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AN M/G/1 QUEUE WITH VACATIONS AND A THRESHOLD

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An M/G/1 Queue with Vacations and a Threshold

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Abstract

We consider a queueing system that differs from the standard M/G/1 only in that at the end of a busy-period the server takes a sequence of vacations; inspecting the state of the queue at the end of each. When the length of the queue exceeds a predetermined level m it returns to serve the queue exhaustively. Only steady-state results are presented.

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Introduction.

1. Consider a queue subject to arrivals that follow a Poisson process with constant rate λ. These arrivals present service requirements that follow the process $\{S_i, i \geq 0\}$, with the S_i all independent and having the cumulative distribution function (cdf) $F(\cdot)$, probability density function (pdf) $f(\cdot)$ and Laplace- Stieltjes transform (LST) $S(\cdot)$. A single server handles them at the order of arrival (this last restriction is immaterial for most of our analysis). In discussing the evolution of the system it will be found convenient to adopt a language that allows the server judgement and volition. At the end of a busy-period the server departs for a "vacation" with a random duration U. When U terminates the length of the queue, X, is inspected. If $X<m$, for some predetermined integer $m>0$, the server goes on an additional vacation, also distributed as U, and so on. Once the queue-length at vacation end reaches (or exceeds) m, the server resumes service until the queue is exhausted, whereupon the vacation process is renewed. We shall evaluate the steady-state distribution of queue-length at a random point in time and request sojourn time, where the main interest is in their dependence on m.

2. Such systems with vacations have been described extensively in the literature, the earliest one being probably [Skinner, 1967] (but see also [Gelenbe and Iasnogorodski, 1980] and additional references there). Recently an interesting observation was made in [Fuhrman, 1984] on the way the number of customers present in the system can be related to the number in a standard M/G/1 system. In [Doshi, 1985] the assumption that all the vacations have the same distribution is relaxed (but they are still assumed independent). As far as we know only the minimal threshold, $m=1$, has been considered.

Analysis.

1. Denote by I the number of customers in the queue when the service is resumed, and note that $I \geq m$ with probability 1. Proceeding exactly as in a class-room treatment of the standard M/G/1 system we observe the chain formed by the queue-length at points that correspond to service completion epochs. Since we later have to consider another embedded chain we shall grace the quantities that pertain to this one with the subscript s. The number of new arrivals, during a period distributed as A will be denoted by $\tilde{A}$, and from the corresponding LST $A(\cdot)$ we obtain the pgf $\tilde{A}(z) = A(\lambda - \lambda z)$. For the steady-state probabilities we can now write

$$\text{Prob (X at departure has length } x) \overset{\text{def}}{=} p_s(x)$$

$$= \sum_{i=1}^{x+1} p_s(i) P(\tilde{S} = x-i+1) + p_s(0) \sum_{i=m}^{x+1} P(I=i) P(\tilde{S} = x-i+1) \qquad x \geq 0 \tag{1}$$

where the latter sum vanishes for $x < m-1$. Hence

$$g_s(z) \overset{\text{def}}{=} \sum_{x \geq 0} p_s(x) z^x = \frac{1}{z}\tilde{S}(z)(g_s(z) - p_s(0)) + p_s(0)\frac{\tilde{S}(z)}{z} I(z)$$

and we get the result

$$g_s(z) = p_s(0)\tilde{S}(z)\frac{I(z)-1}{z - \tilde{S}(z)} \tag{2}$$

Observe that by setting $I(z) = z$ the familiar M/G/1 equation reappears. Substituting $z = 1$ we obtain $p_s(0)$ as the reciprocal of the number of customers served in a busy-period:

$$p_s(0) = (1-\rho)/E[I] \qquad \rho = \lambda E(S) \tag{3}$$

We get the nonsurprising stationarity condition $\rho < 1$.

2. It only remains to determine $I(z)$. Possibly the simplest way is to consider the queue length embedded at a larger set of points, comprising both service and vacation terminations, but *preserving* the "type" of the points. Considering the probability that such a point chosen at random is a vacation-end and that x are then queued, we could write, with obvious notation:

$$p(x,U) = p(0,S)P(\tilde{U}=x) + \sum_{i=0}^{\min(x,m-1)} p(i,U)P(\tilde{U}=x-i) \qquad x \geq 0 \tag{4}$$

and immediately

$$g(z,U) \overset{\text{def}}{=} \sum_{x \geq 0} p(x,U) z^x$$

$$= [p(0,S) + h(z)]\tilde{U}(z); \qquad h(z) \overset{\text{def}}{=} \sum_{i=0}^{m-1} p(i,U) z^i \tag{5}$$

The component probabilities of $h(z)$ can be determined numerically by restricting x to $0 \leq x < m$ in equation (4):

$$p(x,U) = p(0,S)P(\tilde{U}=x) + \sum_{i=0}^{x} p(i,U)P(\tilde{U}=x-i) \qquad 0 \leq x < m. \tag{6}$$

The solution of (6) is immediate by successive substitutions, starting with $p(0,U)$, up to a multiplicative factor – the $p(0,S)$ that appears in the right-hand-side. Let $h_1(z) \overset{\text{def}}{=} \sum_{i=0}^{m-1} p_1(i,U) z^i$ denote the solution obtained when the value of this factor is taken as 1, i.e. $h(z) = p(0,S)h_1(z)$.

For the special case of exponentially distributed vacation durations, $U_i \sim exp(\eta)$, one finds that the $p_1(z,U)$ are all equal (to η/λ), independently of m*.

Considering the relation between $g(z,U)$ and $I(z)$ (or, rather, between the corresponding random variables) we get

$$I(z) = \frac{g(z,U) - h(z)}{g(1,U) - h(1)} = \tilde{U}(z) + h_1(z)[\tilde{U}(z)-1] \tag{7}$$

where equation (5) was used to obtain the last equality.

Surprising as it may seem at first blush, we find that there is no need to evaluate the ratio between $p_s(x)$ and $p(x,S)$ in order to obtain $g_s(z)$. We do not have an adequate intuitive explanation for this convenient fact.

3. Observe that to relate the two one needs the ratio between the expected number of customers served during a (compound) busy-period and the expected number of vacations in an "idle period". This last number, which we denote by $E[J_m]$ has the generating function

$$J(u) \overset{\text{def}}{=} \sum_{m\geq 1} E[J_m]u^m = \frac{u}{1-u}[1-\tilde{U}(u)]^{-1}. \tag{8}$$

The expected queue length at departure epochs (and hence also at random time) is given by the first derivative of $g_s(\cdot)$ at $z=1$:

$$E(X) = \frac{(1-\rho)I'' + (\tilde{S}''+2\rho(1-\rho))I'}{2(1-\rho)I'} \qquad \text{Note: } \tilde{S}'' = \lambda^2 E(S^2) \tag{9}$$

with all the derivatives being taken at $z=1$. Higher moments are similarly available.

The LST of the distribution of the sojourn time H of a customer in this system, assuming FCFS regime, is given in terms of $g_s(z)$, just as for the standard M/G/1 system:

$$H(s) = g_s(1-s/\lambda). \tag{10}$$

The above treatment is entirely straightforward to extend to the case when successive vacation durations are not identically distributed, in the same manner as done in [Doshi, 1985], for the case $m=1$.

References

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* Note that the $p_1(z,U)$ are not probabilities (only $p(z,U)$ are), thus one need not be alarmed when

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$h_1(1) > 1$.