathcal{S}) \) of \( (G,s) \).

From Lemma 6 we have that \( G \) is a parity subdivision of \( K_4 \) \( \iff \) \( G \) is a \( \delta(S) \)-even subdivision of
$K_4$. It follows that there exists a subgraph $\tilde{G} \subset G$ which is $\delta(S)$-even subdivision of $K_4 \Leftrightarrow (G, \bar{s})$ is reducible to $(K_4, \bar{s})$.

Q.E.D.

The following theorem (Theorem 9) is a simple consequence of Lemma 8 and Barahona's result (Theorem 7). It was obtained independently by us [18], in much more complicated way, directly from Seymour's results.

Theorem 9: Let $G=(V,E)$ be a graph, $\text{Co}(G)$ is mengerian $\Leftrightarrow$ there is no $\tilde{G} \subset G$ such that $\tilde{G}$ is a $\delta(S)$-even subdivision of $K_4$.

Proof: Follow immediately from Theorem 7 and Lemma 6.

Q.E.D.

Let $\Delta$ be the sign graph with $|E|=6$, $|V|=3$, such that every 2-edges circuit is negative, i.e. $\Delta$ is the following graph:

Given a circuit $C$ and a path $P=(v_0,e_1,v_1,\ldots,e_k,v_k)$. $P$ is called a chordal path if $v_0 \neq v_k$, $v_0,v_k \in C$, $v_1,\ldots,v_{k-1} \notin C$ and $e_1 \notin C$.

Let $C = \{v_o,e_1,v_1,\ldots,e_k,v_k,v_j\}$ be a circuit where $v_o=v_j$. An arc $A$ is called $(v_o,v_k)$-arc if $A$ is a path with end vertices $v_o$, $v_k$ and $A \subset C$.

Let $G'$ be a family of graphs such that each $G' \in G'$ has the following properties: (a) $G'$ has a circuit $C$ with 3 chordal paths pairwise internally disjoint $P_i$, $i=1,2,3$. (b) For each $P_i=(v_{o_i},e_{1_i},\ldots,e_{k_i},v_{k_i})$ there exists an odd circuit $C_i = P_i \cup [(v_{o_i},v_{k_i})-\text{arc}]$ such that $(C_i \cap C) \subseteq (C_j \cap C)$, $i \neq j$, $1 \leq i,j \leq 3$ and $C_1 \cap C_2 \cap C_3 = \Phi$. For example:
Lemma 10: Let $G=(V,E)$ be a graph. $(G,\bar{s})$ has a minor $\Delta$ if and only if $G$ has a subgraph $G' \in G'$.

Proof: We shall just outline the proof.

I $G$ has the required subgraph $\Rightarrow G$ has the minor $\Delta$.

Let $G'$ be the required subgraph where $C_1, C_2, C_3$ are the 3 odd circuits. Let $(G,\bar{s})$ be a sign graph then every odd circuit in $G$ corresponds to a negative circuit in $(G,\bar{s})$. We shall get the minor $\Delta$ by the following operations:

1. Delete every edge $e \in (E(G)-E(G'))$.
2. Since $C$ and the 3 chordal paths are pairwise internal disjoint $(C_i \cap C) \subseteq (C_j \cap C), i \neq j, 1 \leq i, j \leq 3$ and $C_i \cap C_2 \cap C_3 = \emptyset$, then for each $P_i$ we have an edge $e_i \in P_i$ such that $e_i \notin P_j, j \neq i \in C_j, j = 1,2,3$.

For each circuit $C_i$ we have an edge $e_{3+i} \notin P_j, j = 1,2,3$. It is easy to see that one can contract every $e \in (E(G')-(e_1,...,e_6))$ and get a graph with $|E|=6, |V|=3$. Since the reduction operations keep the circuits sign unchanged we have $\Delta$ as a minor.

II $G$ has a minor $\Delta \Rightarrow$ there is a subgraph $G' \in G'$.

Assume that $\Delta$ was obtained from $G$ by deleting a subset of edges $E_o$ to get $G_o$ and then by a sequence $S$ of contractions and sign changes, (going through a sequence of graphs $G_o,G_1,G_2,...,G_p$ where $(G_p,\bar{s})=\Delta$). W.l.o.g. we assume that $E_o$ is maximal, therefore we do not contract loops.

We shall show that there exist $G' \subseteq G_o$, such that $G' \in G'$, by expanding and sign changes, going backward in the sequence $S$ and collecting edges to a subset of edges $E'$ (at the end $G' = (V(E'),E')$).

First let $E(\Delta)=E'$. At any stage of the expansion we get a new edge $e$. We add $e$ to $E'$ only if $e$ is not a cut edge in $G_{i+1}$ (where $G_{i+1}$ belong to the sequence $S$). We shall consider now the case where $e$ is not a cut-edge. At any stage we expand a vertex of degree (relative to $(V(E'),E')$) 2, 3, or 4. If we expand a vertex of degree 2 then the new added vertex is of degree 2. If we expand a vertex of degree 4 then there are two possibilities: (1) $d(v)$ remains 4 and we add a new vertex of degree 2; (2) $d(v)$ changes
to 3 and we add a new vertex of degree 3.

Clearly, every time we expand a vertex of degree 3, we get a vertex of degree 3 and we add a new vertex of degree 2.

Let $G' = (V'E',E')$ where all the vertices have been exposed.

By using the following 3 facts:

(i) $(G',\mathcal{S})$ has the minor $\Delta$;

(ii) The reduction operations keep the circuits' signs unchanged.

(iii) To every negative circuit in $(G',\mathcal{S})$ correspond an odd circuit in $G'$.

We have that $G' \in G'$.

Q.E.D.

Theorem 11: Let $G = (V,E)$ be a graph. $D(G)$ is mengerian if and only if $G$ has no subgraph $G' \subseteq G'$.

Proof: Follows immediately from Theorem 7 and Lemma 10.

Q.E.D.

Our results give characterizations in terms of the existence of a certain subgraph as opposed to Barahona's results that give characterizations using excluding minor relative to slightly more complicated operations. On the other hand, the beauty of Barahona's result is that it is a finite basis characterization (in terms of Lovasz [9]).
REFERENCES


