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OPTIMALLY CONTROLLED CCD SHIFT REGISTERS
(OPTIMAL INTERCEPTION OF A RECURRENT TRAJECTORY)

by

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Optimally Controlled CCD Shift Registers
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abstract

The question of optimally controlling the shift rate of a CCD-type shift register, when used as a secondary memory device has been treated a few times in the literature. The usual procedure has been to postulate the functional form of the optimal control action, and then evaluate the optimizing parameter under some assumption about the request arrival process. We present a control-theoretic treatment of the problem, derive the form of the optimal policy (which indeed turns out to be nearly identical to the one postulated before) and compute the optimizing parameter for a variety of cases.

Key Words: Optimal control, Bang-bang, CCD shift register
1. Introduction

Consider a shift register containing \( L \) bits which can only be read or written through a port \( P \) - c.f. Figure 1. The register is realized in a technology like that of Charge-Coupled Devices, which implies that the contents of the register must rotate continuously. The speed of rotation need not be constant but is bounded below by the charge 'refresh' requirements and from above by the switching rate of the devices or driving clocks. Let these bounds be \( a \) and \( b \) bits per second, respectively.

We view the register as connected (possibly with many replicas of itself) to a computer that generates occasionally requests to read or rewrite the contents of the register. These operations are assumed to require the entire contents of the register to be conducted opposite the port \( P \), and to commence only when a certain position (signature) reaches the port. We shall denote this signature by bit 0.

Consider the operation of the device controller. Obviously, whenever there is a pending \( I/O \) request, the register contents will be shifted at the maximum speed \( b \) till the request is satisfied. If another request exists at that instant it can immediately be processed with no latency. Using standard queueing-theoretical terminology we could say that only a request that starts a busy period would experience rotational latency, whereas others are serviced as soon as their turn arrives. Clearly, from the instant of arrival of a request finding an 'idle' register (i.e. a request starting a busy period) till the termination of that period, the controller will maximize the benefit to the users, i.e. minimize the delays of their requests, by maintaining a maximal rotational speed.

In this paper we address the problem of determining the optimal operation of the controller which will minimize the latency of the requests that start busy periods. In the sequel we shall assume the arrival process of new requests to be a stationary Poisson process, and then the idle periods are independent and identically distributed. Thus, the optimal operation during a single idle period in conjunction with the trivial optimal operation during the busy period will determine the global optimal policy. At first glance, it seems that continuously maintaining the maximal speed \( b \) is optimal. It will turn out that this is not the case. The reason is not hard to find - envision the position of bit 0 as it circulates. If a request arrives when bit 0 is
in position $s$ (the distance from the port measured in clockwise sense - see Figure 1) the latency will be at least $s/b$. Assuming instantaneous speed changes, we take this as its actual value. Hence, we wish to minimize the position of bit 0 at interception time, figuratively speaking. Therefore, a more promising policy seems to be the policy which applies as high a speed as possible $-b$ - when bit 0 is still far away from the port and the lowest speed $-a$ - when bit 0 is close to the port.

It turned out that this immediate observation covers most of the ground, with some allowance for the nature of the control policies under consideration.

In the next sections we shall formulate the problem, quantify and prove the optimality of the above approach, even in stronger form, using nearly exclusively extreme versions of that policy, where only the rates $a$ and $b$ are used. This type of policy is commonly called a 'bang-bang' policy.

The problem has been treated before in the literature. Sites [4], and Fuller and McGeehearty [1] consider the problem under the assumption that the instant of arrival is uniformly distributed over the duration that the shift register completes a revolution. This however is incompatible with the assumption that the arrival process is a renewal process; though in fact it is the limit of a Poisson arrival process when the rate of arrival approaches zero! In Gelenbe and Mitra [2, Chap. 2] the problem is considered under the assumption of the same arrival process as we do, but their treatment seems to have been derailed by a computational error. All these treatments assume that a 'bang-bang' policy with a single switch-over point is the best policy to adopt. Then one has only to find the optimal change-over point, which the first two references identify correctly, under their assumptions.

In contradistinction we place the problem in the context of control theory, and derive the explicit form of the optimal control. We show that when the rotational rate can be changed at any instant the optimal control is indeed a 'bang-bang' one, with the instant of change-over depending on the rate of arrival. When the rate of rotation can only be changed at a discrete set of points this policy is not necessarily optimal, though an intermediate rate need only be used once per revolution (at most) between two adjacent decision points.
In Section 2 we formulate the problem. In Section 3 we prove that the ‘bang-bang’ policy is $\varepsilon$-optimal for the discrete-time control problem and in Section 4 we show that it is optimal for the continuous-time control problem. The explicit form of the optimal policy is also given. A discussion in Section 5 concludes the paper.

Remark 1.1 For a given control policy and the assumptions detailed in Section 2, the queueing analysis for the distributions of the number of enqueued I/O requests and their overall delay is routine, see [2]. It is a special case of the so-called "a single-server queueing system of walking type".

2. A Controlled Markov Process Model

We shall formulate a model for the shift register as a controlled Markov process in discrete and continuous time.

Assume the geometry of the register as given in Figure 1. The register contents rotate with speed $u$ which can be controlled by the device controller. The speed is constrained to be in the interval $[a, b]$. We assume that the rotational speed can be changed instantaneously. No distinction is made between read and write requests, which arrive according to a Poisson counting process with rate $\lambda$. Hence, at every instant during an idle period, the time until the next request arrival that will initiate a busy period is exponentially distributed with parameter $\lambda$, and in particular, it is independent of the position of bit 0.

Hence, the position of bit 0, measured by the distance from the port in clockwise sense, and denoted by $s$, may be taken as the state of the system.

Let $L$ be the circular length of the register, then $s$ varies in the interval $(0, L]$.

We shall add to the state space a fictitious absorbing state $\ast$, and allow the system to cease operations by entering state $\ast$. Once the system enters this state it "terminates".

At every state $s \neq \ast$, the controller has to determine a rotational speed $u$, $a \leq u \leq b$.

The objective of the controller is to minimize the expected latency during an idle period; this is the same as minimizing the expected value of $s$ from which the system
will enter the state \(*\). Accordingly, we shall define the immediate cost \(C(s,u)\), when the system is in state \(s\), \(s \neq *\) and takes action \(u\) as follows:

If there is an arrival before the next decision epoch then \(c(s,u), s \neq *\), is the state of the system at the moment of that first arrival and zero otherwise. Obviously, \(c(*,u) = 0\).

As long as there is no arrival (the system is still in an idle period) the law of motion (transition probabilities) from any state \(s\) is determined by the rotational speed \(u\) and the standard motion equation - distance covered equals duration times rate. At the moment of the first arrival the system enters the state \(*\) and stays there for ever. A control policy is called admissible if it assigns to each state \(s\), \(s \neq *\) a rate of rotation in the physically realizable range \([a,b]\). For every admissible policy \(\pi\), let \(V(\pi,s)\) be the expected cost of operating the system given that it is currently at state \(s\) and utilizes policy \(\pi\).

Let \(V(s) = \inf_{\pi} V(\pi,s)\).

\(V\) is the value function. A control policy \(\pi^*\) is optimal if

\[ V(\pi^*,s) = V(s) \text{ for every } s \in (0,L]. \]

The problem can be treated as a discrete-time problem as well as a continuous-time one:

In the discrete problem we permit the controller to make decisions only at a set of discrete moments of time. Once a rotational speed is taken at a certain state, the same speed is maintained till the next decision point. A natural set is the one marked out by bit positions, yielding \(L\) decision instants. We shall consider a more comprehensive family of sets though.

In the continuous problem the controller may vary the rotational speed at every moment of time.

**Remark 2.1** Operationally we are only interested in minimizing \(V(L)\), since a busy period is always terminated with \(s = L\). Still we have to embed this in the problem of optimizing the value function for all \(s \in (0,L]\).
3. The Discrete-Time Problem

Let \( N \) be any positive integer, \( \Delta = L 2^{-N} \) and divide the state space interval \((0, L]\) to \(2^N\) equally spaced subintervals, each of size \( \Delta \). This particular choice will be useful in showing the \( \varepsilon \)-optimality of the chosen policy.

The controller is permitted to change the rotational speed only when the system is at one of the states \( s = k \Delta, 0 < k \leq 2^N \). During the intervals \([k \Delta, (k + 1) \Delta)\) the speed is maintained constant.

Every \( \Delta \) defines a discrete-time control problem which will be referred as the \( \Delta \)-problem. The immediate cost, the expected cost under policy \( \pi \) and the value function will be denoted by \( C_\Delta(s, u), V_\pi(s) \) and \( V_\Delta(s) \) respectively.

Let \( \bar{C}_\Delta(s, u) \) be the expectation of \( C(s, u) \) in the \( \Delta \)-problem.

From the definition of \( C_\Delta(s, u) \) in Section 2 and the nature of the Poisson arrival process,

\[
\bar{C}_\Delta(s, u) = \int_0^{\Delta u} \lambda e^{-\lambda t}(s - tu) dt = \Delta e^{-\lambda \Delta u} + \left(1 - e^{-\lambda \Delta u}\right)(s - \frac{u}{\lambda}).
\]

(3.1)

for every \( s \neq \ast \).

Note that by our choice of coordinates \( s \) decreases with time over the range \((0, L]\). Now for every \( \Delta \) and state \( s \neq \ast \), the probability that the system will "terminate" (enter state \( \ast \)) is at least

\[
\alpha \overset{\text{def}}{=} 1 - e^{-\lambda \Delta b} > 0.
\]

(3.2)

Thus, using terminology from [3, p. 56], we have a discounted dynamic programming problem.

Bellman's optimality equations for this problem are:

\[
V_\Delta(s + \Delta) = \inf_{s \in (s, u) \in \mathcal{A}} \{ \bar{C}_\Delta(s + \Delta, u) + e^{-\lambda \Delta u} V_\Delta(s) \}.
\]

(3.3)

For \( s = k \Delta, 0 < k \leq N \) and \( s + \Delta \) is taken modulo \( L \).

Since \( u \) varies on a compact set, \( \bar{C}(s, u) \) is absolutely bounded and uniformly continuous in \( u \). The next theorem follows then from Theorems 6.2 and 6.3 in [3, p. 56].
Theorem 3.1 For every $\Delta$-problem, there exists an optimal stationary control policy which satisfies the optimality equations (3.3). Moreover, if the actions taken by a policy $\pi^*$ satisfy (3.3) then $\pi^*$ is optimal.

In the rest of this section we shall use the simplicity of (3.3) to find the optimal policy and to prove that the 'bang-bang' policy is $\varepsilon$-optimal for small $\Delta$'s.

Lemma 3.1 For every state $s$ and control action $u$

$$V_\Delta(s) - s > \Delta + e^{\lambda\Delta/u} (V_\Delta(s + \Delta) - (s + \Delta)),$$

$s \neq L$

Proof: Substituting from (3.1) and (3.3)

$$V_\Delta(s + \Delta) \leq s + \Delta + (V_\Delta(s) - s) e^{-\lambda \Delta/u} - \frac{u}{\lambda} \left(1 - e^{-\lambda \Delta/u}\right),$$

and the lemma follows by algebraic manipulations.

Note: Since for $s = L$ the value of $s + \Delta$ should be taken as $\Delta$, the proof does not hold there. This will not impede the following analysis.

From Lemma 3.1 we can deduce the following monotonicity properties of the difference $V_\Delta(s) - s$.

Lemma 3.2

(i) If $V_\Delta(s) \leq s$ then $V_\Delta(s + \Delta) < s + \Delta$.

(ii) If $V_\Delta(s) \geq s$ then $V_\Delta(s - \Delta) > s - \Delta$.

Proof: part (i) is immediate a fortiori from Lemma 3.1. Part (ii) is the counterpositive of (i).

Corollary 3.1: Lemma 3.2 implies that the curves $y(s) = V_\Delta(s)$ and $y(s) = s$ intersect at most once.

Clearly, from the interpretation of the cost function $V_\Delta$ as the expected interception coordinate, $V_\Delta(L) < L$ and $V_\Delta(0) > 0$, thus $V_\Delta(s)$ and $y(s) = s$ intersect exactly once. The relation between $y(s) = s$ and $V_\Delta(s)$ is given in Figure 2.

Let $D = [s_0, s_0 + \Delta]$ be the interval containing the intersection, i.e.

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1 We extend $V_k(\Delta)$, $0 < k \leq N$, to the entire interval $(0, L]$ in the natural way which conforms to the definition of $V_\Delta(s)$.
\( V_\Delta(s) > s, \) for \( s < s_0, \) \( V_\Delta(s_0) \geq s_0. \) \hspace{1cm} (3.4) \\
\( V_\Delta(s) < s, \) for \( s \geq s_0 + \Delta, \)

Note that from Lemma 3.1 follows only the monotonicity of \( V_\Delta(s) \) in the interval \((0, s_0 - \Delta].\) However, the monotonicity is not necessary for our results.

For every \( \Delta \)-problem define the following bang-bang policy \( \pi_{bb}(\Delta). \)

Definition 3.1 The policy \( \pi_{bb}(\Delta) \) is the stationary policy which takes the control action \( u(s), \) where
\[
\begin{cases} 
  b & s \in \{ s \mid V_\Delta'(s) < s \} \\
  a & s \in \{ s \mid V_\Delta'(s) \geq s \}
\end{cases}
\] \hspace{1cm} (3.5)

From (3.4) we observe that \( \pi_{bb}(\Delta) \) assigns
\[
\begin{cases} 
  b & s > s_0 \\
  a & s \leq s_0
\end{cases}
\] \hspace{1cm} (3.6)

We now give the form of the optimal policy \( \pi^*(\Delta) \) and show that \( \pi_{bb}(\Delta) \) takes the same action as \( \pi^*(\Delta) \) except possibly at \( s_0 + \Delta. \)

Let
\[
g'(u) = C(s, u) + e^{-\lambda \Delta / u} V_\Delta(s - \Delta),
\] \hspace{1cm} (3.7)

where \( C(s, u) \) is given in (3.1), and \( V_\Delta(s - \Delta) = V_\Delta(L), \) when \( s = \Delta. \)

Lemma 3.3 For every \( u, a \leq u \leq b, \)

(i) If \( s > s_0 + \Delta \) then \( g'(u) < 0, \)

(ii) If \( s \leq s_0 \) then \( g'(u) > 0, \) for \( \Delta \) small enough.

Proof: From (3.1) and (3.7)
\[
g'(u) = e^{-\lambda \Delta / u} \left[ \frac{\lambda \Delta}{u^2} \left( V_\Delta(s - \Delta) - (s - \Delta) \right) - \frac{1}{\lambda} \left( e^{\lambda \Delta / u} - 1 - \frac{\lambda \Delta}{u} \right) \right]
\] \hspace{1cm} (3.8)

(i) When \( s > s_0 + \Delta \) it follows from (3.4), that the right hand side of (3.8), is strictly negative, as required.

(ii) When \( s \leq s_0 \) we have from Lemma 3.1,
\[
V_\Delta(s - \Delta) - (s - \Delta) > \Delta.
\]
Thus, from (3.8) it is sufficient to show that
\[ \frac{\lambda \Delta^2}{u^2} - \frac{1}{\lambda} \left[ e^{\lambda \Delta/u} - 1 - \frac{\lambda \Delta}{u} \right] > 0, \]
or
\[ e^{\lambda \Delta/u} - \frac{\Delta \lambda}{u} - 1 < \left[ \frac{\lambda \Delta}{u} \right]^2. \]  
(3.9)

To see that (3.9) holds for small \( \Delta \)'s, develop \( e^{\lambda \Delta/u} \) as a power series in \( \Delta \), notice that \( u \) is bounded away from zero and that \( \Delta < 3u/2 \lambda \) satisfies relation (3.9).

**Theorem 3.2** If \( \Delta \) is small enough then the optimal policy \( \pi^*(\Delta) \) is the policy which assigns

\[ u^*(s) = \begin{cases} 
  b & s > s_0 + \Delta \\
  u_0 & s = s_0 + \Delta \\
  a & s < s_0 + \Delta 
\end{cases} \]  
(3.10)

where \( s_0 \) is defined by (3.4) and \( u_0 \) is the value which minimizes \( g_{s_0 + \Delta}(u) \) in (3.7).

**Proof:** The theorem follows immediately from Theorem 3.1 and Lemma 3.3. The latter holds for \( a \leq u \leq b \), and the sign of \( g'(s) \) dictates the extreme values specified in (3.10), except at \( s = s_0 + \Delta \), about which Lemma 3.3 is reticent.

**Remark 3.1** When \( \Delta \) is small enough for Theorem 3.2 to hold the optimal control problem becomes an extremum problem with two variables, \( s_0, u_0 \). For every \( s_0, u_0 \) and policy \( \pi \) which acts according to (3.10) the expression for \( V_\Delta(\pi, s) \) is straightforward to compute (see below), and the optimal \( s_0 \) and \( u_0 \) can be directly found. Computational results have indeed shown cases where the optimal \( u_0 \) is not necessarily \( a \) or \( b \). We also could show that when \( \lambda \) is large enough (so that essentially only initial portions of the trajectory of bit 0, after it leaves \( s = L \), count), \( s_0 + 0 \), so that action \( a \) is not used at all. This could be shown from (3.3) to happen also if \( \Delta \) is not small enough.

**Corollary 3.2** For small enough \( \Delta \)'s, the bang-bang policy \( \pi_{bb}(\Delta) \) takes the same control actions as the optimal policy except possibly at the single node \( s_0 + \Delta \).

We now proceed to show that \( \pi_{bb}(\Delta) \) is \( \varepsilon \)-optimal; or in other words, that \( V_\Delta(\pi_{bb}(\Delta), s) - V_\Delta(s) \xrightarrow[\Delta \to 0]{} 0 \).
This is done implicitly, by computing expressions for the value function under policies that have the form of $\pi_b(\Delta)$ and $\pi^*(\Delta)$, with arbitrary values for $s_0$ and $u_0$ (i.e. without actually requiring that these values satisfy the optimality equations).

We shall also use the fact that by increasing $N$ (and decreasing $\Delta$) we add decision points, since $\Delta$ is of the form $2^{-N}$. Therefore, $V_\Delta(s)$ is non-increasing in $\Delta$. Thus, the following limits exist:

$$\lim_{\Delta \to 0^+} V_\Delta(s) = V(s)$$

and

$$\lim_{\Delta \to 0} s_0 = \lim_{\Delta \to 0} s_0(\Delta) = s^*$$

(3.11)

Clearly, $s^*$ is the intersection between the curves of $y(s) = s$ and $V(s)$.

Using the notation $t(s)$ and $s(t)$ to denote the time bit 0 reaches position $s$, and the position it reaches at time $t$ respectively, we have

$$V_\Delta(\pi(\Delta),z) = \frac{t(0)}{\lambda} s(t) + e^{-\lambda t} V_\Delta(\pi(\Delta),L)$$

and in particular,

$$V_\Delta(\pi(\Delta),L) = \frac{T(\pi(\Delta))}{\lambda} s(t) + e^{-\lambda T(\pi(\Delta))} V_\Delta(\pi(\Delta),L)$$

(3.12)

where $T(\pi)$ is the duration of an uninterrupted full revolution of the register under policy $\pi$.

We now compute $V_\Delta(\pi_{bb}^*(\Delta),s)$, using (3.6) and (3.12) to find

$$V_\Delta(\pi_{bb}(\Delta),s) = \begin{cases} s - \frac{a}{\lambda} (1 - e^{-s s_0}) + e^{-\lambda s_0} V_\Delta(\pi_{bb}(\Delta),L) & 0 < s \leq s_0 \\ s - s_0 e^{-\frac{s - s_0}{b} - b (1 - e^{-\frac{s - s_0}{b}}) + e^{-\frac{s - s_0}{b}} V_\Delta(\pi_{bb}(\Delta),s_0)} & s_0 < s \leq L \end{cases}$$

(3.13a)

and denoting $e^{-\lambda L - \frac{s - s_0}{b}}$ by $\alpha$ and $e^{-\lambda s_0}$ by $\beta$ we further find

$$V_\Delta(\pi_{bb}(\Delta),L) = \frac{L - \frac{b}{\lambda} (1 - \alpha) - \frac{a}{\lambda} \alpha (1 - \beta)}{1 - \alpha \beta}$$

(3.13b)

$$V_\Delta(\pi_{bb}(\Delta),s_0) = s_0 + \frac{L \beta - \frac{a}{\lambda} (1 - \beta) - \frac{b}{\lambda} \beta (1 - \alpha)}{1 - \alpha \beta}$$

and

(3.13c)

The same computation for $V_\Delta(\pi^*(\Delta),s)$, using also (3.10) is somewhat more cumbersome, yielding
\[ V_A^*(\Delta, s) = \begin{cases} s - \frac{a}{\lambda} (1 - e^{-\lambda s / a}) + e^{-\lambda s / a} V_A^*(\Delta, L) & 0 < s \leq s_0 \\ s - s_0 e^{-\lambda(s - s_0)/u_0} + e^{-\lambda(s - s_0)/u_0} V_A^*(\Delta, s_0) & s_0 < s \leq s_0 + \Delta \\ s - (s_0 + \Delta) e^{-\lambda(s - s_0 - \Delta)/b} - \frac{b}{\lambda} (1 - e^{-\lambda(s - s_0 - \Delta)/b}) + e^{-\lambda(s - s_0 - \Delta)/b} V_A^*(\Delta, s_0 + \Delta) & s_0 + \Delta < s \leq L \end{cases} \] 

Defining again \( \eta = e^{-\lambda s / a}, \phi = e^{-\lambda s / u_0}, \xi = e^{-\lambda(L - s_0 - \Delta)/b} \), one finds

\[ V_A^*(\Delta, L) = \frac{L - \frac{b}{\lambda} (1 - \xi) - \frac{u_0}{\lambda} \xi (1 - \phi) - \frac{a}{\lambda} \phi (1 - \eta)}{1 - \eta \phi \xi} \]

\[ V_A^*(\Delta, s_0 + \Delta) = s_0 + \Delta + \frac{L \eta - \frac{u_0}{\lambda} \xi (1 - \phi) - \frac{a}{\lambda} \phi (1 - \eta) - \frac{b}{\lambda} \eta (1 - \xi)}{1 - \eta \phi \xi} \]

Comparing (3.13) and (3.14), noting that as \( \Delta \to 0 \) \( \phi \) approaches 1, we obtain

**Theorem 3.3** \( V_A^*(\pi_{bb}(\Delta), s) = V_A^*(s) \to 0 \), uniformly in \( s \).

**Proof** The limit follows from the continuity of the terms that take part in \( V_A^*(\pi_{bb}(\Delta), s) \) and \( V_A^*(\pi^*(\Delta), s) \) as can be seen from (3.13-14). The uniformity follows from the terms being also bounded, and the limit holds for every \( s \) in the interval \( (0, L] \).

We have thus reached our goal. True, the policies \( \pi_{bb}(\Delta) \) (or \( \pi^*(\Delta) \)) still contain a parameter (or two) we have not yet determined. When one solves (3.3) numerically they are naturally obtained as well as the functional values of \( V(s) \), as given generically in Figure 2. The parameters could in principle be also extracted from equation (3.14) (or (3.13)); this will be however far simpler to do in the context of the continuous-time version of the problem we investigate in the next Section. Theorem (3.3) assures us of the quality of the values thus derived, even for discrete-time control.

4. The Continuous-Time Problem

In this section we permit a continuous-time control of the rotational speed, i.e., the controller may change the speed at every state \( s \in (0, L] \).
Let \( \pi \) be a continuous-time control and \( V_\pi(s) \) its expected cost given that the system is currently at state \( s \).

Let \( V(s) = \inf_\pi V_\pi(s) \).

Since the control variable \( u \) is bounded it follows that \( V_\pi(s) \) is continuous in \( s \).

Let \( c(s,u) \) be the infinitesimal generator of \( C_\pi(s,u) \). From (3.1)

\[
\begin{align*}
\mathcal{C}_\pi(s,u) &= \frac{\lambda s}{u(s)}. \quad (4.1)
\end{align*}
\]

For every \( \pi, \Delta > 0 \) and state \( s \) let \( \tau(s,\pi) \) be the travel time from state \( s \) to state \( s - \Delta \) under policy \( \pi \). Then,

\[
V_\pi(s) = \int_{t=0}^{\tau(s,\pi)} \mathcal{C}(s - t, u(s - t)) \, dt + e^{-\lambda \tau(s,\pi)} V_\pi(s - \Delta). \quad (4.2)
\]

Let \( \delta^- V_\pi(s) \) be the left derivative of \( V_\pi(s) \). Dividing (4.2) by \( \Delta \) and letting \( \Delta \to 0 \)

\[
V_\pi(s) = s - \frac{u(s)}{\lambda} \delta^- V(s). \quad (4.3)
\]

Similarly, we obtain the optimality equation

\[
V(s) = \inf_{a \in \mathcal{U}} \left\{ s - \frac{u(s)}{\lambda} \delta^- V(s) \right\}. \quad (4.4)
\]

The optimal control, if it exists, satisfies the optimality equation (4.4).

We show in the next theorem that the 'bang-bang' control policy is the unique solution to (4.4), up to the control action at a single point.

A direct consequence of (4.4) is

**Lemma 4.1** For every state \( s \), \( 0 < s < L \)

\[
\begin{align*}
V(s) > s & \implies \delta^- V(s) < 0, \\
V(s) < s & \implies \delta^- V(s) > 0, \\
V(s) = s & \implies \delta^- V(s) = 0. \quad \square
\end{align*}
\]

Let \( s^* \) be a state satisfying \( V(s^*) = s^* \). Its existence follows from the fact that \( V(L) < L, V(0) > 0 \) and the continuity of \( V(s) \). The last statement of Lemma 4.1 provides
its uniqueness.

**Definition 4.1** Let \( \pi_{bb} \) be the stationary bang-bang policy which takes the control actions \( u^*(s) \), where

\[
  u^*(s) = \begin{cases} 
  b & s > s^* \\
  a & s \leq s^* 
  \end{cases}
\]

**Theorem 4.1** The bang-bang policy \( \pi_{bb} \) is optimal for the continuous time problem.

**Proof:** Since the control variable \( u \in [a,b] \), it is easy to show that the set of all control policies \( \pi \) is a compact metric space. Moreover, \( V_\pi(s) \) is continuous in \( \pi \). Thus, the \( \inf \limits_\pi V_\pi(s) \) is obtained and there exists an optimal policy

From Lemma 4.1 and (4.4) it follows that \( V(s) > s \) for \( s < s^* \) and \( V(s) < s \) for \( s > s^* \). Thus, from (4.4) \( \pi_{bb} \) is the only solution (up to the decision at state \( s^* \)) to the optimality equations.

The existence and the fact that the optimal policy satisfies the optimality equation complete the proof.

Note that \( \pi_{bb} \) is optimal among all possible policies, and we did not restrict ourselves to policies under which the rotation rate and \( V_n(s) \) are smooth.

It is straightforward now to compute explicitly the optimal policy.

Under \( \pi_{bb} \), \( V_{\pi_{bb}}(s) \) is differentiable at any \( s \), except 0 and \( s^* \). Thus, solving (4.3) as a first order differential equation we have for some constants \( C \) and \( D \)

\[
  V(s) = \begin{cases} 
  C e^{-\lambda s/a} + s - \frac{a}{\lambda} & 0 < s < s^* \\
  D e^{-\lambda s/b} + s - \frac{b}{\lambda} & s^* < s < L 
  \end{cases}
\]

(4.5)

The requirement that both branches agree at \( s = s^* \) and the relation \( V(0) = V(L) \) determine \( C \) and \( D \). The latter follows from the continuity of \( V(s) \), and this also implies that \( C \) is a linear function of \( D \) with a positive slope \( e^{-\lambda L/b} \); thus they are minimized together. The value \( s^* \) is the one that minimizes \( V(s) \). From (4.5) we find
Differentiating $D$ with respect to $s^*$ and equating the derivative to zero yields

$$
D = \frac{b-a(\lambda s^*/a-1) + L}{e^{-\lambda s^*/(1/b-1/a)} - e^{-\lambda L/b}}. \tag{4.8}
$$

Consider first an approximate solution, valid for low arrival rates. By dropping from (4.7) terms of order $O\left(\frac{\lambda}{a/L}\right)^3$ we obtain

$$
\left(\frac{L-s^*}{s^*}\right)^2 = \frac{b}{a} \tag{4.8}
$$

This is precisely the value found in [1] upon assuming only bang-bang policies and uniform arrival density over $(0,L)$. This is consistent with the observation that for small $\lambda$, the arrival process becomes close to uniform. Also note that $\lambda < \frac{b}{L}$ is the stability condition for the system when considered as queueing model.

The solution of (4.8) provides

$$
s^* \approx \frac{L}{1+\sqrt{r}} \quad r = \frac{b}{a} \tag{4.9}
$$

and $V(L)$, as given by (3.13b) with $s_0$ replaced by $s^*$ results in

$$
V(L) \approx \frac{1}{2} \frac{s^* s^2 + \frac{L^2-s^* s^2}{b}}{\frac{s^*}{a} + \frac{L-s^*}{b}} + O(\lambda)
$$

and using (4.9), we get after some manipulations

$$
V(L) \approx \frac{L}{1+\sqrt{r}} + O(\lambda) \tag{4.10}
$$

Note the accidental(?) identity of the right-hand-sides of (4.9) and (4.10). The limiting value of (4.10), as $\lambda \to 0$ is precisely the value found by [1]. Brief reflection will convince the reader that for non-zero values of $\lambda$, the probability of interception across like sections of the trajectory of bit 0 increases with $s$, so one should expect then higher values for $V(L)$.

Another instructive quantity is the ratio of $V(L)$ to its value under the naive policy that maintains the rate constantly at its highest value $b$. For low $\lambda$ it provides
$V(L) \approx \frac{1}{2} L + O(\lambda)$, so the relative gain is $(1 + \sqrt{r})/2$. Consider that realistic values of $\sqrt{r}$ are in the range of 10 to 100+.

In the enclosed table we bring some numerical examples obtained by solving (4.7) and substituting in (3.13b), for a range of $\lambda$ and $r$, the values in the row $\lambda=0$ are from (4.9) and (4.10). The two values for each entry are $s^*/L$ above, and $V(L)/L$ below. The values of $\lambda$ are also given in dimensionless form, as $\lambda \frac{b}{L}$, whereas $\frac{b}{L}$ is the maximal input rate consistent with the stability of the system in the queueing-theoretical sense.

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<th>0.0250</th>
<th>0.1250</th>
<th>0.6250</th>
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</table>

5. Discussion

We have demonstrated that the popular notion about the optimal control of a CCD-like shift register is indeed optimal when the control is continuously and instantaneously affected, and arrivals obey the Poisson law. When control actions can only be taken at some discrete set of instants, the bang-bang policy is optimal up to one decision point, but even then it is $\varepsilon$-optimal. We chose to consider such discrete instants that correspond to equally separated positions of the register state descriptor, for technical reasons. It does not appear that a different choice (e.g. instants that are temporally equi-separated) would give a different answer.

Another assumption we have held throughout is that of the time-homogeneous exponential distribution of the time till the next request arrival. Any deviation from that would result in a considerably more complicated structure; but even then we propose the following:
Conjecture: For any time-homogeneous distribution of the time to next request arrival, the optimal continuous-time control is bang-bang, with the number of switching points less than or equal to the number of local maxima of the probability density function. A corresponding result holds for the discrete-time problem.
References


Figure 1 - A CCD shift register

Figure 2 - $V(s)$ vs. $s$