FULL-COMMUTATION AND FAIR-TERMINATION IN EQUATIONAL
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ABSTRACT

In [PF-85] the concepts of fair derivations and fair-termination in term-rewriting systems were introduced and studied. In this paper, we define the notion of fairness in equational term-rewriting systems, where a derivation step is a composition of the equality generated by a (finite) set of equations with one step rewriting using a set of rules. A natural generalization of E-termination (termination of equational term-rewriting systems), namely E-fair-termination is presented. We show that fair-termination and E-fair-termination are the same whenever the underlying rewriting relation is E-fully-commuting, a property inspired by Jouannaud and Munoz' E-commutation property. We obtain analogous results for combined term-rewriting systems.
1. INTRODUCTION

One of the basic motivations behind the definition of a rewrite-rule in term-rewriting-systems is the simplification obtained by rewriting a given term. However, there are rewritings that are intrinsically not simplifying. For example, the rule \( x + y \rightarrow y + x \) defines a rewriting that is not simplifying a term, nevertheless it defines an important relation among terms in a given system - the commutativity. So, the notion of derivation in term-rewriting systems was extended to E-derivation, whereat a derivation step is obtained by composition of an equality generated by a set of equations, with a rewriting using rules. Consider, for example, a system where \(+\) is a binary infix operator with the following properties: commutativity, expressed by the equation \( x + y = y + x \), and absorption of the left identity expressed by the simplifying rule \( x + x \rightarrow x \). The term \( x + e \) is in normal form with respect to the rewriting relation generated by the rule. However, if we allow commutative rewritings, then \( x + e \) rewrites to \( x \). In other words, there is no derivation that starts from \( x + e \), but there is an E-derivation that starts from \( x + e \). Thus, one can easily feel that the concept of reducing terms modulo a set of equations affects the termination property. In this work we prove that it affects also the fair-termination property as defined in [PF-85].

Jouannaud and Munoz introduce in [JM 84] the notion of E-commuting and show that termination of a set of rules \( R \) and termination of this set \( R \) modulo a set \( E \) of equations are the same wherever \( R \) is E-commuting. Bachmair and Dershowitz use in [BD 85] the notion of commutation in order to prove termination of a combined rewrite system \( R_1 \cup R_2 \) by proving termination of \( R_1 \) and \( R_2 \) separately. We introduce the natural generalization of E-termination, namely E-fair-termination, and show that E-commuting of the rewriting relation, together with fair termination are not sufficient for E-fair-termination. We define another property of a rewriting relation, called full-commutation. This property, together with fair termination provide a sufficient condition for E-fair-termination, and analogously connects between fair termination of \( R_1 \) and \( R_1 \)-fair termination of \( R_1 \cup R_2 \).
2. EQUATIONAL TERM-REWRITING SYSTEMS: TERMINATION VERSUS E-TERMINATION

The following definitions are similar to the corresponding definitions in other works, like [JM-84], [JK-84], and [BP-85].

**Definition 1:** An Equational Term-Rewriting System (ETRS) is a pair (S,E) where
- S is a term-rewriting system (F,R) and
- E is a finite set of equations (axioms) of the form α=β where α and β are terms in the free algebra over (F,X).

A "computation" in an ETRS is called an E-derivation, and it combines application of rules together with "application" of equations.

**Definition 2:**
1) \( t \vdash_E t' \) iff there is a substitution σ and an equation \((α=β) \in E\), s.t. \( t = ασ \) and \( t' = βσ \).
2) The reflexive-transitive closure of the relation \( \vdash_E \) is the equivalence-relation denoted by \( =_E \).

The E-rewriting relation, denoted by \( \stackrel{R/E}{\longrightarrow} \), is defined in the following way:

**Definition 3.** \( \stackrel{R/E}{t \longrightarrow t'} \) iff there is a term \( t'' \), s.t. \( t =_E t'' \stackrel{R}{\longrightarrow} t' \).

**Definition 4:** An E-derivation is a finite or infinite sequence of the form
\[
t_1 \stackrel{R/E}{\longrightarrow} t_2 \stackrel{R/E}{\longrightarrow} t_3 \stackrel{R/E}{\longrightarrow} \cdots \stackrel{R/E}{\longrightarrow} t_n \stackrel{R/E}{\longrightarrow} \cdots \quad n \geq 1
\]

The reflexive-transitive closure of \( \stackrel{R/E}{\longrightarrow} \) corresponds to the finite E-derivations, and is denoted by \( \stackrel{R/E}{\longrightarrow} \).
Definition 5:

1) An ETRS (S,E) is terminating iff the TRS S is terminating, that is to say, every derivation (in S) is finite.

2) An ETRS (S,E) is E-terminating iff every E-derivation is finite.

We start with some observations about the connection between termination and E-termination. First we present some notations for describing terms.

Given a term \( t \) in the free algebra over \((F,X)\):

1) \( v(t) \) is the set of variables of \( t \) \( (v(t) \subseteq X) \).

2) The positions within \( t \) are finite dotted lists of natural numbers, i.e. expressions of the form \( n_1.n_2.\ldots.n_k \) for some \( k \geq 0 \). In case \( k = 0 \), we use the notation \( \lambda \) for the empty sequence. The position \( u \) defines the subterm \( t/u \) in the following way:
   i) \( t/\lambda = t \)
   ii) If \( t/u = f(t_1,\ldots,t_n) \), then for every \( j, 1 \leq j \leq n, t/u.j = t_j \).

3) \( t[u\leftrightarrow t'] \) is the term obtained by replacing \( t/u \) by \( t' \) in \( t \). Thus, \( t[u\leftrightarrow t']/u = t' \).

Since every derivation in \( S \) corresponds to an E-derivation in \((S,E)\) (as \( \rightarrow_R \subseteq \rightarrow_{R/E} \)), if a given system is E-terminating, then it is terminating. The converse is not necessarily true.

The following example presents a terminating ETRS that is not E-terminating.

Example 1: Let

\[
F = \{+,s,0\}
\]
\[
R = \{+(s(x),y) \rightarrow +(x,s(y))\}
\]
\[
E = \{+(x,y) = +(y,x)\}
\]

Using Dershowitz' second termination theorem [D-82], one can easily prove that this system is terminating. There is an infinite E-derivation:

\[
+(0,s(s(0))) =^E +(s(0),0) \rightarrow_R +(s(0),s(0)) \rightarrow_R +(0,s(s(0))) =^E .
\]

Thus, the system is not E-terminating.
Jouannaud and Munoz provide in [JM-84] some simple restrictions on the set $E$ of equations, as necessary conditions for the $E$-termination of a system $(S,E)$:

1) If there is an equation $(\alpha = \beta) \in E$, s.t. $\nu(\alpha) \neq \nu(\beta)$ then $(S,E)$ is not $E$-terminating.

2) If there is an equation $(x = t) \in E$, and two different subterm positions $u, v$, s.t. $t/u = t/v \equiv x$, then $(S,E)$ is not $E$-terminating.

One of the important contributions in [JM-84] is a sufficient condition for the $E$-termination of an ETRS $(S,E)$, given that $S$ is a terminating TRS.

**Definition 6:** $R$ *commutes* with a set of equations $E$ (or $R$ is $E$-*commuting*) iff for every $s, s'$ and $t$, there is $t'$, s.t.

\[
\text{if } s' =^E s \quad \frac{+}{R} \rightarrow t \\
\text{then } s' \frac{+}{R} \rightarrow t' =^E t.
\]

This definition assures that if $s'$ is reducible using the $E$-rewriting relation, then it is reducible using the rewriting relation. Moreover, if there is a derivation, from some term that is $E$-equivalent to $s'$, that ends with $t$, then there is a derivation from $s'$ that ends with some term that is $E$-equivalent to $t$. The $E$-commutation of $R$ is described in Figure 1.

**Theorem:** (Sufficiency of $E$-commutation for $E$-termination) [JM-84]

Let $S = (F,R)$ be a terminating TRS. If $R$ commutes with a set of equations $E$, then the system $(S,E)$ is $E$-terminating.

The main idea in the proof is that in case there is an infinite $E$-derivation, then by the $E$-commutation of $R$, each rewriting using rules can be "pushed back" through the preceding equality. Thus, we can get an infinite derivation, contradicting the termination assumption.
In the sequel, we shift the discussion to a new notion of termination, called E-fair-termination. We introduce necessary conditions and a sufficient one for E-fair-termination. These conditions are compared to those stated above for E-termination.

3. FAIRNESS IN ETRS

The notion of fair derivation in a term-rewriting system was introduced in [PF-85], whereby every rewrite rule enabled infinitely often along a derivation is infinitely-often applied along that derivation.

We now introduce the new definition of fairness in ETRS.

**Definition 7:** An E-derivation

\[ d = t_1 \xrightarrow{E} t'_1 \xrightarrow{R} t_2 = t'_2 \xrightarrow{E} t'_3 \xrightarrow{R} t_3 \ldots \]

is a **fair E-derivation** iff it is finite or it is infinite and for every rule \( r \in R \), if \( r \) is applied only finitely often along \( d \), then there is an \( i \geq 1 \), s.t. for every \( t'_j, j \geq i \), \( r \) is not enabled in \( t'_j \).

Note that the checking of enabledness is not done before the "applications" of
equations (on \(t_1, t_2, t_3 \ldots\)) but before the "applications" of rules (on \(t'_1, t'_2, t'_3\ldots\)).

Following are examples of an infinite fair E-derivation and an infinite unfair E-derivation.

**Example 2:** Let

\[ R:: 1) g(0,y,z) \rightarrow g(0,f(y),z) \]
\[ 2) g(0,y,z) \rightarrow g(f(0),y,z) \]
\[ 3) g(f(x),f(y),z) \rightarrow g(f(x),y,z) \]
\[ 4) g(f(x),f(y),z) \rightarrow g(f(x),f(y),f(z)) \]

and \(E = \{g(x,y,z) = g(y,x,z)\}\).

The following infinite E-derivation:

\[
g(f(0),0,0) \overset{E}{=} g(0,f(0),0) \overset{[1]}{\rightarrow} g(0,f(f(0),0) \overset{1]}{\rightarrow} g(f(0),f(f(0),0) \overset{[4]}{\rightarrow} g(f(0),f(f(0),f(0)) \overset{[3]}{\rightarrow} g(f(0);0,f(0)) \overset{E}{=} g(0,f(0),f(0)) \overset{[1]}{\rightarrow} \ldots
\]

is a fair E-derivation, as every rule is infinitely often applied.

Consider the following finite E-derivation:

\[ d = g(f(0),0,0) \overset{E}{=} g(0,f(0),0) \overset{[2]}{\rightarrow} g(f(0),f(0),0) \overset{[3]}{\rightarrow} g(f(0),0,0).\]

By repeating \(d\) infinitely, we get an infinite unfair E-derivation, as the rules (1) and (4) are infinitely often enabled and never applied.

The set of equations affects also the notion of fair termination.

**Definition 8:**

1) An ETRS \((S,E)\) is **fairly-terminating** iff the underlying TRS \(S\) is fairly terminating, that is to say, every fair derivation is finite.

2) An ETRS \((S,E)\) is **E-fairly-terminating** iff every fair E-derivation is finite.

In [P-85] sound and semantically complete proof-rules are introduced, for proving fair termination of a TRS, and E-fair-termination of an ETRS. These proof rules are
based on well-foundedness arguments. Such methods for proving various notions of termination are beyond the scope of this paper, and we omit their presentations.

Claim 1: Figure 2 shows the relations between the set of terminating, E-terminating, fairly-terminating and E-fairly-terminating ETRS's.

Proof: By the definitions, every ETRS that is E-terminating is E-fairly-terminating and terminating, every terminating system is fairly-terminating, and every E-fairly-terminating system is also fairly-terminating.

If the set of rules $R_0$ consists of the single rule $\{a \rightarrow f(a)\}$ then there is an infinite fair derivation $a \rightarrow f(a) \rightarrow f(f(a)) \rightarrow \ldots$, thus for every $F \supset \{f, a\}$ and $E$, $((F, R_0), E) \notin FT$.

If the set of rules is $R_1 = \{a \rightarrow f(a), a \rightarrow b\}$, then the system is obviously not terminating, but there is no infinite fair derivation [PF-85]. For every $F \supset \{f, a, b\}$, $((F, R_1), \varphi) \in EFT-T$.

The system $S_2$ in Example 1 is terminating but not E-terminating. Since this system has a single rule, every infinite E-derivation is a fair E-derivation, thus

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![Figure 2: Termination, E-termination, fair-termination and E-fair-termination.](image-url)
$S_2 \in TEFT$.

The following example presents an ETRS $S_3$, s.t., $S_3 \in (T \cap EFT) - ET$.

**Example 3:** Let

$F = \{+, s, 0\}$

$R = \{+(s(x), y) \rightarrow +(x, s(y)), +(x, y) \rightarrow 0\}$

$E = \{+(x, y) = +(y, x)\}$

This system is terminating (the proof is very similar to the termination proof of the system in example 1.) Since the set of rules, in this example is a subset of the one in example 1, and the set of equations in both systems is the same, this system is obviously not E-terminating. Using an appropriate proof rule [P-85] it can be proved that this system is E-fairly-terminating.

The following example presents an ETRS $S_4$, s.t., $S_4 \in FT - (T \cup EFT)$.

**Example 4:** Let $F = \{a, b, c, d, e\}$

$R = \{a \rightarrow b, b \rightarrow a, a \rightarrow c, d \rightarrow e, e \rightarrow d, d \rightarrow c\}$

$E = \{a = e, d = b\}$.

There is an infinite derivation:

$a \rightarrow_R R b \rightarrow_R R a \rightarrow_R R a \rightarrow_R R a \rightarrow_R R a \rightarrow_R R a \rightarrow_R R a \rightarrow_R R a \ldots$

Consider the following infinite E-derivation:

$a \in^E a \rightarrow_R R e \rightarrow_R R d \rightarrow_R R b \rightarrow_R R a \in^E E \ldots$

Since there is only one rule enabled in the term $a$, and one in term $b$, this E-derivation is fair.

Using an appropriate decision algorithm [PF-85], it can be easily proved that this system is fairly terminating.
4. FAIR TERMINATION VERSUS E-FAIR TERMINATION

We first prove that the simple restrictions on the set E of equations, that were introduced above as necessary conditions for the E-termination of an ETRS, are also necessary conditions for the E-fair termination of an ETRS.

Claim 2: If there is an equation \((a = \beta) \in E\), s.t. \(\nu(a) \neq \nu(\beta)\), then \((S,E)\) is not E-fairly-terminating.

Proof: Let \(R = \{i \to \tau_i \mid 1 \leq i \leq n\}\). Assume \(x \in \nu(a), x \notin \nu(\beta),\) and \(a/u = z\).

As we may instantiate the extra variable in \(a\) to any term, in particular we may choose a l.h.s. \(l_i\), and obtain the following infinite E-derivation that starts from \(\beta\):

\[
\begin{align*}
\beta &= \frac{E}{a[u \leftarrow l_1]} \to \frac{E}{a[u \leftarrow \tau_1]} \to \frac{E}{a[u \leftarrow \tau_2]} \to \frac{E}{a[u \leftarrow \tau_3]} \to \cdots \frac{E}{a[u \leftarrow \tau_n]} \\
&= \frac{E}{a[u \leftarrow \tau_2]} \to \frac{E}{a[u \leftarrow \tau_3]} \to \frac{E}{a[u \leftarrow \tau_4]} \to \cdots
\end{align*}
\]

This is a description of an E-derivation, in which every rule is applied infinitely-often, hence this E-derivation is an infinite fair E-derivation.

Claim 3: If there is an equation \((x = t) \in E\), with two different subterm positions \(u, v\) s.t. \(t/u = t/v = z\), then \((S,E)\) is not E-fairly-terminating.

Proof: Let \(R = \{i \to \tau_i \mid 1 \leq i \leq n\}\).

We construct an infinite E-derivation, starting from some reducible term \(t_0^j\). In other words, there is some rule that is enabled in \(t_0^j\).

For every term \(t_i^j\) \((i \geq 1, j \geq 0)\), we denote by \(t_i^{j+1}\) the term \(t[u \leftarrow t_i^j][v \leftarrow t_i^j]\).

As \((x = t) \in E\), \(t_i^j \vdash \frac{E}{t_i^{j+1}}\) is always true. Since \(t_i^j\) is a subterm of \(t_i^{j+1}\), the set of enabled rules in \(t_i^j\) is a subset of those enabled in \(t_i^{j+1}\). Since the set of rules is finite, for every \(i \geq 1\), there is \(k_i\), s.t. the set of enabled rules in \(t_i^{k_i}\) is equal to the set of enabled rules in \(t_i^{k_i+1}\).

For a given term \(s\), a derivation step from \(t[u \leftarrow s][v \leftarrow s]\), s.t. the rule is applied on the subterm \(t[u \leftarrow s][v \leftarrow s]/u\), is called a derivation step that assures infinity. If the enabled rules in \(s\) are exactly those in \(t[u \leftarrow s][v \leftarrow s]\), then every rule which is enabled in \(t[u \leftarrow s][v \leftarrow s]\), can be applied on the subterm \(t[u \leftarrow s][v \leftarrow s]/u\).
The constructed E-derivation consists of derivation steps that assure infinity, applied on \( t_i^{k+1} \).

We denote by \( t_{i+1}^0 \) some term that is derived from \( t_i^{k+1} \), by applying a rule, that is chosen according to a certain method defined below, in a derivation step that assures infinity.

The E-derivation is in the following form:

\[
\begin{align*}
t_1^0 & \xrightarrow{E} t_1^{k+1} \\
R & \\
t_2^0 & \xrightarrow{E} t_2^{k+1} \\
R & \\
\ldots
\end{align*}
\]

Since the enabled rules in \( t_i^k \) are exactly those in \( t_i^{k+1} \), and \( t_i^k \) is a subterm of \( t_{i+1}^0 \), we get that the set of enabled rules in \( t_i^{k+1} \) is a subset of those enabled in \( t_{i+1}^0 \).

As we already mentioned, the set of enabled rules in \( t_{i+1}^0 \) is a subset of those enabled in \( t_{i+1}^{k+1} \), which are exactly those enabled in \( t_i^{k+1} \).

Thus, joining these two remarks, we get that the set of enabled rules in \( t_i^{k+1} \) is a subset of those enabled in \( t_{i+1}^{k+1} \).

The rules that are applied along the E-derivation are chosen in the following way:

For \( t_i^{k+1} \), the chosen rule to be applied on the derivation step from \( t_i^{k+1} \), is the one that was applied the minimum number of times (possibly not at all) along the E-derivation starting from \( t_i^0 \), from amongst the applications on \( t_1^{k+1} \), \( t_2^{k+1} \) and so on, up to \( t_{i-1}^{k+1} \). In case there are several rules, enabled in \( t_i^{k+1} \), that were applied the same minimum number of times, the rule with the smallest index is chosen.

If the rule \( l_i \rightarrow \tau_i \) is applied along the E-derivation, then there is \( l \geq 1 \), s.t. this rule is enabled in \( t_j^{k+1} \), for every \( j \geq l \). According to the method of choosing the rules to be applied, and since the set of rules is a finite set, this rule is chosen for application infinitely many times. So, by definition of fairness, this E-derivation is an infinite fair E-derivation.

In order to clarify this proof, let us consider the following example:

**Example 5:** Let

\[
R:: 1) \quad \neg \alpha \rightarrow \alpha \\
2) \quad (\alpha \lor \beta) \rightarrow (\neg \alpha \land \neg \beta)
\]
3) \(~(\alpha \land \beta) \rightarrow (\neg \alpha \lor \neg \beta)\)

E:: \alpha \land \alpha = \alpha

Let \(t_1^0 = (\neg (\neg \alpha \lor b)).\)

Only rule (2) is enabled in \(t_1^0\). Since the set of enabled rules in \(t_1^0\) is equal to this in \(t_1^1 = (\neg (\neg \alpha \lor b)) \land (\neg (\neg \alpha \lor b)), k_1 = 0.\) By applying a derivation step (that assures infinity), we obtain the term \(t_2^0 = (\neg (\neg \alpha \land \neg b)) \land (\neg (\neg \alpha \lor b)), \) where both rules (1) and (2) are enabled. Again, since the enabled rules in \(t_1^1 (t_1^1 = t_2^0 \land t_2^0)\) are the same as in \(t_2^0, k_2 = 0.\) The chosen rule to be applied on the next derivation step, from \(t_2^0,\) is (1), and we get the term \(t_3^0.\) The constructed derivation is continued by applying rules (2) and (1) indefinitely.

Next, we prove that the commutation of \(R\) with a set of equations \(E,\) that is a sufficient condition for the \(E\)-termination of a terminating ETRS, is not a sufficient condition for the \(E\)-fair-termination of a fairly terminating ETRS. This is shown using the system in example 4 that is fairly terminating and not \(E\)-fairly terminating.

Proving the commutation of \(R\) with \(E\) is very simple in this case, as we have to check only a finite set of possibilities. For example, if the \(E\)-derivation is \(a \rightarrow^E R d,\) then the term \(b\) satisfies the condition as \(a \rightarrow^R b \rightarrow^E d.\)

The set of equations provides the possibility to "jump" on pathological states like \(a\) or \(d,\) where a fair choice among the enabled rules to be applied imposes termination.

In order to introduce a sufficient condition for \(E\)-fair-termination of a fairly terminating ETRS, we define a stronger version of commutation and prove some lemmas related to it.

**Definition 9:** The derivation \(t_1 \rightarrow^R t_2 \rightarrow^R \ldots \rightarrow^R t_n,\) for \(n \geq 2,\) is a **full-derivation** iff every rule enabled in some \(t_i, 1 \leq i \leq n - 1,\) is applied along this derivation. We denote such full-derivation (that starts from \(t_1\) and ends with \(t_n\)) by \(t_1 \rightarrow^R t_n.\)
Lemma 1: If a TRS $S = (F,R)$ is fairly-terminating, then for every $t$ there is no infinite chain of full-derivations that starts from $t$.

Proof: Assume, by way of contradiction, that there is some $t_1$ and an infinite chain of full derivations that starts from $t_1$: $t_1 \xrightarrow{R} t_2 \xrightarrow{R} t_3 \xrightarrow{R} \cdots$. This derivation is a fair derivation, as every rule enabled infinitely often along it, is enabled in infinitely many full-derivations along this chain. By definition of full-derivation, this rule is applied infinitely many times along the chain. This derivation contradicts the assumption that the given system is fairly-terminating. (Actually, since $\xrightarrow{R}$ is locally finite, for every $t$, there is a natural number $n_t$, s.t. every chain of full-derivations, that starts from $t$, is a concatenation of no more than $n_t$ full-derivations.)

Definition 11: The E-derivation $t_1 =^E t'_1 \xrightarrow{R} t_2 =^E t'_2 \xrightarrow{R} \cdots \xrightarrow{R} t_n$, for $n \geq 2$, is a full-E-derivation if every rule enabled in some $t'_i$, $1 \leq i \leq n-1$, is applied along this E-derivation. We denote such full-E-derivation (that starts from $t_1$ and ends with $t_n$) by $t_1 \xrightarrow{R/E} t_n$.

Definition 12: If $t_i \xrightarrow{R/E} t_{i+1}$, for every $i \geq 1$, then the E-derivation obtained by concatenating all these full-E-derivations is a chain of full-E-derivations.

Lemma 2: For an infinite fair E-derivation
there is $i \geq 1$, s.t. the E-derivation that starts from $t_i$ (a tail of the given one) is an infinite chain of full-E-derivations.

**Proof:** Let $R = \{ \gamma_i \rightarrow \gamma_{i+1} \mid 1 \leq i \leq n \}$.

By the definition of an infinite fair E-derivation, for every rule $\gamma_j \rightarrow \gamma$, if it is applied only finitely often along the given E-derivation, then there is $i_j \geq 1$, s.t. for every $t_k, k \geq i_j$, this rule is not enabled in $t_k$. For the sake of the formal definition of the desired $i$, for every rule $\gamma_j \rightarrow \gamma$, applied infinitely often along the given E-derivation, the corresponding $i_j$ is zero. Let $i = \max_{\gamma_j \in \gamma} i_j$.

We denote by $d_t$ the infinite fair E-derivation that starts from $t_i$. Every rule enabled along $d_t$, is infinitely often enabled (due to the way $i$ was chosen). Thus, by the fairness assumption, this rule is infinitely often applied.

**Claim:** For every $k_i \geq i$, there is $k_2, k_2 > k_i$, s.t. the E-derivation that starts from $t_{k_i}$ and ends with $t_{k_2}$ is a full E-derivation.

**Proof of Claim:** For every rule $\gamma_j \rightarrow \gamma$, if the rule is enabled along the E-derivation $d_t$, then there is $m_j > k_i$, s.t. this rule is applied along the E-derivation that starts from $t_{k_i}$ and ends with $t_{m_j}$ (since such rule is applied infinitely many times along $d_t$). Let $m_j$ be zero for every rule $\gamma_j \rightarrow \gamma$ that is not enabled along $d_t$. So, let $k_2 = \max_{\gamma_j \in \gamma} m_j$. Every rule, enabled along $d_t$, is applied along the E-derivation that starts from $t_{k_i}$ and ends with $t_{k_2}$. Thus, this E-derivation is a full-E-derivation.

The proof of the claim completes the proof of the lemma.

**Definition 13:** $R$ is **fully commuting** with a set of equations $E$ (or $R$ is **$E$-fully-commuting**) iff for every $s$ and $t$, there is $t'$, s.t.

\[
\begin{align*}
\text{if } & \gamma_j \longrightarrow \gamma_j' \in E, \\
\text{then } & s \longrightarrow \gamma_j \longrightarrow t' = E \gamma_j' t.
\end{align*}
\]

This property of $R$ is expressed by the diagram in Figure 3.
Figure 3: A diagram for $E$-full-commutation

Theorem: (Sufficiency of $E$-full-commutation for $E$-fair termination)

Let $S = (F, R)$ be a fairly terminating TRS. If $R$ is fully commuting with a set of equations $E$, then the system $(S, E)$ is $E$-fairly terminating.

Proof: By Lemma 1, the relation $\frac{f}{R}$ is Noetherian.

We first prove, by Noetherian induction on $\frac{f}{R}$, that under the assumptions, for every term $t$, there is no infinite chain of full-$E$-derivations starting from $t$.

Let $t_0 \xrightarrow{f} t_1 \xrightarrow{f} t_2 \xrightarrow{f} \cdots$, be a chain of full-$E$-derivations, starting from $t_0$. By $E$-full-commutation there exists a term $t'_0$ such that $t'_0 = E t_1$ and $t_0 \xrightarrow{f} t'_0$.

As $t'_0 = E t_1$, the rest of the chain of full-$E$-derivations starting from $t_1$ is also a chain of full-$E$-derivations starting from $t'_0$. By the induction hypothesis, this chain is finite, because $t_0 \xrightarrow{f} t'_0$.

By Lemma 2, existence of an infinite fair $E$-derivation implies the existence of an infinite chain of full-$E$-derivations. Hence, the system $((F, R), E)$ is $E$-fairly-terminating.

The following example proves that the $E$-full-commutation property is not a necessary
The commutation of $R_1$ with $R_2$ is described in Figure 4.

The definition is analogous to the definition of commutation between a set of rules and a set of equations. This definition of commutation between two sets of rules, is not
Thus, there must be an infinite derivation in $\mathcal{R}_2$, which implies that $\mathcal{R}_1$ cannot commute with $\mathcal{R}_2$. But, one can easily prove that our definition of commutation implies that of [BD-85].

**Theorem:** (Sufficiency of commutation for termination in combined system) [BD-85]

Let $\mathcal{S}_1 = (F, \mathcal{R}_1)$ and $\mathcal{S}_2 = (F, \mathcal{R}_2)$ be two TRS’s. Assume $\mathcal{R}_1$ commutes with $\mathcal{R}_2$. Then, the combined system $(F, \mathcal{R}_1 \cup \mathcal{R}_2)$ is terminating if and only if $\mathcal{S}_1$ and $\mathcal{S}_2$ both are.

Trivially, if $(F, \mathcal{R}_1 \cup \mathcal{R}_2)$ is terminating, so are $(F, \mathcal{R}_1)$ and $(F, \mathcal{R}_2)$. For the other direction, the main idea in the proof is that in case there is an infinite derivation

$$t_1 \rightarrow^{+} \rightarrow^{+} \rightarrow^{+} \rightarrow^{+} \rightarrow^{+} .$$

then, by the fact that $\mathcal{R}_1$ commutes with $\mathcal{R}_2$, each derivation $\rightarrow^{R_1}$ can once again be "pushed back" through the preceding $\rightarrow^{R_2}$. Thus, there must be an infinite derivation in $(F, \mathcal{R}_1)$.

We would like to consider the results of the previous section about ETRS’s as special cases of the following results dealing with combined TRS’s. As we require in the

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**Figure 4:**
A commutative diagram for commutation of $\mathcal{R}_1$ with $\mathcal{R}_2$.

Technion - Computer Science Department - Technical Report CS0393 - 1986
definition of an E-derivation only finite sequences of consecutive equational replacements, one of the rewriting relations in the combined system, for example $R_2$, is required to be applied repeatedly only a finite number of times. Thus, an $R_1$-derivation in $(F,R_1 \cup R_2)$ is defined in the following way:

**Definition 15:** An $R_1$-derivation is a finite or infinite sequence of the form

\[ t_1 \xrightarrow{R_2} t'_1 \xrightarrow{R_1} t_2 \xrightarrow{R_2} t'_2 \xrightarrow{R_1} t_3 \xrightarrow{R_2} \cdots \]

The following new definitions are similar to those defining fairness in E-derivation and E-fair-termination.

**Definition 16:** an $R_1$-derivation in the system $(F,R_1 \cup R_2)$

\[ d = t_1 \xrightarrow{R_2} t'_1 \xrightarrow{R_1} t_2 \xrightarrow{R_2} t'_2 \xrightarrow{R_1} t_3 \xrightarrow{R_2} t'_3 \xrightarrow{R_1} \cdots \]

is an $R_1$-fair derivation iff it is finite, or it is infinite and for every rule $r \in R_1$, if $r$ is applied only finitely often along $d$, then there is an $i \geq 1$, s.t. for every $t'_j$, $j \geq i$, $r$ is not enabled in $t'_j$.

**Definition 17:** The system $(F,R_1 \cup R_2)$ is $R_1$-fairly-terminating iff every $R_1$-fair derivation is finite.

Thus, one can easily describe a proof rule (similar to the rule introduced in [P-85]), for proving E-fair-termination of TRS) in order to prove $R_1$-fair termination of a system $(F,R)$, where $R_1 \subseteq R$.

As in the case of rewriting modulo equations, the commutation of $R_1$ with $R_2$ is not sufficient for the $R_1$-fair-termination of $(F,R_1 \cup R_2)$, even if the system $(F,R_1)$ is fairly terminating.
Definition 18: An $R_1$-derivation in $(F,R_1 \cup R_2)$

\[
d = t_1 \xrightarrow{R_2} t_1' \xrightarrow{R_1} t_2 \xrightarrow{R_2} t_2' \xrightarrow{R_1} t_3 \ldots \xrightarrow{R_1} t_n, \quad n \geq 2,
\]
is an $R_1$-full derivation iff every rule from $R_1$ enabled in some $t'_i$, $1 \leq i \leq n - 1$, is applied along $d$. We denote such $R_1$-full derivation (that starts from $t_1$ and ends with $t_n$) by $t_1 \xrightarrow{R_1 \cup R_2} t_n$.

Definition 19: $R_1$ is fully commuting with $R_2$ iff for every $s$ and $t$, there is $t'$, s.t.

\[
\text{if } s \xrightarrow{R_1 \cup R_2} t \quad \text{then } s \xrightarrow{R_1} t' \xrightarrow{R_2} t
\]

Using similar lemmas, the proof of the following theorem is just the same as that of the sufficiency of E-full-commutation for E-fair termination theorem.

Theorem: (Sufficiency of full-commutation for fair termination in combined system)

If $R_1$ is fully commuting with $R_2$, then the system $(F,R_1)$ is $R_1$-fairly terminating iff $(F,R_1)$ is fairly terminating.

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