EXTREMAL PROBLEMS ON PERMUTATIONS.
UNDER CYCLIC EQUIVALENCE

by

P. Erdos*, N. Linial** and S. Moran***

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* Hungarian Academy of Sciences
** Institute of Mathematics and Computer Science, Hebrew University, Jerusalem, Israel.
*** Computer Science Dept., Technion - Israel Institute of Technology, Haifa, Israel.
1. INTRODUCTION

Let $\sigma = (a_1, \ldots, a_n) \in S_n$ be a permutation and let

$$[\sigma] = \{(a_j, \ldots, a_n, a_1, \ldots, a_{j-1}) : n \geq j \geq 1\}$$

be the class of all cyclic permutations of $\sigma$. Also for $\pi = (b_1, \ldots, b_n) \in S_n$ denote by $\pi^R$ the permutation $(b_n, \ldots, b_1) \in S_n$.

We also denote by $<\sigma>$ the set $[\sigma] \cup \{\tau^R | \tau \in [\sigma]\}$. For a real function $f : S_n \to \mathbb{R}$ we consider $\tilde{f}$ defined by

$$\tilde{f}(\sigma) = \min \{f(\tau) | \tau \in [\sigma]\}$$

and $\bar{f}$ given by

$$\bar{f}(\sigma) = \min \{f(\tau) | \tau \in <\sigma>\}.$$

Our interest in this article is in finding $\max \{\tilde{f}(\sigma) | \sigma \in S_n\}$ and $\max \{\bar{f}(\sigma) | \sigma \in S_n\}$, for certain functions $f$.

Here we deal with two instances of this general problem:

1) $f(\sigma) = \text{number of inversions in } \sigma = |\{(i, j) | i < j, \sigma(i) > \sigma(j)\}|$

2) $f(\sigma) = \max \{|\sigma(i) - i| | i = 1, \ldots, n\}$.

Our interest in those problems was initiated by studies on the design of electrical circuits for parallel computations [1]. Of course, many other problems suggest themselves that we hope to investigate in the future.

2. COUNTING INVERSIONS

As stated in the introduction we investigate here the function

$$F(n) = \max \min f(\sigma)$$

where $I(\sigma)$ is the number of inversions in $\sigma$, the max is over $\pi \in S_n$ and the min over $\sigma \in [\pi]$.

Theorem 2.1:

$$0.304n^2 + O(n) = \frac{8 - \pi}{16} n^2 - \frac{3n}{2} \leq F(n) \leq \frac{n^2}{9} - \frac{3n - 1}{6} = 0.333n^2 + O(n).$$

Proof: Let us first remark that $F(n) = O(n^2)$ is obvious. Since a permutation of $S_n$ can have at most $\binom{n}{2}$ inversions, $F(n) \leq \binom{n}{2}$. Also for $\pi = (n, n-1, \ldots, 1) \in S_n$ it is easily
verified that

\[ \min_{\sigma \in [\sigma]} I(\sigma) = \frac{n^2}{4} + O(n). \]

The Upper Bound: Let \( \sigma = (a_1, \ldots, a_n) \) and let \( \tau_k = (a_k, \ldots, a_{n}, a_1, \ldots, a_{k-1}) \), \( k \geq n \geq 1 \) be the permutations in \( [\sigma] \). Define variables \( x_{ij}^{(k)} \), \( 1 \leq i < j \leq n, 1 \leq k \leq n \) as follows:

\[ x_{ij}^{(k)} = \begin{cases} 1 & \text{if } a_{k+i} > a_{k+j} \\ 0 & \text{if } a_{k+i} < a_{k+j} \end{cases} \]

Whenever reference is made to \( \tilde{a}_t \) with \( t \in [1, n] \) we mean \( a'_{t} \) where \( t' = t - 1 \pmod{n} \). This convention will be made throughout the article without further notice. Also \( x_{ij} \) stands for \( x_{ij}^{(n)} \).

Note that

\[ I(\tau_k) = \sum_{1 \leq i < j \leq n} x_{ij}^{(k)} \]

We want to find the average of \( I(\tau_k) \) over \( n \geq k \geq 1 \); so let us fix \( 1 \leq i < j \leq n \) and let us calculate

\[ \sum_{k=1}^{n} x_{ij}^{(k)} = \begin{cases} j-i & \text{if } x_{ij} = 0 \\ n-j+i & \text{if } x_{ij} = 1 \end{cases} \]

This is because for \( 1 \leq i < j \leq n \) there are \( n-j+i \) values of \( 1 \leq k \leq n \) for which \( i+k \pmod{n} > j+k \pmod{n} \). Again let us remark that our residue classes \( \pmod{n} \) are 1, ..., \( n \) not 0, ..., \( n-1 \) as usual.

So, we have

\[ \sum_{k=1}^{n} x_{ij}^{(k)} = j-i + (n-2j+2i) x_{ij}. \]

And therefore

\[ \sum_{k=1}^{n} I(\tau_k) = \sum_{k=1}^{n} \sum_{1 \leq i < j \leq n} x_{ij}^{(k)} = \sum_{1 \leq i < j \leq n} (j-i) + \sum_{1 \leq i < j \leq n} (n-2j+2i) x_{ij} \]

\[ = \binom{n+1}{3} + \sum_{1 \leq i < j \leq n} (n-2j+2i) x_{ij} \]

For \( 1 \leq r < s < t \leq n \) we have

\[ 1 = x_{rs} + x_{st} - x_{rt} \quad \text{(1)} \]

Because \( x_{rs} = x_{st} = 1 \) implies \( x_{rt} = 1 \).
Let us sum (1) over all triples \( 1 \leq r < s < t \leq n \). For \( 1 \leq i < j \leq n \) we count \( x_{ij} \) \((j-i-1)\) times in the negative and \( i-1+n-j \) times in the positive sign.

Altogether we get
\[
\left\{ \begin{array}{l}
\sum_{1 \leq r \leq s < t \leq n} 1 \geq \sum x_{rs} + x_{st} - x_{rt} = \sum (n-2j+2i)x_{ij},
\end{array} \right.
\]

Therefore
\[
\sum_{k=1}^{n} I(\tau_k) \leq \left\{ \begin{array}{l}
\frac{n+1}{3} + \frac{n}{3} = \frac{(2n-1)n(n-1)}{6},
\end{array} \right.
\]

and so the average of \( I(\tau) \) over all \( \tau \in [\sigma] \) is at most \( \frac{(2n-1)(n-1)}{6} \). It follows that for every \( \sigma \in S_n \) there is a \( \tau \in [\sigma] \) for which \( I(\tau) \leq \frac{(2n-1)(n-1)}{6} = \frac{n^2}{3} + \frac{3n-1}{6} \), proving the upper bound.

**The Lower Bound:** We want to find a permutation \( \sigma = (a_1, \ldots, a_n) \in S_n \) for which \( I(\tau_k) \) is large for all \( \tau_k = (a_1, \ldots, a_n, a_{k+1}, \ldots, a_{k+n-1}) \in [\sigma] \).

Let us comment first that
\[
I(\tau_{k+1}) - I(\tau_k) = n+1 - 2a_k.
\]

Because on moving from \( \tau_k \) to \( \tau_{k+1} \), \( a_k \) new inversions disappear and \( n-a_k \) new inversions are created.

Let us assume, for simplicity that \( I(\tau_{k+1}) \leq I(\tau_k) \) for all \( n - k > 2 \). That means that for all \( n - 1 \geq k \geq 1 \)
\[
\sum_{j=1}^{k} (n+1-2a_j) \geq 0 \quad (n-1 \geq k \geq 1).
\]

(2)

To simplify our calculations we assume \( n \) to be even, the modifications for odd \( n \) are insignificant. We want to find \( \sigma \) for which \( a_i < a_j \) for \( i < j \) will occur for all \( I \leq \frac{n}{2} < j \).

In other words the numbers in \( [1, n/2] \) will appear in reverse order and so will the ones in \( [n/2+1, n] \). Under this assumption we want to maximize the number of inversions between numbers from these two intervals, while at the same time maintaining (2) valid. This means we set
\[
a_i = \frac{n}{2} - i + 1 \quad \text{for } i \geq 1 \text{ and } a_{i+1} = n.
\]

for some integer \( t_1 \). Of course, we wish to minimize \( t_1 \) so as to maximize the number of
inversions. This must be done subject to the assumption that (2) must hold, which it
certainly does for \( t_1 \geq k \geq 1 \). Let us evaluate the l.h.s. of (2) for \( k = t_1 + 1 \)
\[
\sum_{j=1}^{t_1} (n-2a_j+1) = \sum_{j=1}^{t_1} (2j-1) = t_1^2
\]
And therefore
\[
\sum_{i=1}^{t_1+1} (n-2a_j+1) = t_1^2 - (n-1) > 0.
\]
We, therefore, choose \( t_1 = \left\lfloor (n-1)^{\frac{1}{2}} \right\rfloor \) to meet our goals.

We continue by letting \( a_i = \frac{n}{2} - i + 2 \) for
\[
t_2 + i \geq i \geq t_1 + 2
\]
and \( a_{t_2+1} = n - 1 \). The condition (2) reads
\[
\sum_{j=1}^{t_2+1} (n-2a_j+1) \geq 0.
\]
We group the terms for \( t_1 \geq j \geq 1 \) and those for \( t_2 + 1 \geq j \geq t_1 + 2 \) and the \( (t_1+1) \)-st and
\( (t_2+2) \)-nd term arriving at the inequality
\[
\sum_{j=1}^{t_2} (2j-1) = t_2^2 \geq (n-1) + (n-3) = 2n - 4,
\]
and we choose accordingly \( t_2 = \left\lfloor (2n-4)^{\frac{1}{2}} \right\rfloor \).

In general, where \( t_r = \left\lfloor (rn-r^2)^{\frac{1}{2}} \right\rfloor \), we set \( a_i = n - i + 1 \) \((n/2 \geq r \geq 1) \). This defines
\( n/2 \) of the \( a_i \) \((n \geq i \geq 1) \) the undefined \( a_i \)'s are \( n/2, ..., 1 \) in this order. This construction of \( \sigma \) implies that \( I(\sigma) \leq I(\tau_k) \) for every \( \tau_k \subseteq [\sigma] \). So we have to calculate \( I(\sigma) \): The only situation where \( i < j \) and \( a_i < a_j \) occurs for \( i \leq \frac{n}{2} < j \), and
\[
\left\lfloor \frac{t_r}{r} \right\rfloor \leq \frac{n}{2} \text{ and } a_i < n - r + 1 \rightarrow t_r \text{ for } r = 1, 2, ..., n/2.
\]

Therefore
\[
I(\sigma) = \frac{n}{2} - \sum_{r=1}^{n/2} t_r
\]
But
\[
\sum_{r=1}^{n/2} t_r \leq \sum_{r=1}^{n/2} (1 + (rn-r^2)^{\frac{1}{2}}).
\]
and
\[
\int_{r=1}^{n/2} (r^2 - x^2)^{1/2} \, dr \leq \int_{0}^{n/2} (nx - x^2)^{1/2} \, dx + \frac{n}{2} = \frac{n^2}{16} + \frac{n}{2}
\]

And hence
\[
I(\sigma) = \frac{n}{2} - \frac{\pi n^2}{16} - n = \frac{8-\pi}{16} n^2 - \frac{3n}{2}
\]
establishing the lower bound.

3. MAXIMAL DISTANCE

For \( \sigma = (a_1, \ldots, a_n) \in S_n \), let
\[
D(\sigma) = \max \{ |a_i - i| : i = 1, \ldots, n \}.
\]

In this section we investigate the functions:

\[
G(n) = \max_{\sigma \in S_n} \min_{\tau \in \sigma} D(\tau)
\]

and
\[
H(n) = \max_{\sigma \in S_n} \min_{\tau \in \sigma} D(\tau)
\]

We provide the exact value of \( G(n) \) and an approximate value of \( H(n) \), as described below: Let
\[
\alpha(n) = \min \{ k : k^2 + k - 1 \geq n \},
\]
\[
\beta(n) = \min \{ k : k^2 - k - 4 \geq n \}, \text{ and}
\]
\[
\gamma(n) = \min \{ k : k^2 + k / 2 \geq n \}.
\]

We prove that:

\[
G(n) = n - \alpha(n) \quad [n \geq 1]
\]
\[
n - \beta(n) \leq H(n) \leq n - \gamma(n) \quad [n \geq 8]
\]

The rest of this section is organized as follows: First we present a general result related to \( G(n) \) and \( H(n) \) (Proposition 3.1). Then we use this result to prove the upper bounds on \( G(n) \) (subsection 3.1) and \( H(n) \) (subsection 3.2). We conclude in proving the lower bounds on \( G(n) \) and \( H(n) \) (subsections 3.3 and 3.4).

In investigating the properties of \( D(\sigma) \) it is convenient to deal with the value \( k(\sigma) = n - D(\sigma) \). Let \( \sigma = (a_1, \ldots, a_n) \) and \( k < n \) be given. Then \( a_i \) covers \( \sigma \) if \( |a_i - i| \geq n - k \), and \( \sigma \) is covered if some \( a_i \) covers it. As in the previous section, we denote the
permutations in \([\sigma]\) by \(\tau_1, \ldots, \tau_n, \tau^R_1, \ldots, \tau^R_n\) and the permutations in \(<\sigma>\) by \(\tau_1, \ldots, \tau_n, \tau^R_1, \ldots, \tau^R_n\), where \(\tau_j = (a_j, \ldots, a_n, a_1, \ldots, a_{j-1}), \tau^R_j = (a_{j-1}, a_{j-2}, \ldots, a_1, a_n, \ldots, a_j)\). The proof of the following Proposition follows directly from the definitions, and is omitted.

**Proposition 3.1:**

(a) If \(\alpha_i = k-w+1\) for \(0 < w \leq k\) then \(\alpha_i\) covers the \(2w\)-elements set \(\{\tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i+w}, \tau^R_{i+1}, \tau^R_{i+2}, \ldots, \tau^R_{i+w}\}\). (Recall that if \(t\) is not in \([1,n]\), then \(\tau_t\) is identified with \(\tau_{t'}\), where \(t' = t - 1(\text{mod } n) + 1\)).

(b) If \(\alpha_i = n-k+w\) for \(0 < w \leq k\), then \(\alpha_i\) covers the set \(\{\tau_1, \tau_{i-1}, \ldots, \tau_{i-w+1}, \tau^R_{i+1}, \tau^R_{i+2}, \ldots, \tau^R_{i+w}\}\).

(c) If \(k < \alpha_i \leq n-k\) then \(\alpha_i\) covers \(\phi\).

### 3.1. Upper Bound on \(G(n)\)

We now use Proposition 3.1 to obtain an upper bound on \(G(n)\). To simplify the notations we denote \(\min D(\tau)\) by \(G(\sigma)\), (thus, \(G(n) = \max_{\sigma \in [n]} G(\sigma)\)). Let \(\sigma = (a_1, \ldots, a_n)\) and \(k \leq n\) be given. We shall show that if \(G(\sigma) \geq n-k\) then \(k^2 + k - 1 \geq n\), which, by the definition of \(a(n)\), proves the upper bound.

For \(i = 1, \ldots, n\), let \(v(\alpha_i) = |\{\tau \in [\sigma] : \alpha_i\text{ covers } \tau\}|\). Then, by Proposition 3.1,

\[
\begin{align*}
v(n) &= v(1) = k, \\
v(n-1) &= v(2) = (k-1), \\
&
\end{align*}
\]

\[
\begin{align*}
v(k) &= v(n-k+1) = 1, \\
v(j) &= 0 \text{ for } k < j \leq n-k.
\end{align*}
\]

Let \(\text{FAR}(\sigma) = \{\tau \in [\sigma] : \tau \text{ is covered}\}\). Then if \(G(\sigma) = n-k\), \(|\text{FAR}(\sigma)| = |[\sigma]| = n\).

On the other hand, \(|\text{FAR}(\sigma)| \leq \sum_{i=1}^{n} v(\alpha_i) = 2(1+2+\ldots+k) = k(k+1)\). This means that if \(G(\sigma) \geq n-k\) then \(k(k+1) \geq n\), which gives the upper bound

\[G(n) \leq n - \min[k : k^2 + k \geq n].\]
To improve this bound to
\[ G(n) \leq n - a(n) \] (recall that \( a(n) = \min\{k : k^2 + k - 1 \geq n\} \)),
we show that if \( G(\sigma) \geq n - k \) then for some \( \tau \) in \([\sigma]\) there are \( i1 \) and \( i2, i1 \neq i2 \), such that both \( a_{i1} \) and \( a_{i2} \) cover \( \tau \). Such a permutation \( \tau \) is said to be over covered. Clearly, if some \( \tau \) in \([\sigma]\) is over covered then
\[ |\text{FAR}([\sigma])| \leq \sum_{i=1}^{m} \nu(a_i) - 1 = k^2 + k - 1, \]
which implies the upper bound on \( G(n) \). The next lemma proves that such an over covered permutation must exist.

**Lemma 3.1.1:** If \( G(\sigma) \geq n - k \) then there is a permutation \( \tau \in [\sigma] \) which is over covered.

**Proof:** Assume the contrary. Then each permutation in \([\sigma]\) is covered by a unique \( a_i \) (\( 1 \leq i \leq n \)). Hence \( n = \sum_{i=1}^{m} \nu(a_i) = k^2 + k \), which implies that \( k \leq n/2 \). We say that a permutation \( \tau \) in \([\sigma]\) is of type (S) if the unique \( a_i \) that covers it is not larger than \( k \), and of type (L) otherwise (that is - if that \( a_i \) is larger than \( n - k \)). There are exactly \( (k^2 + k)/2 \) permutations of each type, and hence for some \( j \) in \( \{1, ..., n\} \) \( \tau_j \) is of type (L) and \( \tau_{j+1} \) is of type (S). Let \( q_1 > n - k \) cover \( \tau_j \) and \( q_2 \leq k \) cover \( \tau_{j+1} \). Note that since \( k \leq n/2 \) we must have that \( q_1 \neq q_2 \), and hence \( i \neq j \).

By Proposition 3.1(b) we have that \( i - \nu(a_i) + 1 \leq j \leq i \), and \( a_i \) covers \( \tau_i \) for \( j \leq l \leq i \). Since \( \tau_{j+1} \) is covered by \( a_i \), it cannot be covered by \( a_i \). Hence, \( i \) cannot be greater than \( j \), which implies that it must be equal to \( j \). By similar reasons, using Proposition 3.1(a), we get that \( i = j \). Thus we get that \( i' = j \), and hence \( i \neq i' \) - a contradiction. The lemma follows. \( \blacksquare \)

### 3.2. Upper Bound on \( H(n) \)

Let \( H(\sigma) = \min_{\tau \in [\sigma]} D(\tau) \). Like in the proof of the upper bound on \( G(n) \), we shall show that if for some \( \sigma = (a_1, ..., a_n) \) and \( k \) it holds that \( H(\sigma) \geq n - k \), then \( k^2 + k/2 \geq n \). For \( i = 1, ..., n \), let \( w(a_i) = \|\{\tau \in [\sigma] : a_i \text{ covers } \tau\}\| (\geq 2v(a_i)) \). Then, by Proposition 3.1,
\[ w(n) = w(1) = 2k \]
\[ w(n-1) = w(2) = 2(k-1) \]

\[ w(k) = 2w(n-k+1) = 2; \]

\[ w(j) = 0 \text{ for } k < j \leq n-k \]

Let \( \text{FAR}(\sigma) = \{ \tau \in \sigma : \tau \text{ is covered} \} \) and \( \text{OVER}(\sigma) = \{ \tau \in \sigma : \tau \text{ is over-covered} \} \). Since each permutation in \( \text{OVER}(\sigma) \) is covered by at least two distinct \( \alpha_i \)'s, we have that \( \text{FAR}(\sigma) \leq \sum_{i=1}^{n} w(\alpha_i) = |\text{OVER}(\sigma)| = 2k(k+1) - |\text{OVER}(\sigma)| \).

Also, if \( H(\sigma) \geq n-k \) then \( |\text{FAR}(\sigma)| = |\sigma| = 2n \). Thus we have

**Lemma 3.2.1:** If \( H(\sigma) \geq n-k \), then \( 2k(k+1) \geq 2n + |\text{OVER}(\sigma)| \).

By the above lemma, the upper bound of \( n - \gamma(n) \) on \( H(n) \) follows from the following lemma.

**Lemma 3.2.2:** If \( H(\sigma) \geq n-k \), where \( n \geq 2k \), then \( |\text{OVER}(\sigma)| \geq k \).

**Proof:** Let \( \sigma \in \Sigma_n \) be such that \( H(\sigma) \geq n-k \). Consider the list of indices \( 1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n \) for which \( w(\alpha_{i_j}) > 0 \), and let \( \bar{w}_j \) denote the number \( w(\alpha_{i_j})/2 \).

Consider now the following partition of \( \sigma \) to the \( 2k \) sets \( S_1, \ldots, S_{2k} \) defined by:

\[
S_j = \{ \tau, \tau^R : l \in [t_j, t_{j+1}] \}, \quad (j = 1, \ldots, 2k).
\]

In the definition above, and throughout this lemma, \( [i_{2k}, n] \) means \( [i_{2k}, n] \cup \{1, i_1\} \) if \( i_1 > 1 \), and \( [i_{2k}, n] \) if \( i_1 = 1 \). Also, for \( t > 2k \), \( i_t \) means \( i_{t'} \), where \( t' = t - 2k \). We denote by \( c(S_j) \) the number of distinct permutations in \( S_j \) which are over-covered. The following claim is the main tool used in the proof of this lemma. Though the claim is not surprising, its proof is rather tedious.

**Claim 1:** If for some \( j \), \( c(S_j) + c(S_{j+1}) = 0 \) then \( c(S_{j+2}) + c(S_{j+3}) \geq 2 \).

**Proof of Claim 1** Let \( j \) satisfy the hypothesis of the claim, and denote \( t_j, t_{j+1}, t_{j+2} \) and \( t_{j+3} \) by \( i, t', t'' \) and \( t''' \) respectively. We prove the claim only for the case \( \alpha_i \leq k \), since the proof of the case \( \alpha_i > n-k+1 \) is similar. Let \( \alpha_i = k - \bar{w}_i + 1 \), where \( \bar{w}_i = w(\alpha_i)/2 \).

Then by Proposition 3.1, \( \alpha_i \) covers \( \tau \) for \( i < l \leq i + \bar{w}_i \) and \( \tau^R_m \) for \( i - \bar{w}_i < m \leq i \). In
particular, \( \tau_{i+w} \) and \( \tau^R_{i} \) are covered by \( a_i \), but \( \tau_{i+w+1} \) and \( \tau^R_{i+1} \) are not (since \( k < n \)).

Let \( l \) be such that \( a_l \) covers \( \tau^R_{i+1} \). We consider three cases:

1) \( a_l > n-k+1 \). Then, by Proposition 3.1 and the fact that \( 2k \leq n \), we have that \( i < l < l+w_i \), which implies that \( a_l \) covers also \( \tau^R_{i} \), in contradiction with the assumption that \( \tau^R_{i} \) is not over covered (since \( \tau^R_{i} \in S_v \)).

2) \( a_l \leq k \) and \( l = i' \). This means that \( i = l - w_i < i' < l \). We distinguish between two subcases:

2.1) \( a_l \leq k \). Then \( \tau^R_{i} \) is over covered (by \( a_l \) and \( a_i \)), which contradicts the assumption that \( c(S_v) = 0 \).

2.2) \( a_l \geq n-k+1 \). In this case \( \tau^R_{i+1} \) is over covered (by \( a_l \) and \( a_i \)), and hence \( \tau^R_{i+1} = \tau^R_{i} \), cannot be in \( S_v \) (since \( c(S_v) = 0 \)). This means that \( \tau^R_{i+1} \) is in \( S_v \), and hence that \( i'' = i' + 1 \leq l \). Since \( a_l \geq n-k+1 \), \( a_i \leq k \) and \( l - w_i < i'' < l \), none of \( a_l \) and \( a_i \) covers \( \tau^R_{i} \). We shall use this last fact to show that there is another permutation in \( S_v \cup S_{v'} \), beside \( \tau^R_{i+1} = \tau^R_{i} \), which is over covered. This will prove the claim. We consider three subcases, according to the value of \( m \), for which \( a_m \) covers \( \tau^R_{i} \):

2.2.1) \( m = i'' \). Then it must hold that \( a_{i''} \geq n-k+1 \), and hence \( \tau^R_{i'+1} \) is over covered (by \( a_{i''} \) and \( a_i \)), and clearly \( \tau^R_{i'+1} \in S_v \cup S_{v'} \).

Note that the above argument is valid when ever \( a_l \geq n-k+1 \), and hence we may assume now that \( a_l \leq k \).

2.2.2) \( m = i' \). Then we have that \( i < i' < i'+1 = i'' \leq i'+w_i \), and hence \( \tau^R_{i} \) is over covered (by \( a_i \) and \( a_{i'} \)) - a contradiction to the assumption that \( c(S_v) = 0 \).

2.2.3) \( m \notin \{i, i', i''\} \). Then either \( m < i < i' \leq m+w_m \) and \( a_m \leq k \), or \( m < i' \leq m \) and \( a_m \geq n-k+1 \). In the first case \( \tau_{i+1} \) and \( \tau^R_{i} \) are over covered (the first by \( a_m \) and \( a_i \), the second by \( a_m \) and \( a_{i'} \)), which contrad-
i.e., the assumption. In the second case $\tau_{r+1}$ is over covered (by $a_m$ and $a_r$), and clearly $\tau_r$ is in $S_r \cup S_r$.

3) $a_i \leq k$ and $l = t'. $ Since $c(S_i) = 0$, $a_r$ covers $\tau_{r+1}$ but not $\tau^R_r$. Thus, $t' = i + w_r$.

Since $w_i = w_r$ and $1 \leq w_i, w_r \leq k$, we have that $w_r$ can be either strictly smaller or strictly larger than $w_i$. We consider each of these two possibilities here:

3.1) $w_r < w_i$. Then $\tau_{r+1}$ is over covered (by $a_i$ and $a_r$). This means, by the assumption that $c(S_i) = 0$, that $l + 1 \notin S_i$, hence $l'' = l + 1$. We consider two subcases, according to the value of $a_r$.

3.1.1) $a_r < k$. Then if $w_r = 1$ (i.e., $a_r = k$), $w_r$ must be larger than 1, hence $\tau_{r+1}$ is over covered (by $a_r$ and $a_r$), and thus both $\tau_r$ and $\tau_{r+1}$ are over covered, and the claim follows. If $w_r > 1$ then $\tau^R_r$ is over covered (by $a_r$ and $a_r$) - In contradiction with the assumption of the claim.

3.1.2) $a_r \geq n - k + 1$. Then $\tau^R_r$ is not covered by any of $a_i$, $a_r$, $a_r$, and hence is must be covered by some $a_m$ where $m \notin \{i, i', i''\}$. If $a_m = n - k + 1$ then $m < i < i'' < m + w_m$, and hence $a_m$ covers also $\tau^R_r$, a contradiction. Hence $m - w_m < i'' < m$ and $a_m \leq k$. This implies that $a_m$ covers also $\tau^R_{r+1}$, which is covered also by $a_r$. Hence both $\tau_r$ and $\tau^R_{r+1}$ (which are in $S_r \cup S_r$) are over covered.

3.2) $w_r > w_i$ (i.e., $i + w_r < i + w_i = t'$). Then $\tau_r$ is not covered by $a_i$, neither by $a_r$. The assumption that $c(S_i) = 0$ implies that $\tau_r$ is covered by some $a_m$ where $a_m \geq n - k + 1$ and $m - w_m < i' < m$, which means that $\tau_{r+1}$ is over covered (by $a_r$ and $a_m$). Since $c(S_i) = 0$, this implies that $i'' = i' + 1$, and that $\tau^R_r$ is not covered by $a_r$, neither by $a_m$. We shall use this last fact to show that there must be another permutation in $S_r \cup S_r$, beside $\tau_r = \tau_r$, which is over covered. We consider two cases, according to the value of $i''$:

3.2.1) $i'' = m$ (hence $a_r = n - k + 1$). Then $\tau^R_r$ must be covered by some $a_p$ where $p \notin \{i, i', i''\}$. If $a_p = n - k + 1$ then $p < i' < p + w_p$ and $\tau^R_r$ is over
covered (by $a_p$ and $a_r$) - a contradiction. If $a_p \leq k$ then $p - w_p < i'' < p$, and $\tau_{r+1}$ is over covered (by $a_p$ and $a_r$), and the claim holds.

3.2.2) $i'' \neq m$. Hence $m - w_m < i'' < m$. If $a_r \leq k$ then $\tau_{r+1}$ is over covered (by $a_r$ and $a_m$). If $a_r \geq n - k + 1$ then $\tau_{r+1}$ is not covered by any of $a_t, a_v$ and $a_r$. Let $p$ be such that $a_p$ covers $\tau_{r+1}$. If $a_p \leq k$ then $p - w_p < i'' < p$ and $\tau_{r+1}$ is over covered (by $a_p$ and $a_p$), and the claim holds. If $a_p \geq n - k + 1$ then $p < i'' < p + w_m$ and $\tau_{r+1}$ is over covered (by $a_m$ and $a_r$) - a contradiction. This completes the proof of the claim.

We need one more claim for the proof of the lemma:

Claim 2: Let $B_1, \ldots, B_{2k}$ be $2k$ boxes, each containing $c_i$ balls, and assume that for each $i = 1, \ldots, 2k$, if $c_i + c_{i+1} = 0$ then $c_{i+2} + c_{i+3} \geq 2$. Then $\sum_{i=1}^{2k} c_i \geq k$.

Proof: By induction on the number $t$ of indices $i$ such that $c_i + c_{i+1} = 0$. If $t = 0$ then there are at least $k$ $i$'s such that $c_i \geq 1$ and the claim holds. So assume that for some $i$, $c_i + c_{i+1} = 0$. By the hypothesis of the claim, $c_{i+2} + c_{i+3} \geq 2$. Relocate two balls from boxes $B_{i+2}$ and/or $B_{i+3}$ in $B_{i+1}$ and $B_{i+2}$. This does not change the sum $\sum_{i=1}^{2k} c_i$, and reduces the number of indices $i$ with the above property by at least one, thus the claim follows by induction.

Proof of Lemma 3.2.2: Let $c(S_i) = c_i$. Then $|\text{OVER}(\langle \sigma \rangle)| = \sum_{i=1}^{2k} c_i$, and by Claim 1 the assumption of Claim 2 holds. Hence, by Claim 2, $\sum_{i=1}^{2k} c_i \geq k$. \(\square\)

3.3. Lower Bound on $G(n)$.

First we show that if $n = k^2 + k - 1$ for some positive integer $k$, then there is a permutation $\sigma$ in $S_n$ for which $G(\sigma) = n - k$.

Let $(f_1, \ldots, f_k)$ be the sequence defined by:

$f_1 = 1$;

$f_{i+1} = f_i + k - i + 1$ (for $1 \leq i \leq k - 1$).
(i.e., \( f_k = k(i-1) + (3i-i^2)/2 \)). In particular, \( f_k = (k^2 + k)/2 \).

Similarly, let \((g_1, \ldots, g_k)\) be the sequence defined by:

- \( g_1 = f_k + 2 = (k^2 + k)/2 + 2; \)
- \( g_{i+1} = f_i + k - i + 1 \) (1 ≤ \( i \leq k - 2 \));
- \( g_k = 2 \).

(i.e., for 1 ≤ \( i < k \), \( g_i = k(i-1) + (3i-i^2)/2 \)). In particular, \( g_{k-1} = k^2 + 1 = n - k + 2 \).

Note that for 1 ≤ \( i \leq k \) and 1 ≤ \( j \leq k - 1 \), \( f_i < g_j \), and also that \( g_k = 2 \neq f_k \); it follows that \( f_i \neq g_j \) for all \( i, j \in \{1, 2, \ldots, k\} \).

Let \( \sigma = (a_1, \ldots, a_n) \) be any permutation in \( S_n \) that satisfies the condition *(*):

For 1 ≤ \( i \leq k \), \( \alpha_{f_i} = i \) and \( \alpha_{g_k} = n - k + 1 \). *(*)

Then \( G(\sigma) = n - k \). This follows by the following facts that are easily verified by Proposition 3.1:

1. For \( f_i < j \leq f_{i+1} \), \( \tau_j \) is covered by \( \alpha_{f_i} \) (\( i = 1, \ldots, k - 1 \)).
2. \( \tau_{f_k} + 1 (= \tau_{g_1 - 1}) \) is covered by \( \alpha_{f_k} \) and \( \tau_{g_1} \) is covered by \( \alpha_{g_1} \).
3. For \( g_i < j \leq g_{i+1} \), \( \tau_j \) is covered by \( \alpha_{g_i+1} \) (\( i = 1, \ldots, k - 1 \)).
4. For \( f \in [g_{k-1}, n] \cup \{1\} \), \( \tau_f \) is covered by \( \alpha_{g_k} (= \alpha_2) \). (Note that \( \tau_2 \) is also covered by \( \alpha_2 \), and is the unique permutation in \( \sigma \) which is over covered.)

To prove that the lower bounds of \( n - \alpha(n) \) on \( G(n) \) holds also for \( n \neq k^2 + k - 1 \) we make the following observations:

**Lemma 3.3.1:** If \( n \neq k^2 + k - 1 \) for all positive integers \( k \) then \( \alpha(n-1) = \alpha(n) \); [similarly, if \( n \neq k^2 - k - 4 \) then \( \beta(n-1) = \beta(n) \)].

**Proof:** By the definition of \( \alpha(n) [\beta(n)] \) □

**Lemma 3.3.2:** For all positive integers \( n \), \( G(n-1) = G(n) - 1 \) [\( H(n-1) = H(n) \)].

**Proof:** Define a mapping \( \mu : S_{n+1} \to S_n \) by:

\[ \mu(\sigma') = \mu(b_1, \ldots, b_{n+1}) = \sigma = (a_1, \ldots, a_n) \].
where $\alpha_i$ is defined as follows: let $i_0$ be such that $b_{i_0} = n+1$. Then for $i < i_0$, $\alpha_i = b_i$, and for $i_0 < i < n$, $\alpha_i = b_{i+1}$. It is straightforward to verify that this mapping satisfies the following conditions:

a) $\mu(\{\omega^i\}) = [\mu(\omega^i)] [\mu(\omega^{i+1})]$. (For a subset $T$ of $S_{n+1}$, $\mu(T) = \{\mu(\tau) : \tau \subseteq T\}$.)

b) $D(\mu(\omega^i)) \leq D(\omega^i) - 1$.

By a) and b) above, for all $\sigma \in S_{n+1}$, we have that $G(\mu(\omega^i)) = G(\omega^i) - 1$.

Thus, the lower bound on $G(n)$ for $k^2 + k - 1 \geq n \geq (k-1)^2 + (k+1) - 1$ is proved inductively, where the base of the induction is $n = k^2 + k - 1$ and the correctness for $n \to 1$ follows from the correctness for $n$ by the inequalities:

$G(n+1) \geq G(n) - 1$ [by Lemma 3.3.2],

$= n - \alpha(n) - 1$ [by the induction hypothesis]

$= (n-1) - \alpha(n-1)$ [by Lemma 3.3.1].

3.4: Lower Bound on $H(n)$.

Like in the previous subsection, we show first that if $n = k^2 - k - 4$ for some positive integer $k$, then there is a permutation $\sigma$ in $S_n$ for which $D(\omega^i) = n - k$.

Let $(f_1, \ldots, f_{k-1})$ be the sequence defined by:

$f_1 = 1$;

$f_{i+1} = f_i + k - i \cdot (1 \leq i \leq k - 2)$. (i.e., $f_i = 1 + k(i - 1) + (i - 1)^2)/2$. In particular, $f_{k-1} = (k^2 - k)/2$.

Similarly, let $(g_1, \ldots, g_{k-1})$ be the sequence defined by:

$g_1 = f_{k-1} - 1 = (k^2 - k - 2)/2$;

$g_{i+1} = f_i + k - i \cdot (1 \leq i \leq k - 3)$;

$g_{k-1} = 2$.

(i.e., for $1 \leq i < k - 1$, $g_i = (k^2 - 3k + 2k^2 - i^2 + i - 2)/2$.) In particular,

$g_{k-2} = k^2 - k - 4 = n$. 

It is not hard to verify that for $1 \leq i, j \leq k - 1$, $f_i \neq g_j$. Like in the lower bound for $G(n)$, we claim here that for any permutation $\sigma \in S_k$ which satisfies the condition (**)
below, $H(\sigma) = n - k$:
For $i = 1, \ldots, k - 1$, $a_{f_i} = i$ and $a_{g_i} = n + 1 - i$. (**)
To see this, observe that:

1. For $f_i < j \neq f_{i+1}$, $\tau_j$ is covered by $a_{f_i}$ ($i = 1, \ldots, k - 2$).
2. For $g_i - 1 < j \leq g_i$, $\tau_j$ is covered by $a_{g_i}$ ($i = 2, \ldots, k - 2$).
3. $\tau_{f_i} (= \tau_1)$ is covered by $a_{f_i}$.
4. For $f_i - 1 < j \leq f_i$, $\tau_{f_i}$ is covered by $a_{f_i}$ ($i = 2, \ldots, k - 1$).
5. For $g_i < j \leq g_i + 1$, $\tau_{f_i}$ is covered by $a_{g_i}$ ($i = 1, \ldots, k - 2$).
6. $\tau_{f_i} (= \tau_{R_1})$ is covered by $a_{f_i}(\neq a_1)$.

The proof of the lower bound for $H(n)$ for all $n > 7$ follows by lemmas 3.3.1 and 3.3.2, along the same line of the proof of the lower bound on $G(n)$. The details are omitted.

We conjecture that $H(n)$ is equal to $n - \beta(n)$ (for $n > 7$), though a simple proof of that conjecture may not exist.

REFERENCES


*Mapping Data Flow Graphs on VLSI Processing Arrays*, TR #308, Department of Computer Science, Technion, Haifa.