MESSAGE ROUTING TO TRANSMISSION CHANNELS WITH DIFFERENT RATES

by

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Abstract

Consider \( n \) parallel transmission channels which transmit information with different rates. For a system without buffers, geometric arrival of messages with rate \( p \) and geometric transmission times with rates \( r_1, r_2, \ldots, r_n \), it is shown that

\[
 p \left[ 1 - \prod_{i=1}^{n} \frac{p(1-r_i)}{1-(1-p)(1-r_i)} \right]
\]

is an upper bound on the throughput of the channels over all decentralized deterministic policies.

A new routing policy, the Golden Ratio policy, is suggested and shown to approach a limit which is within at least 98.4% of the upper bound. The Golden Ratio policy is a generalization of the Round Robin policy, when the transmission rates of the channels are different.

Keywords: Communication channels, routing protocols, golden ratio.

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1. Introduction

Consider \( n \) parallel transmission channels which transmit information with different rates. That is, the transmission time of a message through a channel depends on the channel. Every channel can transmit at most one message at any moment of time and each message is transmitted without interruption. Moreover, the channels are capable of blocking themselves from receiving new messages while they transmit; messages are lost if they arrive at blocked channels.

The channels are assumed to be slotted, i.e., the time is divided into equal segments called slots.

Each new message arrives first at a controller which routes it to one of the channels. (In the physical environment, the controller can be an intelligent multiplexer or a transmission node in a network.)

We assume that the controller has no buffer, so messages are routed instantly. We further assume that there is no travel time between the controller and the channels and the transmission of a new message starts at the beginning of a slot.

Suppose that there is a probability \( p \) that a new message arrives at the controller during a time slot and the arrivals are statistically independent (i.e., a Bernoulli arrival process). Let \( R_i (1 \leq i \leq n) \) be the number of slots required for the transmission of a message through channel \( i \). We assume that \( R_i \) is a geometric random variable with probability \( r_i \) for a successful trial. That is, \( \Pr (R_i = j) = r_i (1-r_i)^{j-1}, j = 1,2,3, \ldots ; r_i \) is the transmission rate of channel \( i \). Also, all message lengths are assumed to be independent.

Let \( r = (r_1, r_2, \ldots, r_n) \) and \( V_T (r, \pi) \) be the total expected number of packets successfully transmitted during the first \( T \) slots using the control policy \( \pi \). Define the throughput of the system (under policy \( \pi \))

\[
\bar{V}(r, \pi) = \liminf_{T \to \infty} \frac{V_T (r, \pi)}{T}.
\]

Finally, let
\( V(r) = \sup_{\pi} \bar{V}(r, \pi) \).

\( V(r) \) is the value function. A control policy \( \pi^* \) is optimal for \( r \) if \( V(r, \pi^*) = V(r) \).

For practical reasons, we are interested only in policies which can be implemented without observing the states of the channels (i.e., busy or idle).

The above situation arises at a switching node in a packet-switched communication network where incoming messages must be assigned to one of several outgoing links.

Message routing in a communication network was extensively studied by several authors; see e.g. [Be], [EVW], [FS], [FGK], [Ga], [Se], [Yu].

In most studies, static routing policies were analyzed and the optimal static policies were given by computational algorithms. These algorithms, when implemented, are usually time and space consuming and do not give enough insight about the structure of the optimal policy. This makes the implementation even harder when one wants to approximate the optimal policy to save time and space. The analysis of dynamic policies (policies which, at each moment, base their choice of a route upon the information available) has barely begun and only a simple model consisting of two similar exponential servers was analyzed, [EVW]. In our study we analyze dynamic policies and suggest a simple routing policy which is within at least 98.4% of the upper bound (to the cost criterion).

Two other recent papers [Ha1] and [Ha2] provide deep insight to the routing problem and their results can be used to bound the average delay in a communication network. While other papers consider the average delay as a cost criterion, the throughput of the channels is considered here. (This is the reason for the absence of buffers in the model - a situation where the throughput is most affected by the routing policy). This yields a very tractable model and a link to the average delay criterion via Little's Theorem. Thus one would expect that policies which increase the throughput would decrease the average delay. The routing problem under the average delay cri-
Criterion is the subject of our forthcoming paper.

In a closely related paper, [EVW], the authors considered a continuous time model of two identical exponential servers with an unlimited buffer and a general arrival process. It was shown there that if the queues are not observed, the policy which alternates between the servers (the Round Robin policy) minimizes the expected total time for the completion of all services which arrive before a predetermined horizon $T$. The routing policy which we suggest here, is a generalization of the Round Robin policy for non-identical servers.

In this study we analyze a discrete time routing problem with any number of different channels. In Section 2, the problem is formulated as a Semi-Markov Decision Process and an upper bound on the throughput is given in Section 3. In Section 4 a new routing policy (The Golden Ratio policy) is presented and analyzed. The Golden Ratio policy was originally suggested by Itai and Rosberg in [IR] for a multiple-access channel. This policy was proven to perform well in a multiple-access channel under the throughput cost criterion (see [IR]) and under the long run average delay criterion (see [HR]).

2. A Semi-Markov Decision Process Formulation

Let $(\tau_i - 1) \geq 1$ be the number of the slot during which the $i$-th message arrived, $i = 1, 2, 3, ...$. Clearly, $\tau_i, i \geq 1$, are random slots.

Note that decisions are taken by the controller only at the beginning of slots $2 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq \ldots$.

We shall consider the stochastic process defined by the states of the channels at the beginning of slots $\tau_1, \tau_2, \tau_3, \ldots$ (Define $\tau_0 = 0$).

For every channel $i$ and slot $t$ ($t \leq 1$), define the following random variables:
\( X_i(t) = 1 \) iff channel \( i \) is busy (transmitting a message) during slot \( t \).

\( V(t) = 1 \) iff a new message arrived during slot \( t-1 \).

\( u_i(t) = 1 \) iff the controller routes a message to channel \( i \) at the beginning of slot \( t \).

\( S_i(t) = 1 \) iff a message leaves channel \( i \) at the end of slot \( t-1 \), given that the
the channel was busy during slot \( t-1 \).

For definiteness we assume that each of the channels is busy during the first slot.

From the definition of the model \( \sum_{i=1}^{n} u_i(t) \) equals 0 or 1. Moreover

\[
X_i(t+1) = V(t+1)u_i(t+1) + X_i(t)(1-S_i(t))(1-u_i(t+1)V(t+1)).
\]

\( Pr(V(t)) = p. \)

Since \( R_i \) are geometric r.v.'s, \( S_i(t) \) are independent and

\[
Pr(S_i(t) = 1) = r_i.
\]

Let the immediate reward at slot \( t \), \( w(t) \), be

\[
w(t) = \begin{cases} 
1 & \text{if a new message had arrived during slot } t-1 \text{ and the message} \\
& \text{was routed to an idle channel,} \\
0 & \text{otherwise}
\end{cases}
\]

Thus,

\[
w(t) = \sum_{i=1}^{n} V(t)u_i(t)(1-X_i(t)).
\]

and

\[
V_T(\pi, \nu) = E_\pi \sum_{t=1}^{T} w(t) = \sum_{t=1}^{T} E_\nu (w(t)).
\]

Under any given control policy \( \pi, \mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t)), \ t \geq 1, \) is a Markov Process.

Let \( Y(j) = (Y_1(j), Y_2(j), ..., Y_n(j)), j = 1, 2, 3, ... \) be the Markov Process embedded in \( \mathbf{X}(t) \).
at the random slots \( \tau_j, j = 1,2,3, \ldots \).

\( \mathbf{Y}(j) \) is the state of the channels at decision epoches (right before the message is routed by the controller).

Since we are interested in policies which are not allowed to observe \( \mathbf{Y}(j) \), it cannot be used as a state descriptor in the decision process. The best one can do is to use the probability distribution of \( \mathbf{Y}(j) \) based on the information available to the controller.

As in [IR] and [HR] we need a sufficient statistic for \( \mathbf{Y}(j) \). The usage of the sufficient statistics in this section relies on Stribel's monograph [ST].

Let \( k^{(i)}(t) \) be the number of slots since the controller routed a message to channel \( i \), until time \( t \) (just before a routing decision is made). We have

\[
\begin{align*}
  k^{(i)}(t+1) &= 1 + k^{(i)}(t)(1 - u_i(t)V(t)), \quad t \geq 1 \\
  k^{(i)}(1) &= 1
\end{align*}
\]

(2.5)

Let

\[
\begin{align*}
  \mathbf{u}(t) &= (u_1(t), u_2(t), \ldots, u_n(t)), \\
  \mathbf{u}^{t-1} &= (u(1), u(2), \ldots, u(t-1)) \text{ and} \\
  \mathbf{V}^{t-1} &= (V(1), V(2), \ldots, V(t-1)).
\end{align*}
\]

Also, let \( X_i(t) \) be the value of \( X_i(t) \) right after a slot is ended and before the controller routes a message to the channel.

Since \( R_i \) (1 \( \leq \) \( i \) \( \leq \) \( n \) ) are geometric r.v.'s and \( V(t) \) are independent, the following lemma is obtained from (2.1):
Lemma 2.1

(i) If $u^{-1}, v^{-1}$ are given then $\{X_1(t) = X_2(t) = \ldots = X_n(t) = 1\}$ are mutually independent r.v.'s.

(ii) $\Pr ((X_i(t) = 1) \mid u^{-1}, v^{-1}) = 1 - (1 - \tau_i)^k(1)$ \quad $1 \leq i \leq n$.

Let

$$p_i(k) = 1 - (1 - \tau_i)^k.$$  \hspace{1cm} (2.6)

A Semi Markov Decision Process (SMDP) is defined by the state space $S$, the action space $A = \mathbb{X} A_0$, the law of motion $\eta$ (transition probabilities), the transition times $T$ and the expected immediate reward function $\psi$ (See e.g. [Ro].)

We shall formulate the routing problem as a SMDP.

For every $j$, $Y(j)$ is a Markov process whose transition probabilities depends only on the history of the actions taken by the policy until time $\tau_j$. From Lemma 2.1, the distribution of $Y(j)$ is completely defined by the parameters

$$k(\tau_j) = (k^{(1)}(\tau_j), k^{(2)}(\tau_j), \ldots, k^{(n)}(\tau_j)).$$

Therefore, we consider the state space

$$S = \{k = (k^{(1)}, k^{(2)}, \ldots, k^{(n)}) \mid k^{(i)} = 1, 2, 3, \ldots, 1 \leq i \leq n\}$$

and the action space at state $s \in S$

$$A_s = \{u = (u_1, u_2, \ldots, u_n) \mid u_i = 0, 1; \sum_{i=1}^n u_i = 0, 1\}$$

The transition times (in slots) between consecutive epochs, $T_j = \tau_j - \tau_{j-1}$, $j \geq 1$, are independent geometric random variables, that is

$$\Pr (T_j = k) = p (1 - p)^{k-1}, \quad k \geq 1.$$  \hspace{1cm} (2.7)

Moreover, the $T_j$'s are independent of the policy and the transmission times.
From (2.5) and (2.7) the transition probabilities becomes

\[ q((k^{(1)}+j, k^{(3)}+j, ..., k^{(n)}+j) \mid k,u = (0,0,0)) = p(1-p)^{\ell-1}, \quad (j \geq 1), \]
\[ q((k^{(1)}+j, ..., k^{(n)}+j) \mid k,u = (0,1,0)) = p(1-p)^{\ell-1}, \quad (j \geq 1), \]
\[ q((\ldots) \mid k,u) = 0 \quad \text{otherwise}. \] (2.8)

Finally, from (2.3) and (2.6) the expected immediate reward is

\[ w(k,u) = \sum_{i=1}^{n} u_i p_i(k^{(i)}). \] (2.9)

Thus, from (2.4)

\[ V_p(r,n) = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i p_i(k^{(i)}(\tau_j)). \] (2.10)

From [Ro], Lemma 2.1 (2.6) and (2.7)

\[ \bar{V}(r,n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{n} \sum_{i=1}^{n} \Pr(Y_i(\tau_j)=0) u_i(\tau_j) \]
\[ = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{n} E(\tau_j) \]
\[ = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{n} \Pr(Y_i(\tau_j)=0) u_i(\tau_j). \] (2.11)

**Theorem 2.1** There exists a non-randomized stationary control policy \( \pi^* \) such that

\[ \bar{V}(r) = \sup_{\pi} \bar{V}(r,n) = \bar{V}(r,\pi^*). \]

**Proof:** The proof is based on the theorems in [Ro, Chap. 7].

Let \( \alpha > 0 \) be a discount factor and \( V_{\alpha}(k,\pi_{\alpha}^*) \) the expected total discounted reward incurred when the initial state is \( k \), the discount factor is \( \alpha \) and the optimal policy \( \pi_{\alpha}^* \) is employed.

Let \( \pi_{\alpha}^*(k,k') \) be the non-stationary policy which does the following: when the initial state is \( k \), it pretends that the state is \( k' \) and acts as \( \pi_{\alpha}^* \). Note that from (2.5) it follows that after the first time each channel has already received at least one message, \( \pi_{\alpha}^*(k,k) \) and \( \pi_{\alpha}^* \) take the same actions.
Moreover, for every state \( k \) and \( a \geq 0 \)

\[
V_a(T_1(k), \pi_a^*) - V_a(k, \pi_a^*) \geq V_a(T_1(k), \pi_a^*(T_1(k), k)) - V_a(k, \pi^*) \geq 0.
\]

Moreover, for every state \( k \) and \( a \geq 0 \)

\[
|V_a(k, \pi_a^*) - V_a(1, \pi_a^*)| = V_a(k, \pi_a^*) - V_a(1, \pi_a^*)
\]

\[
\leq V_a(k, \pi_a^*) - V_a(1, \pi_a^*(1, k)) \leq n,
\]

where \( 1 = (1,1,\ldots,1) \).

Now, since the immediate reward is bounded the theorem follows from Theorem 7.7 in [Ro, Chap. 7].

Every policy which permits rejections of new arrivals can be replaced by a better policy which does not. (Replace every rejection by routing the message to some channel.) Therefore, we shall investigate policies which do not permit rejections.

Let \( \pi \) be a stationary policy and \( \omega \) be a realization of the arrivals (\( \omega \) is an element in the sample space of the arrival process).

Further, let \( d_{j}^{(i)}(\omega) - 1; j \geq 1 \) be the number of arrivals between two successive routings to channel \( i \). Note that for a stationary policy, the \( d_{j}^{(i)} \)'s depend on the realization \( \omega \).

This decision process can be observed as a "game" between nature and the decision maker. Nature plays first (although the outcomes are revealed to decision maker one at a time) and the decision maker plays second. Each realization \( \omega \) of a game results in the sequences \( d_{j}^{(i)}(\omega); j \geq 1 \) \( 1 \leq i \leq n \).

We consider policies in which the mechanism is reversed, i.e. the decision maker plays first and then nature. The result of this "game" are deterministic sequences \( d_{j}^{(i)}; j \geq 1 \) \( 1 \leq i \leq n \), which are independent of \( \omega \).
Let \( d^{(i)} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d_j^{(i)} \).

Since \( \frac{1}{d^{(i)}} \) is the long-run proportion of messages which are routed to channel \( i \) and rejections are excluded,

\[
\sum_{i=1}^{\infty} \frac{1}{d^{(i)}} = 1 \tag{2.12}
\]

These policies will be called deterministic splitting policies. Clearly, they are more tractable. It is an interesting and still an open question, under what conditions these policies result the same optimum as stationary policies.

In the next section we shall evaluate the deterministic splitting policies and bound the throughput from above.

3. An Upper Bound

Let \( \pi \) be a deterministic policy, that is, sequences \( d_j^{(i)}; j \geq 1 \) \( 1 \leq i \leq n \) which satisfy (2.12).

For every \( N \), let \( N^{(i)} \) be the number of messages (among the first \( N \)), which are routed to channel \( i \). Clearly,

\[
\sum_{i=1}^{N} N^{(i)} = N, \tag{3.1}
\]

\[
\sum_{j=1}^{N} d_j^{(i)} \leq N.
\]

Let

\[
q_j^{(i)} = \Pr(Y_i(d_1^{(i)} + d_2^{(i)} + \ldots + d_j^{(i)}) = 0).
\]

The sum of the first \( N \) elements in the numerator of (2.11) equals to

\[
\sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \Pr(Y_i(j) = 0) u_i(\tau_j) = \frac{1}{N} \sum_{i=1}^{N} \frac{N^{(i)}}{N} \left[ \frac{1}{N^{(i)}} \sum_{j=1}^{N^{(i)}} q_j^{(i)} \right] \tag{3.2}
\]
Let \( T \) be the \( d \)-convolution of the geometric r.v. \( T' \) whose distribution is given by (2.7).

Note that the transition times (in slots) between two consecutive messages which are routed to channel \( i \), are independent and distributed as \( T_{i,j}^{(k)} \), \( j=1,2,3,... \)

**Lemma 3.1** \( q_j^{(k)} = 1 - \left(1 - \frac{\tau_i}{p + (1-p)\tau_i} \right)^{d_j^{(k)}} \)

**Proof:** From (2.9)

\[ q_j^{(k)} = \Pr(Y_i(d_j^{(0)} + d_j^{(1)} + \cdots + d_j^{(k)}) = 0) = \Pr(T_i \leq d_j^{(k)}). \]

By induction on the values \( d_j^{(k)} \)

\[ \Pr(T_i = k) = \binom{k}{d_j^{(k)-1}} p^{d_j^{(0)}} (1-p)^{k-d_j^{(k)}} \quad k \geq d_j^{(k)}. \]

Thus,

\[ q_j^{(k)} = \sum_{k=d_j^{(0)}}^{d_j^{(k)}} \binom{k}{d_j^{(0)}} p^{d_j^{(0)}} (1-p)^{k-d_j^{(0)}} \frac{\tau_i}{(1-p)(1-\tau_i)} \]

\[ = 1 - \left(1 - \frac{\tau_i}{p + (1-p)\tau_i} \right)^{d_j^{(0)}}. \]

Let

\[ \rho_i = \frac{\tau_i}{p + (1-p)\tau_i}. \]

From the concavity of the function \( f(x) = 1 - (1-p)^x \) (0 \( \leq p \leq 1 \)) and (3.1)

\[ \frac{1}{N^{(k)}} \sum_{j=1}^{N^{(k)}} q_j^{(k)} \leq 1 - (1-\rho_i) \frac{1}{N^{(k)}} \sum_{j=1}^{N^{(k)}} d_j^{(k)} \leq 1 - (1-\rho_i) \frac{N^{(k)}}{N^{(k)}}. \] (3.3)

From (2.11), (3.1)-(3.3), the throughput of a deterministic policy is bounded above by the solution of the following problem:
Theorem 3.1 For every deterministic policy \( \pi \)

\[
\max \left\{ p \sum_{i=1}^{n} x^{(i)} \left[ 1 - (1 - \rho_i)^{x^{(i)}} \right] \right\}
\]

subject to

\[
\sum_{i=1}^{n} x^{(i)} = 1 \quad \text{and} \quad x^{(i)} \geq 0.
\]

Proof: See Theorem 3.1 in [IR].

A straightforward calculation gives the following:

Corollary 3.1 If \( r_i = r, \ i = 1, 2, \ldots, n \), then the round robin policy (the policy which routes to channel \( i \) message \( i (\text{mod} \ n) - 1 \)) obtains the upper bound.

From (2.11) and Lemma 3.1 we have the following theorem.

Theorem 3.2 For every deterministic policy \( \pi \) with \( \delta^{(i)}_j \geq 1 \) \( (1 \leq i \leq n) \)

\[
\bar{V}(r, \pi) = p \lim_{N \to \infty} \sum_{q=1}^{N} \frac{1}{N} \left( \frac{1}{N} \sum_{j=1}^{N} q^{(i)}_j \right)
\]

The routing problem here has a similar mathematical structure as the multiple access channel problem in [IR]. Once this is observed, the results there can be used.

We consider also the asymptotic behavior: when the number of channels \( n \to \infty \), the transmission rate of channel \( i \) (given \( n \) channels) \( r_i(n) \to 0 \) and the total transmission rate \( r = \sum_{i=1}^{n} r_i(n) \) remains fixed.
Notice that
\[
\lim_{n \to \infty} U(r(n)) = p \left( 1 - \lim_{n \to \infty} \prod_{i=1}^{n} (1 - p_i(n)) \right) = p \left( 1 - e^{-\frac{r}{p}} \right).
\] (3.4)

4. The Golden Ratio Routing

Consider deterministic routing policies which can be described by a loop. That is, each sequence \( d_j^{(i)}, j \geq 1 \), is periodic (there is an \( N^{(i)} \) such that \( d_{j+N^{(i)}}^{(i)} = d_j^{(i)} \) for every \( j \)).

In a loop policy, from every subsequent set of \( N = \sum_{i=1}^{n} N^{(i)} \) messages, \( N^{(i)} \) are routed to channel \( i \) and the number of arrivals between two successive routings to channel \( i \) are \( d_j^{(i)}, 1 \leq j \leq N^{(i)} \) (\( N \) is the loop size).

Let \( (n,r(n)) \) be a given system and \( x^{(i)}(n) > 0, i=1,2,\ldots,n, \sum_{i=1}^{n} x^{(i)}(n) = 1 \) be the desirable proportions of routings to each of the channels. (When no confusion arises the argument \( n \) will be omitted.) Also, let \( F_k \) be the \( k \)-th Fibonacci number and \( N_k^{(i)}, i=1,2,\ldots,n \) be integers such that
\[
[x^{(i)}F_k] \leq N_k^{(i)} \leq [x^{(i)}F_k] \tag{4.1}
\]
and
\[
\sum_{i=1}^{n} N_k^{(i)} = F_k
\]
Thus,
\[
\lim_{k \to \infty} \frac{N_k^{(i)}}{F_k} = x^{(i)}. \tag{4.2}
\]
We define the Golden Ratio policy as follows: for each \( k \), the Golden Ratio policy routes \( N_k^{(i)} \) messages from a loop of size \( F_k \) to channel \( i \) and attempts to distribute the routings uniformly over the loop. (The analysis of Section 2 implies that it is desirable to distribute the permission uniformly.)
Open address hashing confronts a similar problem: to distribute keys uniformly over a hash table. The uniformity of the distribution depends on the hash function. It has been shown that multiplicative hashing with Golden Ratio multiplicand, \( \varphi^{-1} = (\sqrt{5} - 1)/2 \approx 0.6180339887 \), distributes the keys most uniformly (Knuth [Kn, vol.1]). The Golden Ratio policy applies some of these results. Fibonacci numbers are related to the golden ratio \( \varphi^{-1} \) by the equation:

\[
F_k = \frac{\varphi^k - (1 - \varphi)^k}{\sqrt{5}}.
\]

Let \( \text{frac}(y) \approx y - \lfloor y \rfloor \), \( a_j = (j \varphi^{-1}) \) and \( A_N = \{ a_j \, | \, j = 0, \ldots, N-1 \} \). The \( t \)-th smallest point of \( A_F \) is identified with the \( t \)-th message of the loop.

**Definition 4.1** The Golden Ratio policy, \( \pi_{\text{GR}(a)} \), is the policy which assigns to channel \( i \) the messages corresponding to the points

\[
\{ a_j \mid \sum_{m=1}^{t} N_k^{(m)} \leq j < \sum_{m=1}^{t+1} N_k^{(m)} \}.
\]

It will be convenient to identify the points 0 and 1, and thus the points \( a_j \) are distributed over a circle \( C \).

**Example 4.1** Suppose \( n = 3 \), \( x^{(1)} = \frac{1}{2} + \varepsilon_1 \), \( x^{(2)} = \frac{3}{8} + \varepsilon_2 \), \( x^{(3)} = \frac{1}{8} + \varepsilon_3 \), where \( \varepsilon_i > 0 \) are arbitrarily small and \( x^{(1)} + x^{(2)} + x^{(3)} = 1 \). Taking \( F_0 = 6 \), \( N_0^{(1)} = 4 \), \( N_0^{(2)} = 3 \) and \( N_0^{(3)} = 1 \), \( \pi_{\text{GR}(a)} \) routes to channel 1 the messages corresponding to 0, \( \varphi^{-1} \), \( \text{frac}(2\varphi^{-1}) \), \( \text{frac}(3\varphi^{-1}) \), to channel 2 the messages corresponding to \( \text{frac}(4\varphi^{-1}) \), \( \text{frac}(5\varphi^{-1}) \) and \( \text{frac}(6\varphi^{-1}) \); and to channel 3 the messages corresponding to \( \text{frac}(7\varphi^{-1}) \). Thus the loop policy keeps routing \( n \) to the channels in the following cyclic order: "1,2,1,3,2,1,2,1".

Let \( \pi_{\text{GR}(a)} \) be a Golden Ratio policy with

\[
N = F_k, \quad N_k^{(i)} = F_k + s_k^{(i)} \quad (0 \leq s_k^{(i)} < F_{k-1}).
\]

Further, let \( f_k \) satisfy
Lemma 4.1 and Theorem 4.1 below are proven in [IR]. (See Lemma 5.2 and Theorem 5.2 there).

Lemma 4.1 For almost all \( k \), \( k_1 = k - j_1 \).

Theorem 4.1 For almost all \( k \),

\[
V(r, \pi_{GR}(k)) = p \left( 1 - \sum_{i=1}^{n}(x^{(i)} - \varphi^{-j_i})(1 - \rho_i)^{j_i} + (x^{(i)} - \varphi^{-j_i})(1 - \rho_i)^{j_i+1} \right)
+ (\varphi^{-j_i+1} - x^{(i)})(1 - \rho_i)^{j_i+2})
\]

In the following theorem we show the asymptotic behavior of the throughput for a large number of channels and a fixed transmission rate \( r = \sum_{i=1}^{n}r_i \).

Theorem 4.2 If \( r_i(n) \to 0 \) and \( \sum_{i=1}^{n}r_i(n) = r \) then

\[
\liminf_{n \to \infty} \lim_{k \to \infty} V(r, \pi_{GR}(k)) = p \left( 1 - \varphi^{-1} \right) e^{-\sqrt{p} \sqrt{r}} - \varphi^{-1} e^{-\sqrt{p} \sqrt{r}}.
\]

For the proof see Theorem 5.3 in [IR].

Denote

\[
V(r, \pi_{GR}) = \lim_{k \to \infty} V(r, \pi_{GR}(k)).
\]

Finding the minimum over \( r \) and \( p \) of the ratio between the \( \liminf_{n \to \infty} V(r(n), \pi_{GR}) \) and \( U(r(n)) \) given in Theorem 4.2 and (3.4) yields the following result about the golden ratio routing policy.
Theorem 4.3 Under our asymptotic conditions, for almost all $n$

$$\frac{\bar{V}(r(n), \pi_{GR})}{U(r(n))} \geq .984.$$
REFERENCES


