A PROGRAM TRANSFORMATION REGARDED AS
A PROOF TRANSFORMATION

by

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Technical Report #371

July 1985
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1. Introduction

In [SFR83] a program transformation was presented, transforming recursive sequential programs into recursive distributed programs, provided that distributed implementations are given for the sequential "pieces" between recursive calls. The use of such a transformation was suggested there as a good methodology for obtaining distributed programs in view of the non-trivial results obtained by applying the transformation to several $k$-selection programs, obtaining new distributed $k$-selection algorithms.

The purpose of this paper is to view the above mentioned transformation as a proof transformation: given a correctness proof for the source sequential program, derive systematically a correctness proof for the distributed program resulting by applying the program transformation. We regard the proof transformation as an interesting exercise in program verification and as an indirect proof of the correctness of the program transformation itself.

In order to carry out such a task, the discussion has to be linked to concrete programming languages for which a deductive proof-system is known. For the sequential programming language we take a simple language with parameterless recursion, similar to the standard languages considered in the context of formal semantics [deB80]. As for the language for distributed programming, we chose a variant of CSP [H78], both because our familiarity with the language and because of the proof system available for it [AFR80]. We use here Apt's new formulation of it [A84], using the proof assumptions paradigm.

As the original CSP had no procedures mechanism, we define an extension, called CSPR, having local parameterless recursive procedures. The extension to parametrized recursion poses no further problems.

Some acquaintance with CSP and the proof system in [AFR80] is needed in
order to understand the paper.

2. Notations and definitions

Define the programming language \text{PL} to include: \textit{skip}, assignments, sequential composition, guarded commands, parameterless recursive procedure declarations (without nested declarations, for simplicity) and procedure \textit{calls}.

Next, define \text{CSPR} as CSP augmented with local parameterless recursive procedures. This is a rather modest extension, where we have recursion within concurrency, as opposed to recursion across concurrency, e.g. [LBC182], where recursive calls may spawn new processes, creating dynamic networks.

In both languages, procedure declarations are denoted by \textcolor{red}{R \leftarrow \text{proc BR end}}., where \textcolor{red}{R} is the procedure name and \textcolor{red}{BR} its body. The keyword \textit{call} is used for procedure calls.

As the assertion language (for both programming languages) we take \text{L}, a first order language with equality. We use \textcolor{red}{a, b, x, y, z} to denote variables of \text{L}, \textcolor{red}{s, f} to denote terms (expressions) of \text{L}, \textcolor{red}{e} to denote a quantifier-free formula of \text{L} and \textcolor{red}{p, q} to denote the formulae (assertions) of \text{L}.

A proof system for partial correctness assertions for \text{PL} is standard (see [AB1]). A proof system for partial correctness assertions for \text{CSPR} is obtained from the corresponding rules for CSP [AFR80, AB4] by the additional rule for (local) recursion appearing in separate process proofs (see figure 1).

\textbf{Explanations:} soundness and completeness of the local recursion rule follow from soundness and completeness of the recursion rule in the sequential context. An assertion \textcolor{red}{\{p_a\} a \{q_a\}} about a communication command \textcolor{red}{a} in the body of a recursive procedure joins the i/o-assumptions set and is, discharged after the cooperation proof [AFR80, AB4]. At execution time such a command \textcolor{red}{a} may be

\begin{align*}
A, \{p \} R \{q \} \models \{p \} BR \{q \}
\end{align*}

\begin{align*}
A \models \{p \} R \{q \}
\end{align*}

\textbf{Figure 1. The (local) recursion rule}
executed several times. The cooperation proof guaranties that if it is executed then its post condition $q_0$ is established. If for some instance of $a$ no matching i/o command is available (leading to deadlock or failure) then $a$ is not executed - without contradiction to the partial correctness assertion. An example of application of the local recursion rule in a partial correctness proof of a simple CSPR program appears in the appendix.

3. The proof transformer

Assume that a sequential PL program $S$ (see figure 2) and a proof of $[p]S[q]$ are given. The assertions $p$ and $q$ may refer to all the variables in $S$. Note that $p$, $q$ are the respective pre and post - conditions associated with the (unique) recursive call to $R$. $NR^0$ denotes the exit part (containing no recursive calls), $e$ denotes the corresponding exit condition and $NR^1, NR^2$ denote the nonrecursive parts of $S$. Handling more than one recursive call (and more than two nonrecursive parts) is similar. Assume a given fixed network with nodes $N_1$ ... $N_n$. We want to assign process $P_i$ to node $N_i$. Let $V = free(p) \cup free(q)$. A distribution $V$ of $V$ is a partition of $V$ into $\{V_i\}_{i=1}^n$. $V_i$ is taken as a local state for $P_i$. Assume the nonrecursive parts of $R$, namely $NR^0, NR^1, NR^2$ and the computation of the exit condition $e$ have (given) distributed implementations in CSPR with respect to distribution $V$: $DNR^k, k=0,1,2$ and $E$, all on the same given network. Also given are the correctness proofs of these distributed implementations. These implementations

\[ S:: \{ p \} R \leftarrow \text{proc} \]
\[ \quad \{ \begin{array}{l}
[ e \rightarrow NR^0 \\
\]
\[ \] \\
\[ -e \rightarrow NR^1; \\
\{p\} \text{call } R \{q\} \\
\{p\} \text{call } R \{q\} \\
\end{array} \]

*Figure 2. The sequential LP program S*
should satisfy the following correctness criteria (see figure 3), where \( I^0, I^1, I^2 \) are some global invariants (GI's) establishing cooperation (as required by the CSP proof system) and without loss of generality (due to the appropriate substitution rules in the CSP proof system), we assume that \( p = \Lambda p_i, \text{free}(p_i) \subseteq V_i \) and \( q = \Lambda q_i, \text{free}(q_i) \subseteq V_i \) and \( e = \text{det}_i, \text{free}(\text{det}_i) \subseteq V_i \).

We construct \( DS \), a complete distributed implementation for \( S \), together with its correctness proof, out of \( E, DNR^0, DNR^1 \) and \( DNR^2 \), again using the same given network. \( DS \) should satisfy \( \{ I^* \land p \} [DS] \{ I^* \land q \} \) with respect to some GI \( I^* \) establishing cooperation. Take \( DS \) to be as shown in figure 4. We do not explicitly define the code for the synchronization operation \( \text{synch}^j \) at this stage. Rather, in the process of proof construction, sufficient conditions for \( \text{synch}^j \) are derived, guaranteeing the validity of the constructed proof. These conditions have to be met by any correct implementation of \( \text{synch}^j \). The main function of \( \text{synch}^j \) is to synchronize the independent recursive calls in the separate processes, to form a correct distributed implementation of the global recursion in \( S \), as suggested in the transformation in [SFR63].

The last steps in proving

\[
\{ p \} E \{ p \land A_{t=1,n} (\text{det}_t = e) \} , \{ p \land e \} DNR^0 \{ q \} ,
\]

\[
\{ p \land \neg e \} DNR^1 \{ p \} , \text{and} \{ q \} DNR^2 \{ q \} \]

are the applications of the parallel composition rule [AFR 80], instantiated as in the figure 5. The last step in the proof of \( DS \) can be taken to be the application of the parallel composition rule instantiated as in the figure 6.

In this application, \( A^* = A^0 \cup A^1 \cup A^2 \cup A_{\text{synch}^0} \cup A_{\text{synch}^1} \cup A_{\text{synch}^2} \).

\[
p = \Lambda_{t=1,n} p_t, q = \Lambda_{t=1,n} q_t \text{ and } A_{\text{synch}^j} \text{ is the i/o assumption set for } \text{synch}^j, j = 0, 1, 2 \text{ respectively.}
\]

---

**Figure 3. Correctness criteria for distributed implementations of nonrecursive parts**
```
{ a } \
end Dr

{ a } DNR { a }

search

{ a } Dr

{ a } DNR { a } De a

search

[]

{ a } DNR { a } De a

search

Dr pro

where

[sa || ... ||0, sa] :: sa
```
\[ \begin{align*}
A^r_t \setminus \{ p_t \} &\models E_t \{ p_t \}, i = 1 \ldots n \quad A^r_t \text{ cooperate w.r.t. } I^*, i = 1 \ldots n \\
\{ p \} &\models E \{ p \} \\
A^0_t \setminus \{ p_t \wedge d_e \} &\models DNR^0 \{ q_t \}, i = 1 \ldots n \quad A^0_t \text{ cooperate w.r.t. } I^0, i = 1 \ldots n \\
\{ p \wedge e \} &\models DNR^0 \{ q \} \\
A^1_t \setminus \{ p_t \wedge - d_e \} &\models DNR^1 \{ p_t \}, i = 1 \ldots n \quad A^1_t \text{ cooperate w.r.t. } I^1, i = 1 \ldots n \\
\{ p \wedge - e \} &\models DNR^1 \{ p \} \\
A^2_t \setminus \{ q_t \} &\models DNR^2 \{ q_t \}, i = 1 \ldots n \quad A^2_t \text{ cooperate w.r.t. } I^2, i = 1 \ldots n \\
\{ q \} &\models DNR^2 \{ q \} \\
\end{align*} \]

Figure 5. Applications of parallel composition rule DNR^i

\[ \begin{align*}
3) A^* \setminus \{ p_t \} &\models DS_t \{ q_t \}, i = 1 \ldots n \quad A^* \text{ cooperate w.r.t. } I^*, i = 1 \ldots n \\
\{ I^* \wedge p \} &\models DS \{ I^* \wedge q \} \\
\end{align*} \]

Figure 6. Application of parallel composition rule to DS

In order to apply 3) it is needed to prove the assertions \( A^* \setminus \{ p_t \} \models DS_t \{ q_t \} \). Each DS_t is a call to a recursive procedure DNR_t which in turn is proved by application of the following instance of the local recursion rule (see figure 7). The proof of the premises of 4) goes as follows:

1. \( \{ p_t \} \models \text{synch}_t^p \{ p_t \} \), \( \text{vars}(p_t) \cap \text{vars}(\text{synch}_t^p) = \emptyset \)
4) $A^s \models \{ p_i \} \text{call} DR_i \{ q_i \} \implies \{ p_i \} BDR_i \{ q_i \}$

$A^s \models \{ p_i \} DR_i \{ q_i \}$

**Figure 7. Applications of the local recursion rule**

2. $A^t \models \{ p_i \} E_i \{ p_i \}, \text{ by } e$.

3. $\{ p_i \land de_i \} \text{synch}_0 \{ p_i \land de_i \}, \text{ vars}(p_i) \cap \text{vars(}\text{synch}_0)=\emptyset$

4. $A^0 \models \{ p_i \land \bar{de}_i \} \text{DNR}_0 \{ q_i \}, \text{ by } 0$.

5. $\{ p_i \land \neg de_i \} \text{synch}_1 \{ p_i \land \neg de_i \}, \text{ vars}(p_i) \cap \text{vars(}\text{synch}_1)=\emptyset$

6. $A^1 \models \{ p_i \land \neg de_i \} \text{DNR}_1 \{ p_i \}, \text{ by } 1$.

7. $A^s \models \{ p_i \} \text{call} DR_i \{ q_i \}, \text{ by recursion assumption}$.

8. $\{ q_i \} \text{synch}_2 \{ q_i \}, \text{ vars}(q_i) \cap \text{vars(}\text{synch}_2)=\emptyset$

9. $A^2 \models \{ q_i \} \text{DNR}_2 \{ q_i \}, \text{ by } 2$.

10. $A^s \models \{ p_i \} BDR_i \{ q_i \}$.

where $A^s = \bigcup_{j \in \{0,1,2\}} A^j \cup \bigcup_{j \in \{0,1,2\}} A^{\text{synch}_j}$.

To complete the premises of 3) cooperation has to be proved for $A^s$. We want to find such a global invariant that allows us to make use of the cooperation proofs from e), 0), 1) and 2).

Consider all the syntactically matching i/o pairs arising in $A^s$. Call "new" those pairs not handled in e), 0), 1) and 2) and the cooperation proofs for $\text{synch}_0, \text{synch}_1, \text{synch}_2$. In the construction of $\text{synch}_j, \text{we require that no new pairs arise from assumptions in } A^s, A^0, A^1 \text{ and } A^2 \text{ with assumptions in } A^{\text{synch}_j}, j \in \{0,1,2\}$ (by means of tagged messages). The only new pairs may originate from assumptions about i/o commands in

1. $A^s, A^0, A^1$ and $A^2$ or

2. i/o commands inside $A^1$ active at different recursion levels - thus not handled in 1). These pairs have passed cooperation vacuously with $I^1$ (and
similarly for \( A^1, A^0 \) and \( A^2 \).

It is \textit{synch} that prevents those communications from actually happening. Thus to handle these pairs in the proof we introduce a derived global invariant \( I^* \), that by expressing the effect of \textit{synch} ensures vacuous cooperation for the "new" pairs. In addition \( I^* \) has to positively simulate \( I^0, I^1, I^2 \) for communications that do occur.

\textbf{Note:} if no "new" pairs arise in \( A^* \) - no \textit{synch} is required. I.e. no matching pairs arise from \( \bigcup_{j \in \{e,0,1,2\}} A^j \) and no i/o pairs passed cooperation vacuously inside \( A^j, j \in \{e,0,1,2\} \). A careful analysis of this situation may lead to optimize the program transformation accordingly, avoiding excessive synchronization. (See the simple example in appendix).

In the next section a specific \textit{synch} implementation is suggested and a global invariant \( I^* \) is presented (see figure 8). The effect of \textit{synch} is expressed by introducing auxiliary variables \( at_i, at_{synch_i} \) and \( l_i, at_i \) indicates the currently executing nonrecursive part, \( at_{synch_i} \) indicates that the execution is in the synchronizing part, and \( l_i \) keeps track of the recursion level.

4. Synch implementation

\textit{Synch} is implemented as a \textit{closed layer} \cite{EF82}, preserving the global invariant \( I^* \). As any execution of a CSP program with a closed layer is equivalent to the execution where all processes are delayed before entering the closed layer and

\[
I^* = \Lambda_{t=1,n} \neg at_{synch_i}
\]

\[
(\Lambda_{t=1,n} \neg at_i = e \land \Lambda_{t=1,n} at_i = 0 \land \Lambda_{t=1,n} at_i = 1 \land \Lambda_{t=1,n} at_i = 2) \land l_1 = \ldots = l_n
\]

\[
\Lambda_{t=1,n} at_i = e \rightarrow I^0
\]

\[
\Lambda_{t=1,n} at_i = 0 \rightarrow I^0
\]

\[
\Lambda_{t=1,n} at_i = 1 \rightarrow I^1
\]

\[
\Lambda_{t=1,n} at_i = 2 \rightarrow I^2
\]

\textbf{Figure 8. The derived global invariant \( I^* \)}. 

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then enter and exit the layer together (by [EP82]) this is sufficient for our needs.

We present below a possible implementation for \textit{synch} in \textit{CSP} (see figure 9). It makes use of some spanning tree \( T \) over the \( N_i \) s. The synchronizing signals are collected up the tree \( \text{(up)} \) and transmitted back down the tree \( \text{(down)} \). For \textit{synch}\(^0\) and \textit{synch}\(^1\) the only difference is in the assignment: \( at_i := 0 \) and \( at_i := 1 \), respectively, and in that the level variable \( l \) need not be updated. For \textit{synch}\(^2\) the difference is in the assignments: \( at_i := 2 \), \( l_i := l_i - 1 \);

For leaf and root nodes the programs are changed correspondingly.

**Note:** Because of the restriction on the structure of bracketed sections in the \textit{CSP} proof system, the above described sections are to be splitted to contain one communication per bracketed section.

What remains to be checked is the preservation of the invariants \( I^0 \), \( I^1 \), \( I^1 \) and \( I^2 \) by the corresponding bodies of \( E \), \( DNR^0 \) and \( DNR^1 \). By the completeness theorem for the proof system of \textit{CSP} it is always possible to find global invariants containing only auxiliary variables. Thus by simple auxiliary variable renaming the variable clashes between \( DNR^i \) and \( I^k \) are prevented.

All the above applies as well to any pattern of simple recursion.

---

The program at some internal node in \( T \)

\[
\begin{align*}
&\text{\texttt{at}synch}_i := \text{true} ; \\
&at_i := e ; l_i := l_i + 1 ; \\
&\text{receive up()} \text{ from all children } ; \\
&\text{send up()} \text{ to parent} ; \\
&\text{receive down()} \text{ from parent} ; \\
&\text{\texttt{send down()} \text{ to all children}; \text{at}synch}_i := \text{false} } \\
\end{align*}
\]

*Figure 9. An implementation of \textit{synch}*. 

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Discussion and future directions

A natural question arising in the context of above is the extension of the proof transformer to handle total correctness proofs. More specifically the question is: how given a sequential recursive program and its total correctness proof and distributed implementations for the nonrecursive parts with their total correctness proofs, to construct a total correctness proof for the whole transformed program. Unfortunately, we have no answer to this question— the main problem being the following: when using the total correctness proof system [AB3] one has to construct termination proofs for each sequential process and then establish cooperation. The termination of the local recursive procedures follows from the termination of the sequential recursive program, but it is not clear how to make use of the termination proof of the sequential procedure in the termination proofs for the local processes.

Another interesting extension follows from the previously made observation that examination of the proof transformer may suggest an algorithmic optimization with respect to given distributed implementations of nonrecursive parts, i.e. by proving that no new semantic matches arise in the complete DS, the synch part is proven redundant. In a forthcoming paper [GSB5] a more general criteria is suggested. Here we view a distributed implementation of each DNR as a stand alone closed layer and suggest a proof rule for proving such closeness. Synch is redundant for any composition of stand alone closed layers.
Acknowledgements

We would like to thank Krzysztof Apt and Orna Grumberg for helpful discussions.

References


Appendix: Deriving a correctness proof for a CSPR factorial program.

We consider a distributed implementation of a factorial procedure \textit{Factorial} on a three processor net: one of the processors (P1) is able to multiply, one (P2) to subtract and the third (P3) to add. To derive the correctness proof of the CSPR implementation of \textit{Factorial} we make use of a sequential procedure proof and the proofs of the distributed CSPR implementations for the nonrecursive parts. Consider the annotated sequential factorial procedure below where consecutive occurrences of assertions \{ \( p \) \} \{ \( q \) \} imply \( p \rightarrow q \) and appeal to the consequence rule.

\textit{Factorial} \(<\leftarrow\text{proc}\)

\[
\{ \text{x} \geq 0 \} \{ \text{x} = 0 \to \{ \text{y} = 1 \} \}
\]

\[
\{ \text{x} \geq 0 \to \{ \text{x} = 1 \} \to \{ \text{y} = \text{x}! \}
\]

\{ \text{y} \cdot (\text{x} + 1) = (\text{x} + 1)! \}

\{ \text{x} = \text{x} \cdot (\text{x} + 1) \}

end \text{Factorial}.

The non recursive parts here are

\[
\text{e} : \text{x} = 0
\]

\[
\text{NR}^0 : \text{y} = 1
\]

\[
\text{NR}^1 : \text{x} = \text{x} - 1;
\]

\[
\text{NR}^2 : \text{x} = \text{x} + 1; \text{y} = \text{y} \cdot \text{x};
\]

Assume the following annotated distributed CSP implementations for the nonrecursive parts, with \( E_i, \text{DNR}_i^0, \text{DNR}_i^1, \text{DNR}_i^2 \) assigned to \( P_i, i=1,2,3 \). For convenience, we omit in the texts the \textit{pre-} and \textit{post-} conditions equal to \textit{true}.

\[ E : \{ \text{x} \geq 0 \}
\]

\[
[ E_1 : \ll \text{P}_2 \cdot \text{x} \gg; \ll \text{P}_3 \cdot \text{x} \gg ] \| E_2 : \ll \text{P}_1 \cdot \text{n} \gg \| E_3 : \ll \text{P}_1 \cdot \text{m} \gg ]
\]

\{ \text{x} \geq 0 \land (\text{n} = 0 = \text{x} = 0 \land \text{m} = 0 = \text{x} = 0) \}

establishing the global invariant: \( I = \text{n} = \text{x} \land \text{m} = \text{x} \)

\[ \text{DNR}_0 : \{ \text{x} \geq 0 \land \text{x} = 0 \}
\]

\[
[ \text{DNR}_0 : \text{y} = 1 ] \| \text{DNR}_0 : \text{skip} ] \| \text{DNR}_0 : \text{skip } \} \{ \text{y} = \text{x}! \}
\]

\[ \text{DNR}^1_1 : \{ \text{x} \geq 0 \land \text{z} = 0 \}
\]

\[
[ \text{DNR}^1_1 : [ \ll \text{P}_2 \cdot \text{x} \gg; \ll \text{P}_3 \cdot \text{x} \gg ] ]
\]

\[
\| \text{DNR}^1_2 : [ \ll \text{P}_1 \cdot \text{n} \gg; \ll \text{n} = \text{n} - 1; \text{P}_1 \cdot \text{n} \gg ] ] \| \text{DNR}^1_3 : \text{skip } ] \{ \text{x} \geq 0 \}
\]

with global invariant: \( I = \text{n} = \text{x} \)

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\[ DNR^2 :: \{ y \cdot (x+1) = (x+1)! \} \]

\[ DNR^3 :: \[ P_2!x; \ P_2?x; \ y := y \cdot x \] \parallel DNR^2 :: \text{skip} \]

\[ \parallel DNR^3 :: \[ P_3!m; \ m := m+1; \ P_3?m \] \{ y = z! \} \]

with global invariant: \( i^2 = m = x \)

We construct the following complete annotated program with \( DS_i \) assigned to processor \( P_i, i = 1, 2, 3. \)

(1): \( DS: \{ x \geq 0 \} [ DS_1 \parallel DS_2 \parallel DS_3 ] \{ y = z! \} \)

where

\( DS_1 :: DR_1 \leftarrow \text{proc} \)

\[ \text{synch}_1^1; \]
\[ \langle P_1!x \rangle; \langle P_3!x \rangle; \]
\[ \{ x = 0 \rightarrow \text{synch}_2^1; y := 1 \{ y = z! \} \}
\]
\[ \{ x \neq 0 \rightarrow \text{synch}_1^1; \{ x > 0 \} \langle P_1!x \rangle \}
\]
\[ \{ z > 0 \} \langle P_2?x \rangle \{ x \geq 0 \} \text{call} DR_1; \{ y = z! \} \]

\[ \text{synch}_2^1; \]
\[ \{ y \cdot (x+1) = (x+1)! \} \langle P_2!x \rangle; \]
\[ \{ y \cdot (x+1) = (x+1)! \} \langle P_3!x \rangle; \{ y \cdot x = z! \} \{ y = z! \} \]

\end DR_1;

\{ x \geq 0 \} \text{call} DR_1; \{ y = z! \}

\( DS_2 :: DR_2 \leftarrow \text{proc} \)

\[ \text{synch}_2^2; \]
\[ \langle P_1?n \rangle; \]
\[ \{ n = 0 \rightarrow \text{synch}_3^2; \text{skip} \}
\]
\[ \{ n \neq 0 \rightarrow \text{synch}_2^2; \]
\[ \langle P_1?n \rangle; \]
\[ \langle n := n - 1; \ P_3!n \rangle; \text{call} DR_2; \text{synch}_2^2; \]

\end DR_2;

\text{call} DR_2
To prove (1), by the parallel composition rule we need to prove:

(2): \( A_1 |\{ x\geq 0 \} DS_1 \{ y = x! \} \)

(3): \( A_2 \models \{ \text{true} \} DS_2 \{ \text{true} \} \)

(4): \( A_3 \models \{ \text{true} \} DS_3 \{ \text{true} \} \)

For (3),(4) the proofs are immediate.

For \( DS_1 \) (i.e. (2)) the procedure call is to be proved by:

(5): \( \{ x\geq 0 \} \text{call } DR_1 \{ y = x! \} \)

To prove (5) we prove the locally recursive \( DR_1 \) procedure using the local recursion rule introduced above. By this rule we need to prove:

(6): \( \{ x\geq 0 \} \text{call } DR_1 \{ y = x! \} |\{ x\geq 0 \} BDR_1 \{ y = x! \} \)

The proof outline for (6) was already presented with the program itself.

To establish the cooperation for (1) the following instantiation of the global invariant \( I^* \) is used:

\[
I^* = \text{true} \land \text{atsynch}_1 \rightarrow (\text{true} \land \text{at} \equiv 1 \lor \text{at} \equiv 0 \lor \text{at} \equiv 1 \lor \text{at} \equiv 2) \land l_1 = l_2 = l_3
\]

\[
\text{true} \land \text{at} \equiv 1 \rightarrow n = x \land m = x
\]

\[
\text{true} \land \text{at} \equiv 2 \rightarrow m = x
\]

Now cooperation needs to be established for the i/o assumptions. The following pairs were already handled in the cooperation proofs for \( E, DNR_0, DNR_1 \) and \( DNR^2 \). As with the help of the auxiliary variables handled by \text{synch} the corresponding invariants \( I^*, j \in \{1, 2\} \) are enabled by the \( I^* \), the proof for them is complete. Other pairs, by \( I^* \), pass cooperation vacuously.

1. \( \{ \text{true} \land I^* \} \ll P_2|x \gg || \ll P_1?n \gg \{ \text{true} \land I^* \} \)
2. \{ \text{true} \land I^* \} \ll P_2!x \gg \| \ll P_1?m \gg \{ \text{true} \land I^* \} \\
3. \{ x>0 \land I^* \} \ll P_2!x \gg \| \ll P_1?n \gg \{ x>0 \land I^* \} \\
4. \{ x>0 \land I^* \} \ll P_2?x \gg \| \ll n;=n-1; P_1!m \gg \{ x \geq 0 \land I^* \} \\
5. \{ y.(x+1)=(x+1)! \land I^* \} \ll P_3!x \gg \| \ll P_1?m \gg \{ y.(x+1)=(x+1)! \land I^* \} \\
6. \{ y.(x+1)=(x+1)! \land I^* \} \ll P_3?x \gg \| \ll m;=m+1; P_1!m \gg \{ y \cdot x = x! \land I^* \}

\textbf{Note:} as no actual "new" pairs arise in \textit{A}\textsuperscript{+}, the \textit{synch} part is redundant in this simple example and leads to excessive synchronization.