FAIRNESS IN TERM REWRITING SYSTEMS

by

S. Porat and N. Francez

Technical Report #367

May 1985
FAIRNESS IN TERM REWRITING SYSTEMS

by
Sara Porat and Nissim Francez
Computer Science Dept.
Technion, Haifa, Israel

Abstract: The notion of fair derivation in a term-rewriting system is introduced, whereby every rewrite rule enabled infinitely often along a derivation is infinitely-often applied. A term-rewriting system is fairly-terminating iff all its fair derivations are finite. The paper presents the following question: is it decidable, for an arbitrary ground term rewriting system, whether it fairly terminates or not? A positive answer is given for several subcases. The general case remains open.
1. Introduction

In this paper we introduce the notion of fairness to the context of term rewriting systems (TRS). Fairness is traditionally studied in the context of programming languages for nondeterminism and concurrency. In this context, most of the interesting problems turn to be highly undecidable [HA, 84]. By shifting the discussion to more abstract models of computation, some insight may be gained in the decidable cases. Previous results [PF, 82, PF, 84] were obtained in the area of context-free grammars, providing an exact characterization of fair termination in that model. By passing to the richer model of TRS, a more interesting notion of enabledness arises, with some interesting consequences with respect to fairness. The ultimate goal of this study is to characterize the border line between the decidable cases of fair termination and the undecidable ones. As _ground_ TRS are known to have a decidable termination property, this model is a natural candidate for being in that border line. In particular since for the "next" powerful model, that of _linear_ TRS [GKM, 83], termination is undecidable. In this paper we provide a partial resolution of this issue by showing that for some special cases of the ground TRS-fair termination is also decidable. In particular, the case of _non-globally-finite_ TRS for which the rewrite rules are not necessarily length-increasing is still open.

A useful property of ordinary termination decision procedures for TRS, namely, the ability to reduce the general question to that of derivations starting with the l.h.s of a rule [DE, 81], is preserved here also.

2. Basic Definitions and Notations

In this section we introduce the basic definitions regarding fairness in the context of TRS and state the basic problem.

**Definition:** A Term Rewriting System (TRS) $S = (F, X, R)$ consists of:

- $F$: a finite set of symbols called _function symbols_.
- $X$: an infinite set of _variables_.
- $R$: a finite set of _Rewrite Rules_ (RR) of the form $l ightarrow r$, where $l$ and $r$ are _terms_ over $(F, X)$.

In case $X = \emptyset$ we say that $S$ is _ground_.

Note that all function symbols have variable arity.

We assume that RRs are always well formed in that the collection of variables in $r$ is contained in that of $l$. We use the standard definitions of _subterm_ and _substitution_. We use $\tau$ to range over terms (written in prefix notation) and $\sigma$ to range over substitutions.

**Definition:**

1. An RR $l \rightarrow r$ is _enabled_ in a term $\tau$ iff there exist a subterm $\tau'$ of $\tau$ and a substitution $\sigma$ such that $\tau' = l\sigma$.
2. A term $\tau$ is _enabled_ in a term $\tau'$ iff $\tau'$ is a subterm of $\tau$ and there exist an RR $l \rightarrow r$ and a substitution $\sigma$ such that $\tau' = l\sigma$.

**Definition:** $\tau \rightarrow \tau'$ if there is an RR $l \rightarrow r$ enabled in $\tau$ and $\tau'$ is obtained by replacing an occurrence of the subterm $l\sigma$ in $\tau$ by the term $r\sigma$. (In this case we say that $\tau'$ is obtained by _applying_ the rule $l \rightarrow r$ on $\tau$.)

**Definition:** An _$S$-derivation_ is a (finite or infinite) sequence.
Definition: A TRS $S$ is **terminating** iff it does not admit any infinite $S$-derivation.

Remark: It is known [HL 78] that termination is an undecidable property for general TRS. On the other hand termination is decidable for ground TRS [HL 78, DE 81, DT 84].

We now come to the new notions related to fairness. The nondeterminism displayed by a TRS is two folded. Both the choice of the rule to be applied (from the enabled RRs) and the choice of the subterm to be replaced (from the enabled subterms) are made nondeterministically. We consider here one aspect of this nondeterminism only.

Definition: An $S$-derivation $d$ is **rule-fair** (abbreviated to fair) iff it is finite, or it is infinite and every RR enabled infinitely often along $d$ is also infinitely often applied along $d$.

The concept of rule fair, as defined here, is similar to the definition of a strongly fair computation in [F 85], in contrast with the definition of an unconditionally fair computation in [F 85] (or an impartial computation in [LPS 81]); within an infinite unconditionally-fair computation every direction (in our context every rule) is applied infinitely often. In order to clarify this point, let's consider the following example.

Example:

Let

\[ R = \{ a \to f(a), b \to c \}. \]

The infinite derivation

\[ d = < a \to f(a) \to f(f(a)) \to \ldots > \]

is fair as the first rule is infinitely often applied and the second one is never enabled.

The dual definition of subterm fairness of a derivation seems to be less natural. First note that the set of all enabled terms can be infinite in contrast to the finite set of enabled rules. Though we are not going to consider here subterm fairness, the following examples will point out the difference between the two notions of fairness and the problem in defining subterm-fairness.

Examples:

1) \( R = \{ a \to f(a), a \to b \}. \)

The infinite derivation

\[ d = < a \to f(a) \to f(f(a)) \to f(f(f(a))) \to \ldots > \]

is not rule-fair (the second rule is never applied, though enabled in every term along the derivation). Since in every term along $d$ only one subterm is enabled, in every possible definition of subterm-fairness this derivation will be subterm-fair.

2) \( R = \{ a \to g(a, g(a)), g(x, y) \to b \}. \)

The infinite derivation

\[ d = < a \to g(a, g(a)) \to g(a, b) \to \ldots > \]
is rule-fair (both rules are infinitely often enabled). Along the derivation, different subterms of the form \( g(x, y) \) are enabled, but the second rule is applied by replacing those occurrences (subterms) so that the new term will have at least one \( a \) as a subterm (so, the derivation can continue). It seems there is some sort of unfairness that depends on the structure of the subterm.

**Definition:** A TRS \( S \) is *fairly terminating* iff it does not admit infinite fair derivations, i.e. its only fair derivations are the trivially-fair ones, the finite ones.

Clearly, as general TRS are known to be as powerful as Turing machines, it follows [HA 84] that their fair termination is highly undecidable.

Following are examples of a non fairly-terminating TRS and of a fairly-terminating one.

**Example:** (a non-fairly-terminating TRS)
Let 
\[
R = \{ f(a, b, x) \rightarrow f(x, x, b), b \rightarrow a \}.
\]
The infinite derivation 
\[
d = f(a, b, b) \rightarrow f(b, b, b) \rightarrow f(a, b, b) \rightarrow f(b, b, b) \rightarrow \cdots
\]
is fair as both rules of \( R \) are applied infinitely often.

**Example:** (a fairly terminating TRS)
Let 
\[
R = \{ f(x, x) \rightarrow f(a, x), b \rightarrow a, b \rightarrow c \}.
\]
It is possible to prove the fair termination of this TRS by the methods discussed in the sequel. Intuitively, after some finite number of applications of the third rule (in independent subterms) all the rules become disabled.

The central issue addressed in this paper is the decidability of the problem of fair termination of ground TRS. As we mentioned, ground TRS gave rise to several decidable properties, e.g. confluence, global finiteness and acyclicity (and hence, termination) [DT 84], etc. Thus, it seems interesting to find out whether fair termination is still in the same class.

In the next section, we show that it suffices to consider derivations starting in a l.h.s of an RR in order to characterize the presence of fair derivations. In the following section this characterization is used to obtain partial results regarding the central problem.

**3. Fair derivations in ground TRS**

We start with some observations about the derivations in ground TRS. First we present the notation for describing positions within terms, needed in order to refer in some unique way to subterms. We take as positions finite dotted lists of natural numbers, i.e. expressions of the form \( n_1, n_2, \ldots, n_k \) for some \( k \geq 0 \). In case \( k = 0 \), we use the notation \( \lambda \) for the empty sequence. The position \( u \) defines, for each term \( T \), the subterm \( T/u \) in the following way:
1) \( \tau / \lambda = \tau \).
2) If \( \tau / u = f(\tau_1, \ldots, \tau_n) \), then for every \( j, 1 \leq j \leq n \), \( \tau / u_j = \tau_j \).

Consider a derivation
\[ d = \langle \tau_1 \rightarrow \ldots \rightarrow \rangle \]
and let \( u_i^d \) be the position of the subterm of \( \tau_i \) replaced in the \( i \)th step. Let
\[ U^d = \{ u_i^d \mid i \geq 1 \} \]
\( \)namely, the set of positions of replacements along \( d \). Clearly, \( U^d \neq \emptyset \). We omit the superscript \( d \) when clear from context. Let \( U^d_{\text{min}} \) be the subset of \textit{minimal} positions in \( U \) (these having no proper prefix in \( U \)).

Example: Consider a ground TRS with the only rule \( a \rightarrow f(a) \) and the infinite derivation
\[ f(a) \rightarrow f(f(a)) \rightarrow \ldots \rightarrow f^i(a) \rightarrow \ldots \]
Here \( U = \{ 1, 1, 1, 1, 1, \ldots \} \) and \( U_{\text{min}} = \{ 1 \} \).

\( U_{\text{min}} \) is always a finite set with cardinality depending on \( \tau_1 \).

A special kind of derivations, described next, plays an important rôle in the sequel, in that it allows the restriction to derivations starting in l.h.s of RRs.

An \( S \)-derivation is called \textit{major} if for some \( i, i \geq 1, \tau_i = \tau \) and \( \tau_{i+1} = r \) where \( l \rightarrow r \) is the rule applied at that stage. In other words, the whole term is replaced at some stage of a major derivation.

Consider an \( S \)-derivation \( d \), where \( |U_{\text{min}}| = m \).
The \( m \) \textit{induced major derivations} are obtained from \( d \) as follows:

If \( U_{\text{min}} = \{ \lambda \} \) then \( d \) itself is an induced major derivation. Otherwise, let \( U_{\text{min}} = \{ w_1, \ldots, w_m \} \). The \( i \)th induced derivation starts from a term \( \tau_i / u_i \). Consider a step \( \tau_i \rightarrow \tau_{i+1} \) in \( d \). There is some position \( u_j \in U_{\text{min}} \) which is the prefix of \( u_i \) in \( U_{\text{min}} \). That step induces a step in the induced derivation starting in \( \tau_i / u_j \). The same rule is applied but this time attributed to position \( u_i \), where \( u_i = u_j \).

From the construction it follows that each of these induced derivations is also major.

We now relativize fairness to subsets of rules.

\textbf{Definition:} For \( R' \subseteq R \), an infinite \( S \)-derivation \( d \) is \( R' \)-fair if every RR in \( R' \) is infinitely-often applied if infinitely-often enabled along \( d \).

Let \( k_r \) be the number of arguments of the outermost function symbol of \( \tau \) \( (k_r > 0) \) in case \( \tau = f(\tau^1, \ldots, \tau^k) \), for some \( f \in F \).

\textbf{Theorem:} (general restriction)

There is an infinite fair \( S \)-derivation iff there are \( m \leq |R| \) infinite \( S \)-derivations \( d_1, \ldots, d_m \) each starting in a l.h.s of some RR, and for every RR \( l \rightarrow r' \), if it is only finitely often applied along every \( d_i \), then it is also only finitely often enabled along every \( d_i \).

\textbf{Proof:}

(i) Assume such \( m \) infinite derivations. Note that each \( d_i \) is \( R_i \)-fair, for some cover \( \{ R_i \mid 1 \leq i \leq m \} \) of \( R \). For \( m = 1 \) the claim is trivial. Thus assume \( m > 1 \). As each given derivation is fair w.r.t some subset of rules, we construct an interleaving of the given derivations which is fair w.r.t \( R \) itself. The
given derivations are all the induced major derivations of that interleaving. Let \( k = \max_{r \in R} \max_{t_0} k_t \), \( f \) - some function symbol, \( t_0 \) - a l.h.s. of some rule that is applied infinitely often along some \( d_i \).

The interleaved derivation \( d \) starts with the following term \( \tau \), which embeds as arguments of \( f \) the initial terms of the \( m \) given derivations.

\[
\tau = \begin{cases} 
    f(\tau_1, \tau_2, \ldots, \tau_m) & m > k \\
    f(\tau_1, \tau_2, \ldots, \tau_m, t_0) & m \leq k
\end{cases}
\]

where \( \tau_1, 1 \leq j \leq m \) is the first term of the given \( d_j \).

The intuitive description of the interleaving is as follows. It consists of segments of length \( m \), in which one step is taken from each \( d_i \) and arranged in a round-robin. The first segment consists of all the corresponding first steps in the given derivations, and, in general, the \( n \)’th segment consists of the corresponding \( n \)’th steps of the given derivations. The structure of \( \tau \) is exactly the one needed to support such an interleaved derivation.

Thus, in the \( j \)'th step in the \( i \)'th segment, the rule applied is the one applied in the \( i \)'th step of \( d_j \). The corresponding position is \( j \cdot \frac{m}{k} \).

Every \( RR \) that is enabled infinitely often along \( d \) is infinitely often enabled along some \( d_i \). (The structure of \( \tau \) prevents the case in which a rule is enabled infinitely often along \( d_i \), but only finitely often along every \( d_i \).) So, by the given assumption, there is some \( d_j \) where this rule is infinitely often applied. So, the resulting derivation \( d \) is \( R \)-fair.

(only-if)

Assume an infinite fair \( S \)-derivation \( d \) is given. Consider its induced major derivations.

For every \( RR \in R \) that is infinitely often applied along \( d \), there is an induced major derivation \( d_{RR} \) where this rule is infinitely often applied. So, we match an infinite induced major derivation of \( d \) to every such \( RR \), and we get a set of \( m \leq |R| \) infinite derivations. Every such derivation \( d_{RR} \) defines an infinite derivation \( d'_{RR} \) (actually, a tail of it) that starts from a l.h.s. of some rule. For every rule, if it is infinitely often enabled along every \( d'_{RR} \), then by the construction of the matching, this rule is infinitely often enabled along \( d \). So, by the fairness assumption, it is also infinitely often enabled along every \( d'_{RR} \). To see this, note that if a rule is enabled on a term along an induced derivation, it is enabled on the corresponding superterm along the original derivation. Hence, the resulting set satisfies the desired condition.

Remark: Assume there is a set of \( m \leq |R| \) infinite \( S \)-derivations \( d_1, \ldots, d_m \) each starting in a l.h.s. of some \( RR \). Each \( d_i \) is \( R_i \)-fair, for some cover \( \{ R_i \} \), \( 1 \leq i \leq m \) of \( R \). Such a set does not imply the existence of an infinite fair \( S \)-derivation. Consider the following counter example:

\[
R :: 1) \quad a \rightarrow f(a) \\
2) \quad g(a,b) \rightarrow c \\
3) \quad a \rightarrow g(a,b)
\]

The infinite derivation

\[
d_1 = < a \rightarrow f(a) \rightarrow f(g(a)) \rightarrow \ldots >
\]

is \( \{1,2\} \)-fair, and the infinite derivation
\[ d_2 = \langle a \rightarrow g(a, b) \rightarrow g(g(a, b), b) \rightarrow \ldots \rangle \]

is \(3\)-fair. One can prove that the given TRS is fairly-terminating.

The results of this theorem do not apply to any subset of terms over \( F \). For example, if the subset restrict a function symbol to a fixed arity (in which case \( f(\tau_1, \tau_2, \ldots, \tau_n) \) is a term only if \( f \) is of arity \( n \)), the theorem is no more true. This point will be discussed in the last section.

4. Decidability of fair termination of ground TRS

As mentioned in the introduction, we do not have yet a positive settlement of the decidability problem of fair termination of ground TRS in the general case. However, from the theorem in the previous section, a positive settlement of some special cases is obtained.

**Definition:** A TRS is **globally finite** iff for every term \( \tau \), the set of terms \( \{\tau', \tau \rightarrow \tau'\} \) is finite.

We use GFGTR\(S \) for globally finite ground TRS.

**Remark:** The property of global finiteness is decidable for ground TRS [DT 84].

**Theorem:** (decidability of fair termination for GFGTR\(S \))

It is decidable whether an arbitrary, GFGTR\(S \) \( S \) is fairly terminating.

**Proof:**

Consider some fixed enumeration of the RRs in \( R \),

\[ l_i \rightarrow \tau_i, \quad i = 1, \ldots, n. \]

We associate with \( S \) a directed graph \( G_S = (V_S, E_S) \). Both the nodes and the edges are labeled. The nodes in \( V_S \) are terms. Each node \( i \) is labeled by the set \( \{i \} \) of the indices of the rules enabled on that term. There is an edge in \( E_S \) from \( \tau_1 \) to \( \tau_2 \), labeled \( i \) if \( \tau_1 \rightarrow \tau_2 \) by applying the \( i \)-th rule. The set of nodes of \( G_S \) is constructed iteratively.

First, \( \{\tau_i | 1 \leq i \leq n\} \) is included in \( V_S \). At each step of the iteration, add to the current \( V_S \) the set \( V' = \{\tau' | \tau' \rightarrow \tau, \tau \in V_S\} \). By the global finiteness, this iteration eventually stops, and the graph obtained is finite.

There is a directed cycle in \( G_S \) iff there is an infinite \( S \)-derivation starting from a \( l.h.s \) of some RR.

A cycle is called \( R^-\text{-fair} \) iff for every \( RR = l_i \rightarrow \tau_i \in R^-\) if there is a node \( v \) on the cycle labeled by an index set containing \( i \), then there is an edge \( e \) on the cycle labeled by \( i \). In other words, every RR enabled on the cycle is applied along the cycle.

A cycle in \( G_S \) defines an \( S \)-derivation \( d \) that starts from a \( l.h.s \) of some rule, and for every \( j \) so that \( j \) is contained in an index set labeling some node in the cycle, \( l_j \rightarrow \tau_j \) is infinitely often enabled along \( d \); and for every \( j \) so that \( j \) is labeling some edge in the cycle, \( l_j \rightarrow \tau_j \) is infinitely often applied along \( d \).

An infinite \( S \)-derivation \( d \) starting in a \( l.h.s \) of some rule defines a cycle in \( G_S \), and for every \( j \) so that \( l_j \rightarrow \tau_j \) is infinitely often enabled along \( d \), there is a node labeled by an index set containing \( j \) in the cycle; and for every \( j \) so that \( l_j \rightarrow \tau_j \) is infinitely often applied along \( d \), there is an edge labeled by \( j \) in the cycle.
By the general restriction theorem $S$ is not fairly terminating iff there are $m \leq n$ cycles, and for every $RR = l_i \rightarrow v_i \in R$, if in every cycle there is no edge labeled by $i$, then in every cycle there is no node labeled by an index set containing $i$.

The presence of the required cycles is decidable by simple graph-theoretic considerations.

We now present some examples to the constructions discussed above.

Examples:

1) $R:: 1) a \rightarrow b$
2) $b \rightarrow a$
3) $a \rightarrow c$

The graph $G_S$ is shown in figure 1. In $G_S$ there is only one cycle in which no edge is labeled by 3, but there is a node labeled by $\{1,3\}$. So, by the decidability theorem for GFGTRS this system is fairly terminating.

2) $R:: 1) f(a) \rightarrow f(c)$
2) $h(f(c)) \rightarrow h(f(a))$
3) $a \rightarrow b$
4) $g(f(b)) \rightarrow g(f(a))$

The graph $G_S$ is shown in figure 2. In the graph $G_S$ of this example there are two cycles. For every $i, 1 \leq i \leq 4$, there is an edge in some cycle labeled by $i$. By the decidability theorem for GFGTRS the system is not fairly terminating.

3) In this example, we show the insufficiency of the procedure described in the proof of the decidability theorem in case there is no global finiteness. In such a case, the graph $G_S$ is infinite (see figure 3) and infinite fair derivation does not imply the presence of a cover by a finite number of partially-fair cycles. The rules in this example are:

![Diagram](image)

Figure 1: the graph of a fairly terminating GFGTRS
Figure 2: disjoint partially-fair cycles covering $R$

$R::$

1) $a \rightarrow f(a)$

2) $a \rightarrow b$

3) $b \rightarrow a$

The system is obviously not fairly terminating. There are infinitely many cycles in the graph. In every cycle there is no edge labeled by 1, but there is a node labeled by $\{1,2\}$.

Figure 3: a graph for a non globally finite TRS
Definition: The \textit{depth} \( \text{dep}(\tau) \) of a ground term \( \tau \) is defined by

1) \( \text{dep}(f) = 1 \), for some \( f \in F \)

2) \( \text{dep}(f(\tau_1, \ldots, \tau_n)) = \max \text{dep}(\tau_i) + 1 \).

We use UN-TRS (unary/nullary TRS) for TRS in which \( k_i \leq 1 \) and \( k_r \leq 1 \) for every \( l \rightarrow r \in RR \).

Definition: A ground TRS is called \textit{(length-)increasing} iff each rule \( l \rightarrow r \) satisfies \( \text{dep}(l) \leq \text{dep}(r) \).

Theorem: (decidability of fair-termination of increasing UN-TRS)

It is decidable whether an increasing UN-TRS is fairly terminating.

Proof:

Let the RRs be numbered by \( 1, \ldots, n \). Let \( m = \max \text{dep}(l_i) \). The set \( T_m = \{ \tau \mid \text{dep}(\tau) \leq m \} \) is finite. We again consider a labeled graph \( G_S = (V_S, E_S) \) in order to characterize the behaviour of \( S \). The set of nodes is \( V_S \subseteq T_m \). A node \( \tau \in T_m \) is labeled by an index set \( I \), the set of indices of all RRs enabled on \( \tau \).

The intuitive idea behind the construction of \( G_S \) is that an \( S \)-derivation \( \tau_1 \rightarrow \tau_2 \rightarrow \cdots \) satisfies \( \text{dep}(\tau_i) \leq \text{dep}(\tau_{i+1}) \) for all \( i \geq 1 \). Thus, in a term of depth greater than \( m \), only the subterm of depth not greater than \( m \) need to be considered as subterms of greater depth may never become enabled.

Define a \textit{transition function} \( \delta: T_m \times R \rightarrow T_m \).

Consider \( \tau \in T_m \) and \( \tau' \rightarrow \tau'' \).

If \( \tau' \in T_m \) then \( \delta(\tau', \bar{t}) = \tau' \).

If \( \tau' \notin T_m \) then \( \delta(\tau', \bar{t}) = \tau'' \), the subterm of \( \tau' \) the depth of which is exactly \( m \).

There is an edge in \( E_S \) from \( \tau_1 \) to \( \tau_2 \) labeled by \( t \) if \( \delta(\tau_1, t) = \tau_2 \).

Again, the set of nodes \( V_S \) is constructed iteratively. First, \( \{ l_i \mid 1 \leq i \leq n \} \) is included in \( V_S \). At each step of the iteration, add to the current \( V_S \) the set \( V' = \{ \tau'' \mid \delta(\tau'', t) = \tau', \tau'' \in V_S, 1 \leq i \leq n \} \). By the finiteness of \( T_m \), this iteration eventually stops.

Every cycle in this graph defines an infinite derivation and vice versa. The presence of an infinite fair derivation in the TRS depends on the presence of the appropriate cycles like in the proof of the theorem for GFGTRS.

We now come back to the TRS in example 3, which is not fairly terminating but could not be shown as such by the GFGTRS decidability theorem.

Example: Here \( m = 1 \), \( T_m = \{ a, b \} \). The graph is shown in figure 4. Since this graph contains an \textit{R}-fair cycle, the system is not fairly terminating.

Example:

We cannot simply apply the procedure described in the last proof to decide whether an arbitrary increasing TRS is fairly terminating.
In a term of depth greater than 1, only subterms of depth 1 need to be considered, but there is no bound for the number of such subterms.

5. **Fixed arity**

In this section we assume that every function symbol \( f \) has a fixed arity \( \text{ar}_f \). We use \( F\text{-TRS} \) for TRS that satisfies this assumption.

**Theorem:** (restriction for unary/nullary functions in \( F\text{-TRS} \))

If for every \( f \in F \), \( \text{ar}_f \leq 1 \) then:

there is an infinite fair \( S \)-derivation iff there is an infinite fair \( S \)-derivation starting in a l.h.s of some RR.

The proof is very similar to that of the general restriction theorem, so it is omitted.

In case there is \( f \in F \) with \( \text{ar}_f > 1 \), the situation is quite different. Suppose there are \( m \leq |R| \) infinite derivations, each starting in a l.h.s. of a rule, and for every \( RR = l \rightarrow r \), if it is only finitely often applied along every given derivation, then it is also only finitely often enabled along every given derivation. The existence of such derivations does not necessarily imply the existence of an infinite fair derivation.

**Example:** Consider again the TRS \( S = (F', \phi, R) \) the graph of which is described in figure 2.

Let \( F' = \{ a, b, c, h, f, g, f' \} \)

where

\[ \text{ar}_a = \text{ar}_b = \text{ar}_c = 0, \quad \text{ar}_h = \text{ar}_f = \text{ar}_g = 1, \quad \text{ar}_{f'} = 2. \]

Let \( R \subseteq R' \), and for every two terms \( \tau \) and \( \tau' \) that are nodes in \( G_S \)

\( f'(\tau, \tau') \rightarrow a \in R' \).
One can prove that the F-system $S' = (F', \phi, R')$ is fairly terminating though we have the two cycles in the graph $G_S$ as in $G_S$, defining two derivations satisfying the desired condition.

By adding a nullary function symbol $f_0$ to the set $F$ we can achieve again the sufficiency of the condition.

**Theorem (general restriction for FTRS)**

Let $S = (F, \phi, R)$ and $S' = (F \cup \{ f_0 \}, \phi, R)$ where $f_0 \notin F$ and $\alpha r f_0 = 0$.

If there is $f \in F$ with $\alpha r f > 1$ then there is an infinite-fair $S'$-derivation if there are $m \leq |R|$ infinite $S$-derivations $d_1, \ldots, d_m$, each starting in a l.h.s of some RR, and for every RR $t + r$, if it is only finitely often applied along every $d_i$, then it is also only finitely often enabled along every $d_i$.

**Proof:**

(if) For $m = 1$ the claim is trivial. Thus assume $m > 1$. As in the general restriction theorem, we construct an interleaving of the given derivations. The interleaved derivation $d$ starts now with the term $\tau$, which embeds as arguments of $f$ at various depths the initial terms of the $m$ given derivations. As $\alpha r f > 2$ is possible, the positions of the arguments from the third onwards are padded with the new term $f_0$. We use the abbreviation $T_0(b)$ for

\[
\begin{align*}
\alpha r f - b & \text{ times, } b \in \{1, 2\}, \\
\tau &= f(f(\ldots f(\tau_1, T_0(1)), \tau_2, T_0(2)), \ldots), \tau_m, T_0(2)) \\
&\text{where } \tau_j, 1 \leq j \leq m \text{ is the first term of the } j\text{-th given } d_j.
\end{align*}
\]

We abbreviate the position $1 \ldots 1$ as $1^k$.

Again, the interleaved derivation consists of segments of length $m$. In the $j$'th step in the $i$'th segment, the rule applied is the one applied in the $i$'th step of $d_j$, but now the corresponding position is

\[
\begin{align*}
1^m . u_i^d_1 & \quad j = 1 \\
1^m - j . 2 . u_i^d_j & \quad j > 1
\end{align*}
\]

The function symbol $f_0$ acts as a filler so, again, every RR that is enabled infinitely often along $d$ is infinitely often enabled along some $d_i$. Thus, the resulting derivation $d$ is $R$-fair.

(only-if)

The set of $m$ infinite derivations constructed exactly as in the (only-if) proof of the general restriction theorem consists of $S$-derivations and satisfies the desired condition.

**Remark:** All the decidability theorems stated above can be easily modified to apply also to TRS containing a ground r.h.s and, possibly, variables in the l.h.s. The construction of the deciding graphs will start from the $r_4$'s instead of the $l_4$'s.
Acknowledgment: Conversations with Nachum Dershowitz were helpful. The part of the second author was partially supported by the fund for the promotion of research, the Technion.
References:


[DT 84] N. Dauchet, S. Tison: "Decidability of confluence for ground term rewriting systems", (undated manuscript), Lille university.


