A PUMPING LEMMA FOR MONOTONOUS SUPER-NETS LANGUAGES

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Abstract

A pumping lemma for languages recognized by a restricted type of Super-nets is presented. This type of Super-nets, called *monotonous*, differs from the usual Super-nets in the absence of restricting places, inhibiting arcs and λ-transitions. The lemma can be used to prove languages to be unrecognizable by monotonous Super-nets. An immediate corollary is the application of the lemma to other known types of Petri-nets. It is notable that the pumping lemma is very similar in structure to the one for regular languages.

1. Introduction

Pumping lemmas are a most useful tool in formal languages theory for proving languages not to belong to a certain class of languages, the one to which the pumping lemma at hand refers. Such well known pumping lemmas exist for regular sets and for context-free languages (see [HU] for a full description of these lemmas).

The concept of a Petri-net was introduced in order to model systems involving concurrency, and literature on this subject can be found, for example, in [B], [CV], [DB] and [P]. Various modifications of the basic petri-nets have been proposed in order to expand their modeling power, among which are the introduction of inhibitor arcs, OR-logic transitions, Boolean type places, and places with finite capacity ([AF], [H], [JV]; [P], [W], [Y], [YB]). The Super-net, introduced in [EY], includes all these extensions, plus some more. In [EY], various special types of Super-nets are defined, and then compared by means of classes of languages associated with them. Both labeled Super-Nets and non-labeled ones are discussed there ("labeled" means that letters from some finite alphabet are associated with the transitions, ([H], [J], [P])).

Here, we consider a restricted type of labeled Super-nets in which everything is allowed except restricting places, inhibiting arcs and λ-transitions. For these nets, which are termed *monotonous*, we introduce a pumping lemma, similar in structure to the one for regular languages. As a corollary, we have that the lemma is applicable to many of the special types of Super-nets defined in [EY].
In section 2.4 we repeat the definition of Super-nets and their languages as it appears in [FY]. In section 5 the pumping lemma is phrased. Section 6 includes the proof of the pumping lemma, and in section 7 we show an example of how the lemma can be used to prove that a language cannot be recognized by a monotonous Super-net.

2. Super-Nets

We denote by \( \mathbb{N} \) the set of non-negative integers.

**Definition.** A SUP-Net (super-net) is a 4-tupla \( N=(P,T,V,K) \), where

1. \( P \) and \( T \) are finite sets of places and transitions, respectively,
2. \( P \cap T = \emptyset, P \cup T \neq \emptyset \),
3. \( V \) is a function,
   \[ V: (P \times T) \cup (T \times P) \rightarrow \omega \cup \{ I, E, L \} \]

   Note. \( I, E, L \) are symbols indicating 'Inhibiting', 'Emptying', and 'Logical' arcs, respectively.

4. \( V(T \times P) \subseteq \omega \),
5. \( K \) is a function,
   \[ K: P \rightarrow \{ \omega \} \cup \omega \times \{ A, R \} \]

   Note. \( A, R \) are symbols indicating 'Absorbing', and 'Restricting' places.

If \( K(p) = \omega \), we say that the place \( p \) has infinite capacity. If \( K(p) = (k, A) \) or \( K(p) = (k, R) \) we say that the place \( p \) has finite capacity \( k \in \omega \). We denote by \( k(p) \) the capacity (\( \omega \) or \( k \in \omega \)) of the place \( p \).

**Definition.** A Marked SUP-Net is a pair \( S=(N,M) \), where \( N \) is a SUP-Net and \( M \) is a marking of \( N \), i.e. a function \( M: P \rightarrow \omega \), satisfying the condition

\[ (\forall p \in P)[k(p) \in \omega \rightarrow M(p) \leq k(p)] \]
A marked SUP-Net $S=(P, T, V, K, M)$ is represented graphically as follows:

1. places are represented by circles (O);
2. each node $p$ is labeled by $p/K(p)$;
3. A transition $t$ is represented by a bar, labeled by $t$;
4. the place $p \in P$ is connected by a directed arc to the transition $t \in T$, iff $V(p,t)\neq 0$; the arc is labeled by $V(p,t)$;
5. the transition $t \in T$ is connected by a directed arc to $p \in P$, iff $V(t,p)>0$; the arc is labeled by $V(t,p)$;
6. the integer $m=M(p)$ is written inside the circle representing $p$; usually, one does not write 0 inside the circle.

An example of a marked SUP-Net is shown in Fig. 1.

**Definition.** Let $S=(P, T, V, K, M)$ be a marked SUP-Net. We define a function $W: P \times T \rightarrow \omega$ as follows:

$$W(p,t) = \begin{cases} V(p,t) & \text{if } V(p,t) \in \omega, \\ 0 & \text{if } V(p,t) = I, \\ M(p) & \text{if } V(p,t) = E, \\ 1 & \text{if } V(p,t) = E \land M(p) > 0, \\ 0 & \text{if } V(p,t) = L \land M(p) = 0. \end{cases}$$

**Definition.** Let $S=(P, T, V, K, M)$ be a marked SUP-Net. A transition $t \in T$ is enabled iff the following conditions are satisfied:

1. $(\forall p \in P)[V(p,t) \in \omega \rightarrow M(p) \geq V(p,t)]$,
2. $(\forall p \in P)[V(p,t) = I \rightarrow M(p) = 0]$,
3. $(\forall p \in P)[V(p,t) = E \rightarrow M(p) > 0]$,
4. $(\exists p \in P) V(p,t) = L \rightarrow (\exists p \in P)[V(p,t) = L \land M(p) > 0]$,
5. $(\forall p \in P)[K(p) \in (\omega, R) \rightarrow M(p) + V(t,p) - W(p,t) \leq k(p)]$.

**Definition.** Let $S=(P, T, V, K, M)$ be a marked SUP-Net and $t \in T$, an enabled transition
We define the marking $M$ of $N=(P,T,V,K)$ as follows:

$$(\forall p \in P) M'(p) = \min[M(p) + V(t,p) - W(p,t), k(p)].$$

We say that $M'$ is obtained from $M$ by firing $t$ (notation: $M[t>M']$). If there exists $t \in T$ such that $M[t>M'$ we write $M \models M'$. Frequently, it is convenient to represent a marking $M$ by the vector $(M(p_1), M(p_2), \ldots, M(p_n))$, where $P = \{p_1, \ldots, p_n\}$.

For the example of Fig. 1 we obtain the following 'firing sequence',

$(0.3,1)[t_2>(0.5,1), (0.5,1)[t_2>(0.6,1), (0.6,1)[t_5>(0.6,0).$
In this section we represent various types of nets as special cases of SUP-Nets.

**Definition.** The following table defines various types of nets as special cases of SUP-Nets, by restricting $\text{range}(V)$ and $\text{range}(K)$.

<table>
<thead>
<tr>
<th>Type of Net</th>
<th>$\text{range}(V)$</th>
<th>$\text{range}(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GP-Net</td>
<td>$\omega$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>P-Net</td>
<td>${0,1}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>L-Net</td>
<td>${0,1} \cup {I}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>E-Net</td>
<td>${0,1} \cup {E}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>L-Net</td>
<td>${0,1} \cup {L}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>A-Net</td>
<td>${0,1}$</td>
<td>$\omega \times {A}$</td>
</tr>
<tr>
<td>R-Net</td>
<td>${0,1}$</td>
<td>$\omega \times {R}$</td>
</tr>
</tbody>
</table>

**4. Super-nets languages**

With a given marked SUP-Net $S = (P, T, V, K, M)$ we associate a language $L(S)$ over the alphabet $T$.

**Definition.** Let $N = (P, T, V, K)$ be a SUP-Net, and let $w \in T^+$, i.e. $w$ is a finite string of transitions $w = t_1 t_2 \cdots t_r$. $w$ is called a *firing sequence* of the marked SUP-Net $S = (N, M)$ iff there exist markings $M_1, \ldots, M_r$ such that

$$M[t_1 > M_1, M_1[t_2 > M_2, \ldots, M_r-1[t_r > M_r].$$

In this case we write $M[w > M_r]$, and say that $M_r$ is reachable from $M$. We also write
$M[\lambda > M]$ for every marking $M$, where $\lambda$ denotes the empty sequence.

**Definition.** Let $S=(P,T,V,K,M)$ be a marked SUP-Net. We define its language $L(S)$ as follows:

\[ L(S) = \{ z \in T^* \mid (\exists M') M[\lambda > M'] \}. \]

In this paper we are also concerned with labeled Super-Nets and their languages.

**Definition** A labeled SUP-Net is a triple $\Gamma=(S,\Sigma,\eta)$, where $S=(P,T,V,K,M)$ is a marked SUP-Net, $\Sigma$ is a finite alphabet and $\eta$ is a mapping $\eta : T \rightarrow \Sigma \cup \{ \lambda \}$. $\Gamma$ is called $\lambda$-*free* iff $\eta(T) \subseteq \Sigma$. The language of $\Gamma$ is defined by

\[ L(\Gamma) = \eta(L(S)) = \{ \eta(z) \mid z \in L(S) \}. \]

If $M[w > M']$ where $w \in T^*$ we also write $M[\eta(w) > M']$, and say that $M'$ is obtained from $M$ by *firing* $\eta(w)$. Clearly every marked SUP-Net $S$ is also a labeled $\lambda$-free SUP-Net $\Gamma=(S,\Sigma,\eta)$ where $\Sigma=T$ and $(\forall t \in T)(\eta(t)=t)$.

**Definition.** Let $L$ be a language over some finite alphabet $\Sigma$. We say that $L$ is $GP$-*realizable* iff $L=L(\Gamma)$ for some marked GP-Net $S=(P,T,V,K,M)$ with $T=\Sigma$. $L$ is $LGP$-*realizable* iff $L=L(\Gamma)$ for some labeled $\lambda$-free GP-Net $\Gamma=(S,\Sigma,\eta)$. $L$ is $L^GP$-*realizable* iff there exists an arbitrary labeled GP-NET $\Gamma$ with $L=L(\Gamma)$. We denote by $GPL$, $LGPL$, $L^GPL$ the sets of all GP-realizable, LGP-realizable, and $L^GP$-realizable languages, respectively.

In a similar way we associate sets of languages with the other types of SUPER-Nets defined above.

Evidently, all the languages defined above are prefix languages.

5. The pumping lemma

The result we are about to present deals with a restricted kind of labeled SUP-Nets, which we now define.
Definition. A monotonous SUP-Net is a labeled-\(\lambda\)-free SUP-Net, having no restricting places, and no inhibiting arcs, i.e range \(N\) \(\subseteq \omega \cup \{E,L\}\) and range \((K) \subseteq \omega \cup \omega \times \{A\}\). We denote by MSL the family of languages recognized by monotonous SUP-Nets.

Remark. The MSL class contains all languages recognized by \(\lambda\)-free SUP-Nets having no inhibiting arcs, allowing restricting places, since such places can be simulated by ordinary ones (see Theorem 5.2 (a) in [EY]). Restricting places are not allowed in monotonous SUP-Nets in order to simplify the proof of the pumping lemma. We shall discuss the other restrictions later.

The theorem can now be stated.

Theorem (pumping lemma for petri nets languages). Let \(L\) be a language in MSL. Then there exists a constant \(n\), such that whenever \(z \in L\) and \(|z| \geq n\), \(z\) can be written as \(z = uvw\), in such a way that the following conditions hold:

(a) \(|uw| \leq n\).
(b) \(|u| \geq 1\).
(c) For all \(i \geq 1\) \(uv^i w \in L\).

6. Proof of the theorem

In order to proceed, we need some results concerning vectors of non-negative integers. Let \(\omega^n\) denote the set of all vectors over \(\omega\) of length \(n\). For vectors \(v_1, v_2 \in \omega^n\) we write \(v_1 \geq v_2\) iff for all \(1 \leq i \leq n\), \(v_1(i) \geq v_2(i)\). \(v_1 \leq v_2\) is used similarly. \(v_1 \not\geq v_2\) and \(v_1 \not\leq v_2\) are used for the negations of the above notions. The following two lemmas recall well-known results.

Lemma 1. Let \(n_1, n_2, n_3, \ldots\) be an infinite sequence of non-negative integers. Then there exists an infinite increasing sequence of integers \(i_1 < i_2 < i_3 \ldots\) such that \(n_1 \leq n_{i_1} \leq n_{i_2} \leq n_{i_3} \ldots\).

Proof. Let \(i_1\) be the index of the first occurrence of the minimal element of \(\{n_1, n_2, n_3, \ldots\\}\) in the given sequence. In general, let \(i_j\) be the index of the first occurrence of the minimal element of \(\{n_{j-1}+1, n_{j-1}+2, n_{j-1}+3, \ldots\}\) in the given
Lemma 2. Let \( v_1, v_2, v_3, \ldots \) be an infinite sequence of vectors from \( \omega^n \). Then there exists an infinite increasing sequence of integers \( t_1 < t_2 < t_3 \ldots \) such that \( v_{t_1} \leq v_{t_2} \leq v_{t_3} \ldots \).

Proof. By lemma 1, construct an infinite subsequence of \( \{v_i\}_{i=1}^\infty \) in which the first coordinate is nondecreasing. Again by lemma 1, construct from this subsequence another one, in which the second coordinate is also nondecreasing, and so forth. The original indices of the vectors in the final (n-th) subsequence form the desired sequence \( t_1, t_2, \ldots \).

In the sequel, we shall need Konig's infinity lemma ([K]).

Lemma 3. (Konig's infinity lemma). Let \( T \) be an infinite directed tree. If the out-degree of every vertex in \( T \) is finite, then there exists an infinite path in \( T \).

Let \( A \) be a set and let \( Q \subseteq A \times A \) be a binary relation. \( Q \) is said to be finitely branching if for all \( a \in A \) the set \( \{ b \mid (a, b) \in Q \} \) is finite. Let \( R \) be a binary relation over \( \omega^n \). An \( R \)-sequence is a finite sequence of vectors from \( \omega^n \), \( v_1, v_2, \ldots, v_l \), such that for all \( 1 \leq i \leq l-1 \), \( (v_i, v_{i+1}) \in R \). \( l \) is the length of the sequence.

Lemma 4. Let \( v_1 \in \omega^n \) be a vector; and let \( R \) be a finitely branching binary relation over \( \omega^n \). Then, there exists a constant \( k \) such that for all \( R \)-sequences of length greater or equal to \( k \) starting with \( v_1, v_1, v_2 \ldots v_l \), there are \( 1 \leq i \leq l \) such that \( v_i \leq v_j \).

Proof. Assume, by way of contradiction, that for all \( k \), there exists an \( R \)-sequence \( v_1, v_2, \ldots, v_l \) with \( l > k \), in which for all \( 1 \leq i < j \leq l \), \( v_i \not\leq v_j \). Construct a directed tree \( T \) having vectors from \( \omega^n \) as nodes as follows: The root of \( T \) is \( v_1 \). The direct descendents of a node \( v \), to which the path from the root is \( v_1, v_2, \ldots, v_k \), are exactly the vectors \( v' \) such that \( (v, v') \in R \) and \( v' \not\leq v_i \) for all \( 1 \leq i \leq s \). If a vector is to be a descendent
of more then one node, it is duplicated, so that the tree structure is kept (thus, formally, the nodes of T are not vectors from \( \omega^n \), but rather, they are labeled by such vectors).

By the assumption, there are paths of length greater or equal to \( k \) in \( T \) for all \( k \), hence \( T \) is infinite. Also, since \( R \) is finitely branching, the out-degree of each node is finite. Thus, by Konig's infinity lemma, there is an infinite path in \( T \), \( v_1-v_2--- \). By the construction, we have that, for all \( i < j \), \( v_i \neq v_j \) and therefore, there could not exist an infinite sequence of integers \( i_1 < i_2 < i_3 \cdot \cdot \cdot \) such that \( v_{i_1} \leq v_{i_2} \leq v_{i_3} \cdot \cdot \cdot \) - a contradiction to lemma 2.

\[ \square \]

We now return to petri nets. Given a SUP-Net \( N=(P;T,V,K) \), one can regard a marking of \( N \) as a vector from \( \omega^{|P|} \), as we do in the following lemma.

**Lemma 5.** Let \( N=(P,T,V,K) \) be a SUP-Net. Then the relation:

\[ R = \{ (M,M') \mid M,M' \text{ are markings of } N, \text{ and } M \models M' \} \]

is finitely branching.

**Proof.** Trivial, since for a given marking \( M \), the set \( \{ M' \mid M \models M' \} \) can contain at most \(|T|\) elements, corresponding to the activation of each transition in \( T \).

\[ \square \]

The next lemma makes use of the monotonocity of a labeled SUP-Net.

**Lemma 6.** Let \( \Gamma=(S,\Sigma,\eta) \) be a monotonous SUP-Net, and let \( M_1, M_2 \) be markings of \( \Gamma \) such that \( M_1 \models M_2 \) (again, markings are regarded as vectors). If for some marking \( \overline{M}_1 \) and \( z \in \Sigma^* \), \( M_1[z] > \overline{M}_1 \), then there exists a marking \( \overline{M}_2 \) such that \( M_2[z] > \overline{M}_2 \) and \( \overline{M}_1 \leq \overline{M}_2 \).

**Proof.** Since \( M_1[z] > \overline{M}_1 \), there exists a firing sequence in \( S, y \), such that \( \eta(y)=z \) and \( \overline{M}_1[y] > \overline{M}_1 \) (in \( S \)). It follows immediately from the definition, that in the absence of restricting places and inhibiting arcs, whatever is firable in a given marking is also such in
any marking greater or equal to it. Thus $M_2[x>M_2$ for some $M_2$, which implies that $M_2[x>M_2$. It is also immediate from the definition that $M_2>M_2$.

\[ \square \]

We now complete the proof of the theorem. Let $L$ be $L(\Gamma)$ for a monotonous SUP-Net $\Gamma=(P,T,V,K,M,\Sigma,\eta)$. Let $R$ be the relation over $\omega^{|P|}$ defined as before, i.e. $R=\{ (M',M') \mid M',M' \text{ are markings of } \Gamma, \text{ and } M'\triangleright M'' \}$. By lemma 5, $R$ is finitely branching, and thus, by lemma 4, there exists a constant $k$ such that for all $R$-sequences of length equal or greater to $k$ starting at $M_i$: $M_1(\triangleright M),M_2, \cdots M_i$, there are $1\leq i<j\leq k$ such that $M_i\leq M_j$.

We claim that $n=k-1$ satisfies the requirements of the pumping lemma. To see that, let $z=a_1a_2 \cdots a_i$ be some word in $L$ with $l=n$. Since $L$ is accepted by $\Gamma$, which is $\lambda$-free, there exist $M_1,M_2, \cdots M_{i+1}$ with $M_1=M$, such that $M_i[t_1>M_2[t_2]> \cdots [t_i>M_{i+1}$ for some $t_1, \cdots t_i$ in $T$, satisfying that $\eta(t_\alpha)=a_\alpha$ for all $\alpha$. The sequence $M_1, \cdots M_{i+1}$ is an $R$-sequence, and so, Since $l+1\geq k$, there are $1\leq i<j\leq k$ such that $M_i\leq M_j$. Thus, let $i<j$ be such indices, with the additional property that $j$ is minimal, i.e. for all other $i'<j'$ satisfying that $M_i\leq M_{j'}$ we have $j\leq j'$ (it is easy to see, that such $i,j$ always exist).

Now, fix $u=\eta(t_1 \cdots t_{i-1})$, $v=\eta(t_i \cdots t_{j-1})$ and $w=\eta(t_j \cdots t_i)$. Clearly, $z=uvw$. We now verify that the three conditions hold.

(a) $(|uv| \leq n)$. Assume, by contradiction, that $|uv|>n$. Since $|uv|=j-1$ and $n=k-1$, we have that $j>k$. The sequence $M_1 \cdots M_k$ is an $R$-sequence of length $k$ and thus, by the definition of $k$, there exist $1\leq i'<j' \leq k$ such that $M_i \leq M_{j'}$, but then, $j'<j$ - a contradiction to the minimality of $j$.

(b) $(|v| \geq 1)$. This is clear, since $i<j$ and $\Gamma$ is $\lambda$-free.

(c) (For all $i\geq 1$) $uv^i \in L$) By lemma 6, we have that $M_j[t_1 \cdots t_{j-1}>M'$ for some $M' \geq M_j$ (since $M_i[t_1 \cdots t_{j-1}>M_j$ and $M_i \geq M_j$). Again by lemma 6, $M_1[t_1 \cdots t_j>$. By summarizing the above conclusions, we get that $M_1[t_1 \cdots t_{j-1}t_i \cdots t_{j-1} \cdots t_i>$, and so $uv^i \in L$. This scheme can be extended to all $i \geq 1$ using induction.

This completes the proof of the pumping lemma.
As an immediate corollary we have,

**Corollary.** The pumping lemma also holds for PL, GPL, EL, LL, AL, RL, LPL, LGPL, LED, LLL, LAL and LRL (i.e. the lemma can be rephrased with any of the above families instead of MSL).

**Proof.** Immediate, since all these families are contained in MSL.

### 7. Applications of the pumping lemma

The above pumping lemma provides us with a standard tool to prove that a given language \( L \) cannot be recognized by any monotonic SUP-Nets.

**Example.** We show that \( L = \text{pref}\left( \{ a^k b^k c \mid k \geq 0 \} \right) \) is not in MSL.

Assume, by contradiction, that \( L \in MSL \). Let \( n \) be the constant of the pumping lemma. Choose \( z = a^n b^n c \). Clearly, \( z \in L \) and \( |z| \geq n \). By the assumption, \( z \) can be written as \( z = uvw \) with conditions (a)-(c) holding. Now, from (a) and (b) we have that \( u, v, w \) are of the form \( u = a^{t_1} \), \( v = a^{t_2} \), and \( w = a^{t_3} b^k c \) where \( t_3 \geq 1 \). By (c), for \( i = 2 \), \( uv^2w = a^{t_1 + 2t_2 + t_3} b^k c \in L \) - a contradiction since \( t_1 + 2t_2 + t_3 > n \).

### 8. Summary

We have presented a pumping lemma for monotonic Super-nets languages, providing us with a standard tool for proving languages not to be in MSL.

Monotonic Super-nets differ from the ordinary ones in having three restrictions. The absence of restricting places is not necessary, as mentioned before. Not allowing inhibiting arcs is necessary, as can be seen from the \( I \)-net shown in Figure 2. This net recognizes the language \( \text{pref}(\{ a^k b^k c \mid k \geq 0 \}) \), which was shown in the previous section not to satisfy the requirements of the pumping lemma. We do not know whether not allowing \( \lambda \)-transitions is necessary, and this issue still remains to be closed.
It would be useful to find a pumping lemma for a less restrictive type of Super-nets. Note that finding one for an unrestricted Super-net is unlikely, since it is known that the class of languages accepted by the Super-nets is the class of all the RE languages.

Another possible work is to try to "refine" the lemma, so that it can be used to separate between more limited types of Super-nets. For example, finding a pumping lemma for LRL, which is not applicable to LPL, could help proving languages in LPL not to be in LRL.
REFERENCES


Definition. A monotonic SUP-Net is a labeled $\lambda$-free SUP-Net, having no restricting places, and no inhibiting arcs, i.e. $\text{range}(V) \subseteq \omega \cup \{E,L\}$ and $\text{range}(K) \subseteq \{\infty\} \cup \omega \times \{A\}$. We denote by $\text{MSL}$ the family of languages recognized by monotonic SUP-Nets.

Remark. The $\text{MSL}$ class contains all languages recognized by $\lambda$-free SUP-Nets having no inhibiting arcs, allowing restricting places, since such places can be simulated by ordinary ones (see Theorem 5.2 (a) in [EY]). Restricting places are not allowed in monotonic SUP-Nets in order to simplify the proof of the pumping lemma. We shall discuss the other restrictions later.

The theorem can now be stated.

Theorem (pumping lemma for petri nets languages). Let $L$ be a language in $\text{MSL}$. Then there exists a constant $n$, such that whenever $z \in L$ and $|z| \geq n$, $z$ can be written as $z = uvw$, in such a way that the following conditions hold:

(a) $|uv| \leq n$.

(b) $|v| \geq 1$.

(c) For all $i \geq 1$, $uv^i w \in L$.

6. Proof of the theorem

In order to proceed, we need some results concerning vectors of non-negative integers. Let $\omega^n$ denote the set of all vectors over $\omega$ of length $n$. For vectors $u_1, u_2 \in \omega^n$ we write $u_1 < u_2$ iff for all $1 \leq i \leq n$, $u_1(i) \leq u_2(i)$. $u_1 \leq u_2$ is used similarly. $u_1 < u_2$ and $u_1 \neq u_2$ are used for the negations of the above notions. The following two lemmas recall well-known results.

Lemma 1. Let $n_1, n_2, n_3, \ldots$ be an infinite sequence of non-negative integers. Then there exists an infinite increasing sequence of integers $i_1 < i_2 < i_3 \ldots$ such that $n_1 \leq n_2 \leq n_3 \ldots$.

Proof. Let $i_1$ be the index of the first occurrence of the minimal element of $\{n_1, n_2, n_3, \ldots\}$ in the given sequence. In general, let $i_j$ be the index of the first occurrence of the minimal element of $\{n_{j-1}, n_{j-1+2}, n_{j-1+3}, \ldots\}$ in the given