EXTRAPOLATION METHODS FOR DIVERGENT OSCILLATORY INFINITE INTEGRALS, THAT ARE DEFINED IN THE SENSE OF SUMMABILITY

by

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EXTRAPOLATION METHODS FOR DIVERGENT OSCILLATORY INFINITE INTEGRALS THAT ARE DEFINED IN THE SENSE OF SUMMABILITY

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ABSTRACT

In a recent work by the author an extrapolation method, the \( W \)-transformation, was developed, by which a large class of oscillatory infinite integrals can be computed very efficiently. The results of this work are extended to a class of divergent oscillatory infinite integrals in the present paper. It is shown in particular that these divergent integrals exist in the sense of Abel summability and that the \( W \)-transformation can be applied to them without any modifications. Convergence results are stated and numerical examples given.
1. INTRODUCTION

In some problems in physics and engineering one encounters oscillatory infinite integrals that do not exist in the ordinary sense due to their integrands' not being integrable at infinity. These integrals may, however, exist in the summability sense and represent physical quantities, and one would like to know their numerical values. One such integral was given in [3], and it is

\[ \int_{0}^{\infty} x e^{-y} \exp(-y \sin \alpha) [2(c^2 + x^2)J_0(x \cos \alpha) - 2x \sec \alpha J_1(x \cos \alpha)] dx, \]

where

\[ y = (x^2 + R^2/4)^{1/2}, \quad c = Ry + R^2/2 \]

and \( R \) is a positive constant, with \( \alpha = 0 \). This integral arises in fluid mechanics in the study of particle interaction in a slow viscous flow. The angle \( \alpha \) relates the particle position to the flow direction and \( R \) is the Reynolds number. See the references in [3].

When \( \alpha = 0 \) the integrand in the integral above belongs to the set \( B_{tl} \), a general family of oscillatory functions, which we define precisely in Section 2. The set \( B_{tl} \) is closely related (and complementary) to another set of functions that we shall term \( B_c \) that was considered in a recent work [7]. The difference between the two sets is that functions in \( B_c \) are integrable at infinity, whereas those in \( B_{tl} \) are not. In fact if \( f(x) \) is in \( B_{tl} \), then \( f(x)/x^p \), for some positive integer \( p \), is in \( B_c \). (For \( \alpha \neq 0 \), the integrand above is in \( B_c \).) In [7] we developed an extrapolation procedure, the \( W \)-transformation, for numerically evaluating the infinite integrals \( \int_{0}^{\infty} f(t) dt, \alpha > 0, f \in B_c \). In the present work we consider the problem of numerically evaluating the divergent integrals \( \int_{a}^{\infty} f(t) dt, \alpha > 0, f \in B_{tl} \), that are defined in the sense of Abel summability. We show that the \( W \)-transformation of [7] can be used in evaluating these integrals efficiently.
To give an idea about the kind of integrands we shall consider, we end this section with a description of the subset $\tilde{B}_d$ of the set $B_d$. We start this with the following definition of the set $A^{(\gamma)}$.

**Definition 1.1** We say that a function $a(x)$, defined for $x>a\geq 0$, belongs to $A^{(\gamma)}$, if it is infinitely differentiable for all $x>a$, and if, as $x\to\infty$, it has a Poincare-type asymptotic expansion of the form

$$a(x) \sim z^{\gamma} \sum_{i=0}^{\infty} \alpha_i / x^i,$$  \hspace{1cm} (1.1)

and all its derivatives, as $x\to\infty$, have Poincare-type asymptotic expansions, which are obtained by differentiating the right hand side of (1.1) term by term.

As consequences of Definition 1.1, we have

1. $A^{(\gamma)} \supset A^{(\gamma-1)} \supset A^{(\gamma-2)} \supset \cdots$.
2. If $\alpha \in A^{(\gamma)}$ and $\beta \in A^{(\gamma)}$, then $\alpha \beta \in A^{(\gamma+\theta)}$, and if, in addition $\beta \not\in A^{(\gamma+\theta)}$, then $\alpha/\beta \in A^{(\gamma-\theta)}$.
3. If $\alpha \in A^{(0)}$, then $\alpha$ is infinitely differentiable for all $x>a$ up to and including $x=\infty$, although not necessarily analytic at $x=\infty$.

**Definition 1.2** The subset $\tilde{B}_d$ of $B_d$ is the collection of all functions $f(x)$ that are defined for $x>a\geq 0$ and are expressible in the form

$$f(x) = \exp(i\phi(x))h(x),$$  \hspace{1cm} (1.2)

where

1. $\phi(x)$ is a real function in $A^{(m)}$, $m$ being a positive integer,
2. $h(x)$ is a (complex) function in $A^{(\gamma)}$ for some $\gamma \geq m-1$.

We require $\gamma \geq m-1$ so that $f(x)$ is not integrable at $x=\infty$. (For $\gamma<m-1$, $f(x)$ is integrable at infinity and belongs to $B_\infty$.)

**Example:** $f(x) = \exp[i(x^3+\sqrt{x^4+2x^3+x})(x+2/\sqrt{x^2+1})^{7/3}]$. Here $m=3$ and $\gamma=7/3$. 


Let us also define

\[ F(x) = \int_a^x f(t) \, dt \quad (1.3) \]

and

\[ I[f] = \int_a^x f(t) \, dt. \quad (1.4) \]

In the next section we show that \( I[f] \) exists in the Abel summability sense, and we derive an asymptotic expansion for its "tail" \( \int_x^\infty f(t) \, dt \) as \( x \to \infty \). This asymptotic expansion is valid also for all functions \( f(x) \) in the set \( B_d \), which we define precisely in the next section. Based on this asymptotic expansion, we devise an extrapolation method for computing \( I[f] \), and this method turns out to be the \( W \)-transformation of \( [7] \). We recall that the \( W \)-transformation produces approximations to \( I[f] \) by using a small number of the finite integrals \( F(x_l), l=0,1,\ldots \), for some carefully selected values of \( x_l \). In Section 3 we give some numerical examples to illustrate the use of the extrapolation procedure. A convergence result is also given.

2. THEORY

Let \( f(x) \) be in \( B_d \). Then \( f(x) \) is expressible as

\[ f(x) = \exp(i\psi(x))h(x), \quad (2.1) \]

where \( \psi(x) \) is real and belongs to \( A^{(m)} \) for some positive integer \( m \), and \( h \in A^{(\gamma)} \) for some \( \gamma \geq m - 1 \). Define

\[ f_\varepsilon(x) = e^{-\varepsilon x}f(x), \quad \varepsilon > 0, \quad (2.2) \]

and let

\[ F_\varepsilon(x) = \int_a^x f_\varepsilon(t) \, dt. \quad (2.3) \]

and consider
\[ I[f_{\varepsilon}] = \int f_{\varepsilon}(t) dt, \]  

(2.4)

which exists in the ordinary sense.

It can be shown that \( f_{\varepsilon}(x) \) satisfies the homogeneous linear first order differential equation

\[ f_{\varepsilon}(x) = s(x)f'_{\varepsilon}(x), \]  

(2.5)

with

\[ 1/s(x) = i\varphi'(x) - \varepsilon + h'(x)/h(x). \]  

(2.6)

Since \( h'/h \in A^{(-1)} \), \( \varphi' \in A^{(m-1)} \), \( m \geq 1 \), and \( \varphi(x) \) and \( \varepsilon \) are real, we see that \( 1/s \in A^{(m-1)} \) and \( 1/s \notin A^{(m-2)} \), for all \( \varepsilon \) including \( \varepsilon = 0 \). Thus \( s \in A^{(-m+1)} \) for all \( \varepsilon \geq 0 \).

Substituting (2.5) in the integral

\[ I[f_{\varepsilon}] - F_{\varepsilon}(x) = \int f_{\varepsilon}(t) dt, \]  

(2.7)

and integrating by parts once, we obtain

\[ \int_{\varepsilon} f_{\varepsilon}(t) dt = -s(x)f_{\varepsilon}(x) - \int s'(t)f_{\varepsilon}(t) dt. \]  

(2.8)

Next defining

\[ s_1(x) = s(x), \quad s_k(x) = s(x)s_{k-1}(x), \quad k = 2,3,..., \]  

(2.9)

substituting (2.5) in the integral on the right hand side of (2.8), and integrating by parts \( N-1 \) times, we obtain

\[ \int f_{\varepsilon}(t) dt = \left[ \sum_{k=1}^{N} (-1)^k s_k(x) f_{\varepsilon}(x) + (-1)^N \int s_N'(t)f_{\varepsilon}(t) dt \right]. \]  

(2.10)

Now \( s_1 = s \in A^{(-m+1)} \) implies that \( s_2 = ss' \in A^{(-2m+1)} \), and, in general, \( s_k \in A^{(-2m+1)} \) for all \( \varepsilon \geq 0 \). The integrand \( s_N'(t)f_{\varepsilon}(t) \) of the integral on the right hand side of (2.10) is of the form \( \exp(-\varepsilon t + i\varphi(t))\hat{h}(t) \), where \( \hat{h}(t) = h(t)s_N'(t) \) and \( \hat{h} \in A^{(n-m)} \) for all \( \varepsilon \geq 0 \). Therefore, the integral \( \int s_N'(t)f_{\varepsilon}(t) dt \) exists in the ordi-
nary sense when \( \varepsilon = 0 \) provided \( \gamma - Nm < m - 1 \), or \( N > -1 + (\gamma + 1)/m \equiv \nu \). In addition, this integral is absolutely convergent when \( \varepsilon = 0 \) provided \( \gamma - Nm < 1 \) or \( N > \nu + 1 \).

Also for \( 0 \leq \varepsilon \leq \varepsilon_0 \), for some fixed \( \varepsilon_0 \), and \( x \) sufficiently large, we can show that there exists a function \( M(t) \) that is independent of \( \varepsilon \), and is in \( A^{(\gamma - Nm)} \), such that \( |\hat{h}(t)| \leq M(t) \) for all \( t \geq x \). Let \( N > \nu + 1 \). Then all the conditions of Theorem 25.14 in [2, p. 352] are satisfied, and consequently

\[
\lim_{\varepsilon \to 0+} \int_0^\infty s_N(t) f(t) dt = \int_0^\infty \lim_{\varepsilon \to 0+} s_N(t) f(t) dt.
\]  

(2.11)

Combining (2.7), (2.10), and (2.11), we finally have

\[
\lim_{\varepsilon \to 0+} I[f_\varepsilon] = F(x) + \left[ \sum_{k=1}^N (-1)^k \lim_{\varepsilon \to 0+} s_k(x) \right] f(x) +
\]

\[
+ (-1)^N \int_0^\infty \lim_{\varepsilon \to 0+} s_N(t) f(t) dt.
\]  

(2.12)

thus proving that \( I[f] \) is defined in the Abel summability sense.

Let us denote \( \lim_{\varepsilon \to 0+} s_k(x) = q_k(x) \). As mentioned in the previous paragraph \( q_k \in A^{(-k\nu + 1)} \). We now reexpress (2.12) in the form

\[
\lim_{\varepsilon \to 0+} I[f_\varepsilon] = F(x) + \left[ \sum_{k=1}^N (-1)^k q_k(x) \right] f(x) + (-1)^N \int g(t) dt,
\]  

(2.13)

where \( g \in B_c \) and is of the form

\[
g(x) = \exp(i\beta x) r(x), \quad r \in A^{(\gamma - Nm)}.
\]  

(2.14)

From Theorem 2.2 in [7] \( \int_0^\infty g(t) dt = g(x) \Lambda(x) \), \( \lambda \in A^{(-m+1)} \).

\[
\int_0^\infty g(t) dt = g(x) \lambda(x), \quad \lambda \in A^{(-m+1)}.
\]  

(2.15)

Thus, (2.13) becomes

\[
\lim_{\varepsilon \to 0+} I[f_\varepsilon] = F(x) + \left[ \sum_{k=1}^N (-1)^k q_k(x) + (-1)^N \mu_N(x) \right] f(x).
\]  

(2.16)

It is easy to see that the expression inside the square brackets is in \( A^{(-m+1)} \). With this observation we now state the main result of this section.
Theorem 2.1: Let \( f \in B_d \) be expressible as in the first paragraph of this section. Then \( \int_a^\infty f(t)dt, a \geq 0, \) is defined in the sense of Abel summability, and for \( x > a, \) there exists a function \( \beta(x) \) in \( A^{(0)} \) such that

\[
\lim_{\varepsilon \to 0^+} I[f] = F(x) + x^{-m+1}\beta(x)f(x). \tag{2.17}
\]

Remark 1: Surprisingly, the result in (2.17) is identical to the one that was obtained for \( f \in B_\gamma \) with \( \gamma < m - 1, \) see Theorem 2.2 in [7]. Thus for all \( \gamma \) we have

\[
I[f] = F(x) + x^{-m+1}\beta(x)f(x), \quad \beta \in A^{(0)}, \tag{2.18}
\]

where \( I[f] \) is to be interpreted as the value of \( \int_a^\infty f(t)dt \) in the sense of Abel summability whenever \( f \in B_d. \)

From this point on we follow closely the treatment given in [7]. \( \vartheta \in A^{(m)} \) implies

\[
\vartheta(x) \sim x^m \sum_{i=0}^{\infty} \vartheta_i / x^i \text{ as } x \to \infty. \tag{2.19}
\]

We can then express \( \vartheta(x) \) in the form \( \vartheta(x) = \vartheta(x) + \Delta(x), \) where

\[
\vartheta(x) = \sum_{i=0}^{m-1} \vartheta_i x^{m-i}, \quad \Delta(x) \sim \sum_{i=0}^{\infty} \vartheta_{m+i} / x^i \text{ as } x \to \infty. \tag{2.20}
\]

Notice that \( \vartheta(x) \) is a polynomial of degree \( m, \) and \( \vartheta(x) \) is in \( A^{(0)} \) since \( \Delta \in A^{(0)}. \)

Thus (2.18) can be re-expressed in the form

\[
I[f] = F(x) + x^{-m+1}e^{i\vartheta(x)}\beta'(x), \quad \beta' \in A^{(0)}, \tag{2.21}
\]

where \( \beta'(x) = x^{-m+1}e^{i\vartheta(x)}\beta' \).

We now define the set \( B_d \).

Definition 2.1: \( B_d \) is the set of all functions \( f(x) \) that can be expressed in the form

\[
f(x) = \sum_{j=1}^r f_j(x), \tag{2.22}
\]

where each \( f_j(x) \) is of the form
such that

(1) $u_j(x)$ is either $e^{ix}$ or $e^{-ix}$ or any linear combination of these (like $\cos x$ or $\sin x$);

(2) $\phi_j \in A^{(m)}$ for all $j$, and $\phi_j(x) = \phi_{j'}(x) = \phi(x)$ for $j \neq j'$;

(3) $h_j \in A^{(\gamma)}$ such that $\gamma_j - \gamma = \text{integer for } j \neq j'$ (hence $h_j \in A^{(\gamma)}$ for all $j$, where $\gamma = \max\{\gamma_1, \ldots, \gamma_r\}$, and $\gamma \geq m - 1$. (If $\gamma < m - 1$, then $f \in B_c$.)

Comparing with $B_c$ (see Lemma 2.1 in [7]), we see that $B_c \cup B_d$ is simply $B_c$ with no restrictions on the $\gamma_f$.

The result in (2.21) can now be extended to functions in $B_d$.

**Theorem 2.2:** Let $f \in B_d$ with the notation of Definition 2.1. Then $I[f]$ is defined in the sense of Abel summability, and

$$I[f] = F(x) + x^{\gamma-m+1} \left[ \cos(\phi(x))b_1(x) + \sin(\phi(x))b_2(x) \right],$$

where $b_1, b_2 \in A^{(0)}$.

**Remark 2:** Theorem 2.2 is an extension of Lemma 2.1 in [7] in the sense that Theorem 2.1 is an extension of Theorem 2.2 in [7]. Thus, (2.24) is valid for all $\gamma_j$ provided $I[f]$ is interpreted as $\int_a^x f(t) dt$ in the sense of Abel summability. The $W$-transformation of [7] is based solely on (2.24), thus it can be used for all functions $f(x)$, whether in $B_c$ or in $B_d$.

For the sake of completeness, we shall briefly recall the main points of the $W$-transformation.

Let $x_0$ be the smallest zero of $\sin(\phi(x))$ greater than $a$ so that $x_0$ is a root of the equation $\phi(x) = q\pi$ for some integer $q$. Then determine $x_0 < x_1 < x_2 < \ldots$, where $x_i$ is a root of $\phi(x) = (q + i)\pi$. (In a similar manner, we can reverse the roles of $\sin$ and $\cos$, starting with $x_0$ as the root of $\cos(\phi(x)) = 0$ or $\phi(x) = (q + 1/2)\pi$.)
The proof of this theorem is 'very similar to those of Theorems 4 and 5 and

\[ (2.25) \]

and solve the system of linear equations

\[ (2.26) \]

for \( \tilde{W}_n^{(j)} \), the approximation to \( I[f] \). The \( \tilde{W}_n^{(j)} \) can be computed very efficiently in

a recursive manner by the \( \tilde{W} \)-algorithm of [6], which is summarized below: Let

\[ M_k^{(s)} = F(x_k) / \psi(x_k), \quad N_k^{(s)} = 1 / \psi(x_k), \quad s = 0, 1, ..., \]

Compute for \( s = 0, 1, ..., \) and \( k = 0, 1, ..., \)

\[ M_k^{(s)} = \left( M_k^{(s)} - M_k^{(s+1)} \right) / \left( x_k^{-1} - x_{k+1}^{-1} \right) \]

\[ N_k^{(s)} = \left( N_k^{(s)} - N_k^{(s+1)} \right) / \left( x_k^{-1} - x_{k+1}^{-1} \right) \]

\[ \tilde{W}_k^{(s)} = M_k^{(s)} / N_k^{(s)} \].

We finally state convergence results on the \( \tilde{W}_n^{(j)} \) for two types of limiting

processes that have been designated Process I and Process II in [6] and [7]. In

Process I \( n \) is fixed and \( j \to \infty \), while in Process II \( j \) is fixed and \( n \to \infty \).

**Theorem 2.3:** For Process I

\[ I[f] - \tilde{W}_n^{(j)} = O \left[ x_j^{-\gamma - m - n} \right] = O \left[ (\gamma - m - n) / m \right] \quad \text{as } j \to \infty, \]

while for Process II

\[ I[f] - \tilde{W}_n^{(j)} = O \left[ n^{-\mu} \right] \quad \text{as } n \to \infty, \quad \text{any } \mu > 0. \]

The proof of this theorem is very similar to those of Theorems 4 and 5 and

their corollary in [6], the only additional factors being that for \( f \in B_d \)

\[ \max_{j \leq j \leq j + n + 1} |\psi(x_j)| = x_{j+1}^{-m+1} = \begin{cases} 0 & \text{as } j \to \infty \end{cases} \]

\[ \max_{j \leq j \leq j + n + 1} |\psi(x_j)| = x_{j+1}^{-m+1} = \begin{cases} 0 & \text{as } n \to \infty \end{cases} \]

\[ (2.29) \text{ implies that Process I converges provided } n > \gamma - m, \text{ while } (2.30) \text{ implies} \]

that Process II always converges and much more quickly than Process I.
3. NUMERICAL EXAMPLES

In this section we apply the $W$-transformation to several integrals whose integrands are in $B_d$. The transformation is implemented using the $W$-algorithm. Only the approximations $W_n^{(0)}$ are tabulated. The computations for these examples were done in double precision arithmetic on the IBM-370 computers at Technion, Haifa and NASA Lewis Research Center, Cleveland, Ohio.

Example 3.1:

$$I = \int_0^\infty \exp[i\varphi(x)]\varphi(x)\varphi'(x)dx = \exp[i\varphi(0)][-1+i\varphi(0)].$$

For $\varphi \in A^{(m)}$, $m>0$ an integer, the integrand $f(x) = \exp[i\varphi(x)]\varphi(x)\varphi'(x)$ is in $B_d$. Numerical results were obtained for the choice $\varphi(x) = x^2-2\sqrt{x^2+x+1}$ so that $\varphi(0) = 0$ and $I = -1$. For this choice of $\varphi(x)$ we have $m = 2$, $\varphi(x) = x^2 + 2x$, and $\gamma = 3$.

The $z_i$ are taken to be consecutive zeros of $\sin(\varphi(x))$. Hence $z_i = -1 + \sqrt{1+(l+1)^2}$ and $\varphi(z_i) = (-1)^l z_i^2$, $l = 0, 1, 2, ...$. The $W$-transformation with these $z_i$'s is applied to the real and imaginary parts of this integral, namely to the integrals $I_1 = \int_0^{z_1} f_1(x)dx$ and $I_2 = \int_0^{z_2} f_2(x)dx$ respectively, where $f_1(x) = \cos(\varphi(x))\varphi(x)\varphi'(x)$ and $f_2(x) = \sin(\varphi(x))\varphi(x)\varphi'(x)$. Note that application of the $W$-algorithm to the original (complex) integral $I = \int_0^\infty f(x)dx$ produces exactly the same approximations for $I_1$ and $I_2$. That is to say, if we denote the approximations to the integrals $I$, $I_1$, and $I_2$ obtained by using the $W$-transformation by $W_n^{(j)}[f]$, $W_n^{(j)}[f_1]$, and $W_n^{(j)}[f_2]$ respectively, then $W_n^{(j)}[f] = W_n^{(j)}[f_1] + iW_n^{(j)}[f_2]$.

The numerical results for $I_1$ and $I_2$ are given in Table 3.1.

Before giving Examples 3.2 and 3.3 we would like to recall that for any $\nu$ and for $\varphi \in A^{(m)}$, $m>0$ an integer, the Bessel functions $J_\nu(\varphi(x))$ and $Y_\nu(\varphi(x))$ are expressible in the form $\eta_1(x)\cos(\varphi(x)) + \eta_2(x)\sin(\varphi(x))$, where $\eta_1, \eta_2 \in A^{(-m/2)}$ and...
\( \bar{\Omega}(x) \) is as defined in (2.20). (See the example following Lemma 2.1 in [7].)

**Example 3.2:**

\[ I_p = \int_0^\infty x^{2p} J_0(x) \, dx. \]

By what has been said in the previous paragraph, for \( p \geq 1/4 \), \( f(x) = x^{2p} J_0(x) \) is in \( B_d \) with \( m=1 \), \( \bar{\Omega}(x) = x \) and \( \gamma = 2p - 1/2 \). The \( x_l \) are chosen to be consecutive zeros of \( \sin x \). Thus, \( x_l = (l+1)\pi \) and \( \psi(x_l) = (-1)^l x_l^{2p-1/2} \), \( l = 0,1,... \). \( I_p \) was computed using the \( W \)-transformation for \( p=1 \) and \( p=2 \), for which, \( I_1 = -1 \) and \( I_2 = 9 \). The results of the computations for \( I_1 \) and \( I_2 \) are given in Table 3.2.

**Example 3.3:** The integral in the first paragraph of Section 1 with \( \alpha = 0 \), namely

\[ I = \int_0^\infty \frac{2x}{cy} \left[(c^2+x^2)J_0(x) - xJ_1(x) \right] \, dx. \]

Here, \( c, y \in A^{(1)} \), and, by what has been said prior to Example 3.2, \((c^2+x^2)J_0(x) - xJ_1(x) \) is of the form \( \omega_1(x) \cos x + \omega_2(x) \sin x \) with \( \omega_1, \omega_2 \in A^{(3/2)} \) so that \( m=1 \) and \( \bar{\Omega}(x) = x \). As in Example 3.2, \( x_l = (l+1)\pi \), thus \( \psi(x_l) = (-1)^l x_l^{1/2} \), \( l=0,1,... \). Table 3.3 contains the numerical results obtained for \( I \) with \( R = 0.1, 1, \) and \( 10 \). For the sake of completeness we mention that, for \( \alpha \neq 0 \), the integrand of this integral is in \( B_e \) with \( \bar{\Omega}(x) = x \cos \alpha \). The \( W \)-transformation can be applied to it with \( x_l = (l+1)\pi \sec \alpha \) and \( \psi(x_l) = (-1)^l x_l^{1/2} \exp(-x_l \sin \alpha) \), \( l=0,1,... \), see [7].

### 4. CONCLUDING REMARKS

In this work we have shown that divergent infinite oscillatory integrals of functions in the set \( B_d \) can be computed very efficiently by using the \( W \)-transformation. The \( W \)-transformation was originally designed to accelerate the convergence of a class of convergent infinite oscillatory integrals of functions in the set \( B_e \). The sets \( B_e \) ad \( B_d \) are complementary in the sense that if a function
$f(x)$ is in $B_d$, then $f(x)/x^p$, for some positive integer $p$, is in $B_c$.

The subject of computation of divergent integrals does not seem to have received much attention with the exception of a few recent works like [1] and [3]. In both of these works, based on numerical testing, it is concluded that if the oscillatory integral $\int_a^x f(z)dz$ is expressed as an infinite series $\sum_{j=0} u_j$, where

$$u_j = \int_{x_j}^{x_{j+1}} f(z)dz,$$

and $x_0=a$ and $x_j, j=1,2,...$, are the consecutive zeros of $f(x)$ greater than $a$, then application of convergence acceleration methods to $\sum_{j=0} u_j$ may produce good results for $\int_a^x f(z)dz$ even when this is a divergent integral defined in the Abel summability sense. In fact [1] demonstrates the use of the Euler and iterated Shanks [5] transformations like $e_1, e_2^2, e_3^3$, etc., and [3] demonstrates the use of the higher order Shanks transformations (or the $\varepsilon$-algorithm [9]) and the Levin [4] transformations. All the integrals, convergent or divergent, dealt with in both [1] and [3] have integrands in $B_c$ or $B_d$.

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<td>6</td>
<td>-0.1000000030821927D 01</td>
<td>0.1456733791697413D 07</td>
</tr>
<tr>
<td>7</td>
<td>-0.999999987003277D 00</td>
<td>-0.6276491325060319D 09</td>
</tr>
<tr>
<td>8</td>
<td>-0.10000000020746D 01</td>
<td>0.5368566940970118D 10</td>
</tr>
<tr>
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<td>-0.100000000000778D 00</td>
<td>-0.1601551103268190D 11</td>
</tr>
<tr>
<td>10</td>
<td>-0.100000000000004D 01</td>
<td>0.582960821128176D 13</td>
</tr>
<tr>
<td>11</td>
<td>-0.100000000000069D 01</td>
<td>0.46684922110266D 13</td>
</tr>
<tr>
<td>12</td>
<td>-0.100000000000071D 01</td>
<td>0.4023335702912757D 13</td>
</tr>
</tbody>
</table>

**TABLE 3.1** $w_n(0)$ for the real and imaginary parts $I_1 = -1$ and $I_2 = 0$

<table>
<thead>
<tr>
<th>n</th>
<th>$w_n(0)$ for $I_1$</th>
<th>$w_n(0)$ for $I_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.165323622758453D 01</td>
<td>-0.1260549269695D 00</td>
</tr>
<tr>
<td>1</td>
<td>-0.102958793239956D 00</td>
<td>-0.042038026620777D 01</td>
</tr>
<tr>
<td>2</td>
<td>-0.999947313856609D 00</td>
<td>0.0457056402650254D 02</td>
</tr>
<tr>
<td>3</td>
<td>-0.999657260248673D 00</td>
<td>0.9046401058467052D 01</td>
</tr>
<tr>
<td>4</td>
<td>-0.10000000121626740D 00</td>
<td>0.899988983220464D 01</td>
</tr>
<tr>
<td>5</td>
<td>-0.99999998716675824D 00</td>
<td>0.99997695356624D 01</td>
</tr>
<tr>
<td>6</td>
<td>-0.999999943795554D 00</td>
<td>0.900000141022153D 01</td>
</tr>
<tr>
<td>7</td>
<td>-0.100000000379949D 00</td>
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</tr>
<tr>
<td>8</td>
<td>-0.999999996961305D 00</td>
<td>0.8999999978832987D 01</td>
</tr>
<tr>
<td>9</td>
<td>-0.999999999991695D 00</td>
<td>0.900000000018098D 01</td>
</tr>
<tr>
<td>10</td>
<td>-0.100000000000193D 01</td>
<td>0.899999999963019D 01</td>
</tr>
<tr>
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<td>-0.1000000000000113D 01</td>
<td>0.89999999981573D 01</td>
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<tr>
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<td>-0.1000000000000101D 01</td>
<td>0.900000000000193D 01</td>
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**TABLE 3.2** $w_n(0)$ for the integrals $I_1$ and $I_2$ in Example 3.2.
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<th>n</th>
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<th>$w_n^{(0)}$ for $R=1$</th>
<th>$w_n^{(0)}$ for $R=10$</th>
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<tr>
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<td>-0.1477911294169409D 01</td>
<td>0.260711912227270D 01</td>
</tr>
<tr>
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<tr>
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</tr>
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</tr>
</tbody>
</table>

**TABLE 3.3** $w_n^{(0)}$ for $R=0.1, 1, 10$ for the integral in Example 3.3.