INTERACTIVE SYSTEMS WITH A FAST LOCAL GROWTH: DETERMINISTIC CASE

by

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ABSTRACT

A property typical to fastly growing parallel systems is discussed and studied in the framework of L system theory. This property is true of the members of a large subclass of the IL class, called bounded systems, which grow very rapidly, in a certain well defined sense. It is shown that every deterministic bounded system is equivalent to a coding of a DOL system and to an EQL system. Since this equivalence is effective, some properties which are formally undecidable for general DIL systems can be shown to be decidable for deterministic bounded systems.

Key Words: L, Lindenmayer, Developmental Systems,
1. INTRODUCTION

Suppose that a growing parallel system is growing in a mode such that the "distance" between any two of its "parts" is increasing in a rate which is greater than the maximal speed of any communication signal transferred between them. In this case the "parts" of the system cannot communicate with each other. This intuitively described phenomenon can be described precisely using IL systems as a model. (For notations and terminology of L systems see chapter 2 or [Rozenberg (1974)]).

Any substitution of a letter in a string produced by a $<k,\lambda> L$ system can be influenced by $k$ adjacent letters on its left and $\lambda$ on its right. Hence, the speed of any signal moving from left to right, during the development of a string is at most $k$ letters per derivation step and at most $\lambda$ letters in the opposite direction. Let us denote $d_{ij}$ the number of letters separating two letters $i,j$ in a string produced by the system (the "distance" between $i$ and $j$), let $\alpha_i, \alpha_j$ be the strings substituted for $i,j$, respectively, in the current derivation step and let $N$ be a constant integer.

Suppose that the following condition holds:

For any string produced by the system and for any two letters $i,j$ in the string, $d_{ij} > N$ implies that $\alpha_i, \alpha_j$ are separated after the derivation producing the next string by at least $d_{ij} + \max(k,\lambda)$ letters.

It follows that the letter $i$ and all letters derived from it cannot communicate (influence or being influenced by) $j$ and all letters derived from it. Hence, the range of interaction cannot exceed $N$ letters, independently of the number of derivation steps. As a result, the production power of the system is significantly reduced and the language produced by such deterministic systems is shown to be EOL.

This paper deals with (mainly deterministic) L systems (DIL) having the above property and called R-bounded systems. $R$ is an integer constant of the system used as an alternative to the number $N$, and defined later, within the definition of a more convenient for analysis, equivalent property (R-boundedness). The class of deterministic bounded systems is a large subclass of DIL systems reducible constructively to EOL systems producing the same languages. This reduction can be used to show that some properties undecidable in the class of DIL systems are decidable within this subclass.
The nondeterministic case is studied in [Raz (1983)].

The idea of bounding the interaction in IL systems is studied also in [Cullik, Karhumäki (1979)]. Two types of propagating \( <1,1> \) systems are introduced here:

1) s-G2L systems which are a special case of bounded IL systems 
(5-BG \( <1,1> \) in our notations, where G stands for "growing").

2) e-G2L in which the interaction is bounded by two mechanisms. The first one is a fast local growth as in bounded systems. The second one is by introducing "context-free" rules, i.e. rules in which the rewriting is an invariant of the context.

The main results there are based on the fact that the languages of these systems can be described by L systems without interactions like in the present work. However, the kind of systems and the technics are different and hence some of the results are similar but not the same.

The full extent of the relationship between the e-G2L class and the bounded IL class has not been studied yet.

Chapter 2 of this paper represents notations and definitions. Chapter 3 includes results characterizing R-bounded systems. Chapter 4 introduces applications of these results. Appendix A contains an example of a \( 4 \)-bounded DIL system. The proofs of theorems appear in Appendix B.

Some of the results represented here appeared without proofs in [Raz (1978)]. This paper is a revision of [Raz (1980)].

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2. DEFINITIONS AND NOTATIONS

Definition 1: A \( <k, \lambda> \) system, \( k, \lambda \geq 0 \), is a structure
\[ G = \langle \Sigma, P, w \rangle \]
where \( \Sigma \) is a finite alphabet.
\( P \) is a set of production rules each with the following structure:
\[ (\alpha, \beta) \rightarrow \gamma \]
where \( \alpha \in \Sigma^* \), 
\[ \alpha \in \Sigma^i \Sigma^j \quad i+j = k \]
\[ \beta \in \Sigma^m \gamma^n \quad m+n = \lambda \]
\[ \gamma \in \Sigma^* \]
and \( \gamma \) is a symbol not in \( \Sigma \).

For each such \( \alpha \beta \in \bigcup \Sigma^i \Sigma^j \times \Sigma \times \Sigma^m \gamma^n \) there is at least
\[ i+j = k \]
\[ m+n = \lambda \]
one rule in \( P \).
\[ \beta \in \Sigma^* \] is the initial word or axiom.

Definition 2:
1) An IL system, a system with interactions, is a \( <k, \lambda> \) system where \( k+\lambda > 0 \).
2) An OL system, a system without interactions, is a \( <k, \lambda> \) system where \( k = \lambda = 0 \).

Remark: In the case of OL systems, the \( \alpha \) and \( \beta \) of definition 1 are empty strings. For this reason we will use the simpler notation \( \alpha \beta \gamma \) (rather than \( (\alpha, \beta) \rightarrow \gamma \)) when dealing with OL systems.

Definition 3: Let \( w \in \Sigma^* \), \( w = a_1 a_2 \ldots a_m \), \( a_i \in \Sigma \) \( i = 1, 2 \ldots m \).
1) The length of \( w \) is \( m \) and is denoted \( |w| \).
2) The length- \( k \) prefix of \( w \) is defined and denoted by \( \text{PRE}_k(w) = a_1 a_2 \ldots a_k \).
3) The length- \( k \) suffix of \( w \) is defined and denoted by \( \text{SUF}_k(w) = a_{m-k+1} \ldots a_m \).
Definition 4: Let \( G \) be a \( <k, \lambda> \) system. The relation \( (\alpha) a_1' a_2' \ldots ' a_r(\beta) \rightarrow_{\lambda} \alpha_1' \alpha_2' \ldots ' \alpha_r, \ r \geq 1 \), is a generalized rule of \( G \) iff

\[
\begin{align*}
(\alpha) a_1 (\text{PRE}_k(a_2, a_3, \ldots, a_r, \beta)) & \rightarrow \alpha_1 \\
& \vdots \\
(\text{SUF}_k(\alpha a_1 a_2 \ldots a_{i-1})) a_i (\text{PRE}_k(a_{i+1}, \ldots, a_r, \beta)) & \rightarrow \alpha_i, \quad i = 2, 3, \ldots, r-1 \\
& \vdots \\
(\text{SUF}_k(\alpha a_1 \ldots a_{r-1})) a_r(\beta) & \rightarrow \alpha_r
\end{align*}
\]

are production rules of \( G \).

Remark: Sometime we shall drop some pairs of corresponding apostrophes from both sides of the relation \( \rightarrow \) (at the cost of losing information of course) and denote the above generalized rule as

\[
(\alpha) \beta_1' \beta_2' \cdots ' \beta_t(\beta) \rightarrow \gamma_1' \gamma_2' \cdots ' \gamma_t, \quad 1 \leq t \leq r
\]

where

\[
\begin{align*}
\beta_1 & = a_1 a_2 \ldots a_j \\
\vdots & \vdots \\
\beta_i & = a_i a_{i+1} \ldots a_j \\
\gamma_1 & = \alpha_1 a_2 \cdots \cdots \alpha_j \\
\vdots & \vdots \\
\gamma_i & = \alpha_j a_{j+1} \cdots \cdots a_{j+1} \\
& i = 2, \ldots, t
\end{align*}
\]

and \( j_t = r \).

Definition 5: The language \( L(G) \) of a \( <k, \lambda> \) system \( G \) is defined recursively as follows:

1) The initial word \( \omega \in L(G) \).

2) \( w_1 \in L(G) \) and \( (g^k) w_1 (g^\lambda) \rightarrow w_2 \) implies \( w_2 \in L(g) \).

When the left hand side of the implication in 2) holds, we say that \( w_1 \) directly derives \( w_2 \) in \( G \). Notation: \( w_1 \rightarrow^* w_2 \).

Let \( \rightarrow^* \) be the reflexive-transitive closure of \( \rightarrow \). When \( w_1 \rightarrow^* w_2 \), we say that \( w_1 \) derives \( w_2 \) in \( G \).
Definition 6: Let \( \alpha_1 a_2 \ldots a_r(\beta) \rightarrow \alpha'_2 \alpha'_2 \ldots \alpha'_r \) be a generalized rule of \( G \) and let \( \alpha a_1 \ldots a_r \) be a subword of a word in \( g^k \triangleleft L(G) g^k \), then \( \alpha a_1 \ldots a_r \) produces \( \alpha'_2 \alpha'_2 \ldots \alpha'_r \) in \( G \).

Definition 7: A \( \langle k, l \rangle \geq L \) system \( G \) is \( R \)-bounded (or \( k, l \)-system) if every subword with length \( R-1 \) of a word in \( g^k \triangleleft L(G) g^k \) produces only words of length \( R-1 \) or longer. It is bounded if it is \( R \)-bounded for some \( R \).

Remarks:
1) It is clear by definition 6 that \( R > k + l + 1 \).
2) It can be verified easily that the property described in chapter 1 exists in an \( IL \) system iff the system is \( R \)-bounded for some \( R \).

Definition 8: An active subword in a \( \langle k, l \rangle \geq L \) system \( G \) is a subword of a word in \( g^k \triangleleft L(G) g^k \).

Definition 9: An active rule in an \( IL \) system \( G \) is a production rule which acts at least once in a derivation of a word in \( L(G) \).

Definition 10: The \( R \)-shifts of a word \( w = a_1 a_2 \ldots a_m \in \Sigma^* \) is the word \( S_R(w) \in [[\Sigma^R]]^* \) defined as follows:

\[
S_R(w) = \begin{cases} 
[a_1 a_2 \ldots a_R] [a_2 a_3 \ldots a_{R+1}] \ldots [a_{R+1} \ldots a_{R+1}] \ldots & \text{if } m \geq R \\
\epsilon & \text{if } m < R \quad (\epsilon \text{ is the empty word}).
\end{cases}
\]
Definition 11: The OL system $G'$ associated with an $R$-B $< k, \ell > L$ system $G$ is defined as follows: Let $G = \langle \Sigma, \mathcal{P}, \omega \rangle$. Then $G' = \langle \Sigma', \mathcal{P}', \omega' \rangle$ where:

1. $\Sigma' = \{[w] | w \text{ is an active subword of } G \text{ with length } R\}$
2. Let $[w] \in \Sigma'$, $w = \alpha \beta \gamma$, $|\alpha| = k$, $|\gamma| = \ell$, $\alpha \in \Sigma$, and let $\beta a(\gamma) \rightarrow \beta_1 \beta_2 \alpha_1$ be a generalized rule of $G$, where $|\beta_2| = R - 1$. (The existence of such $\beta_2$ is guaranteed by $G$ being $R$-bounded. See definition 7.)

Define

$$
\delta_w = \begin{cases} 
g^k \beta_1 \beta_2 \alpha_1 g^\ell & \text{if } \alpha = g^k \gamma = g^\ell \\
g^k \beta_1 \beta_2 \alpha_1 & \text{if } \alpha = g^k \gamma \neq g^\ell \\
\beta_2 \alpha_1 g^\ell & \text{if } \alpha \neq g^k \gamma = g^\ell \\
\beta_2 \alpha_1 & \text{if } \alpha \neq g^k \gamma \neq g^\ell 
\end{cases}
$$

Then $P' = \{[w] \rightarrow S_R(\delta_w) | [w] \in \Sigma'\}$.

3. $\omega' = S_R(g^k \omega g^\ell)$.

Remark: If $|g^k \omega g^\ell| < R$ then $S_R(g^k \omega g^\ell) = \varepsilon$. In this case we should take as axioms for $G'$ all the words $S_R(w_i)$, $i = 1, 2, \ldots, p$ where $w_i \in g^k L(G)g^\ell$, $|w_i| \geq R$, and $w_i$ is produced in $G$ from a word smaller than $R$. The number of such words ($p$) is finite, and $G'$ is a FOL system (a system with a finite number of axioms).

Definition 12: A $< k, \ell > L$ system $G$ is deterministic ($D < k, \ell > L$) if and only if $(\alpha)a(\beta) \rightarrow \gamma_1$, $(\alpha)a(\beta) \rightarrow \gamma_2$ being production rules in $G$ implying $\gamma_1 = \gamma_2$. 
3. RESULTS

The extent of the subclass of bounded systems is demonstrated by the following theorem:

**Theorem 1:** A sufficient condition for a \( \langle k, l \rangle \) system, \( k \geq 1 \), to be R-bounded for every \( R \geq 2(k+1)+1 \) is that every active rule \((a)\alpha(B) \rightarrow \gamma\) has \( |\gamma| > 1\).

**Remark:** The condition above is not necessary for R-boundedness. See the example in Appendix A.

**Theorem 2:** An R-\( \langle k, l \rangle \) system is an R'-B\( \langle k, l \rangle \) system for \( R' = 2R-(k+1) > R \) and hence for infinitely many values of \( R' \).

**Theorem 3:**
1) It is decidable whether an IL system is R-bounded for a given \( R \).
2) All active subwords of length not greater than \( R \) of an R-bounded IL system can be found (and hence all its active rules).

However, because of the strong computing power of IL systems (identical to the power of Turing machines - see [Herman, Rozenberg (1975b)]) we have the following result:

**Theorem 4:** It is undecidable for a given DIL system whether there exists an integer \( R \) such that the system is R-bounded.

An immediate result from the definition of an associated system (definition 11) is the following:

**Theorem 5:** The associated system of a deterministic R-BIL system (R-BDIL) is deterministic (DOL).

The relationship between the languages of an R-bounded DIL system and its associated system is described by the following two theorems:

**Theorem 6:** Let \( G \) be a deterministic R-\( \langle k, l \rangle \) system (R-BD \( \langle k, l \rangle \)) and let \( G' \) be its associated system, then

\[
L(G') = \{ S_R(g, w^2) | w \in L(G) \}.
\]
Remark: In the nondeterministic case \( L(G') \) contains also additional words not in the form \( S_R(g^k w g^k) \).

**Theorem 7:** Let \( G \) be an \( R-\text{BD}<k,\lambda> \) system and let \( G' \) be its associated system. Then there is a homomorphism \( h \) such that
\[
L(G) = h(L(G'))
\]
where \( h \) is defined as follows:

For any \( x \in \Sigma' \), \( x = [\alpha \beta \gamma], |\alpha| = k; |\gamma| = 2; a \in \Sigma \)
\[
h(x) = \begin{cases} 
\alpha & \text{if } \alpha \neq g^k \\
\beta a & \text{if } \alpha = g^k 
\end{cases}
\]
(see the notations of definition 11).

A homomorphism \( h \) is a *coding* if \( |h(a)| = 1 \) for each letter \( a \) in the domain.
By a small modification of the associated system \( G' \) we can create a new system \( G'' \) with the following properties:

**Theorem 8:** Let \( G \) be an \( R \)-bounded DIL system, then a DOL system \( G'' \) can be built such that
\[
L(G) = C(L(G''))
\]
where \( C \) is a coding.

**Corollary:** \( L(G) \) is an EOL language (a language produced by an extended DOL system; see [Ehrenfeucht, Rozenberg (1974)].)
Remark:

If $g^k L(G)g^l$ contains words smaller than $R$, these words are not expressed in the associated system (see remark for definition 11). In this case we should exclude these words from $L(G)$ in theorems 6,7,8, by replacing $L(G)$ by $L(G) - \{w \mid |g^k w g^l| < R\}$. However, as the number of these words is finite, and as the class EOL is closed under union (see (Rozenberg (1974))), the corollary of theorem 8 holds even in this case.
4. APPLICATIONS

Using theorem 6 and the decidability of the equivalence problem for DOL systems (see [Culik, Fris (1976)]) we get:

**Theorem 9**: The equivalence problem for R-BD < k,ε> L systems is decidable (systems with the same R,k,ε).

This problem is undecidable for DIL systems [Vitanyi (1974)].

**Remark**: Theorem 9 can be extended to the general bounded DIL class.

Any word in the language L(G) of a DIL system G directly derives exactly one word. Hence the system defines an infinite sequence of words:

\[ w = W_0 \xrightarrow{G} W_1 \xrightarrow{G} \ldots \xrightarrow{G} W_n \xrightarrow{G} \ldots \]

The sequence of lengths \(|W_0|, |W_1|, \ldots, |W_n|, \ldots\) defines the growth function of G, \(f_G(n)\). In the class of DIL systems the type of \(f_G(n)\) (e.g., logarithmic, polynomial, exponential) is undecidable (see [Vitanyi (1974)]). In the case of R-BDIL systems it is obvious that the growth is exponential, but in addition we can derive an explicit expression for \(f_G(n)\). In the case of DOL systems the Parikh vector \(\Pi(n)\) for \(W_n\) (a vector such that each of its coordinates is the number of occurrences in \(W_n\) of a specific letter from the alphabet) can be computed by the formula

\[ \Pi(n) = \Pi(0) A^n \]

where \(A\) is a (constant) transition matrix from \(\Pi(i)\) to \(\Pi(i+1)\). Hence, the growth function is defined by

\[ f(n) = \Pi(0) A^n n \]

where \(n = (1,1,\ldots,1)^T\). (see [Herman, Rozenberg (1975a)])

Using theorems 5, 6, 7 we get:
Theorem 10: Let G be an R-BD $<k, \lambda>_{L}$ system and let $G'$ be its associated one. Then

1) The Parikh vector of $w_{n}$, the n-th word derived by G, is

$$\pi_{G}(n) = \pi_{G'}(0) A_{G'}^{n} B$$

where $\pi_{G'}(0)$ is the Parikh vector of $w_0 = \omega$ the initial word of $G'$.

$A_{G'}$ is the transition matrix of $G'$.

B is a transformation matrix from Parikh vectors of words in $L(G')$ to Parikh vectors of words in $L(G)$, and defined as follows:

Let $x \in \Sigma', x = [\alpha \beta \gamma]_{\alpha, \beta, \gamma} = k, |\gamma| = \lambda, a \in \Sigma$.

Then for any $a' \in \Sigma$:

If $\alpha \neq g^k$ then $B_{xa} = 1$

If $\alpha = g^k$ then $B_{xa} = 0$ if $a' \neq a$.

If $\alpha = g^k$ then $B_{xa} = \text{number of occurrences of } a' \text{ in } \beta a$.

(\text{B}_{xa} \text{ is the element of B with indices } x, a.)

(2) The growth function of G is:

$$f_{G}(n) = \pi_{G'}(0) A_{G'}^{n} \eta + R - (k+\lambda+1)$$

where $\eta = (1, 1, \ldots, 1)^T$.

(see an example in Appendix A).
REFERENCES


__________, (1975b), ____________, Chapter 17.


APPENDIX A

AN EXAMPLE: A 4-BD <1,0> L SYSTEM

Let $G = \langle \Sigma, P, \omega \rangle$ be a D <1,0> L system where $\Sigma = \{a, b\}, \omega = a a a b$.

Suppose the $P$ includes the following production rules:

\begin{align*}
(g) a & \rightarrow a a a b \\
(b) a & \rightarrow a a a \\
(a) a & \rightarrow a a b \\
(a) b & \rightarrow \epsilon \quad (\epsilon \text{ is the empty word})
\end{align*}

Using these rules the first words in $g^1(G)g^0$ will be:

- $g^0 = \epsilon$
- $g^1 = \epsilon a a a b$
- $g^2 = \epsilon a a a b a a a b$
- \hspace{1cm} \vdots

By the algorithm described in the proof of theorem 3 for $R = 4$ it follows that the above rules are the only active rules (and hence it does not matter what other rules are chosen) and that $G$ is 4-BD <1,0> L system.

The associated DOL system $G'$ is constructed as follows:

The active subwords of length 4, $w$ (for letters in $\Sigma'$), the appropriate generalized rule in $G$ and $\delta_w$ for each are:
The initial word of G' is \( \omega' = S_4(\omega) = [g a a a] [a a a b]. \)

The production rules in G', \([w] \rightarrow S_4(\delta_w)\) are:

1. \([g a a a] \rightarrow [g a a a] [a a a b] [a a a b] [a b a a] [b a a b] [a a b a] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b]
2. \([a a a a] \rightarrow [a a b a] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b]
3. \([a a a b] \rightarrow \epsilon\)
4. \([a a b a] \rightarrow [a a b a] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b]
5. \([a b a a] \rightarrow [a a a a] [a a a b] [a a a b] [a b a a] [b a a b]
6. \([b a a a] \rightarrow [a a b a] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b] [a a a b] [a b a a] [b a a b]
7. \([b a a b] \rightarrow \epsilon\)

Keeping the above order of letters in \(\Sigma'\) we have:

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
3 & 0 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A'_{G'} = \begin{bmatrix}
1 & 0 & 1 & 2 & 2 & 0 & 2 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ \Pi_{G'}(0) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad f_{G'}(0) = 2 + 4 - 2 = 4 \]

\[ \Pi_{G'}(0) A_{G'} = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 2 \end{bmatrix} \quad f_{G'}(1) = 8 + 4 - 2 = 10 \]

\[ \Pi_{G'}(0) A_{G'}^2 = \begin{bmatrix} 1 & 4 & 3 & 4 & 4 & 2 \end{bmatrix} \quad f_{G'}(2) = 20 + 4 - 2 = 22 \]
APPENDIX B

PROOFS OF THE THEOREMS

Proof of Theorem 1

Let G be the system and let \( w = a_1 a_2 \ldots a_{R-1} \) be any subword with length \( R-1 \) of a word in \( g^L(G)g^L \). Suppose

\[
(a_1 a_2 \ldots a_k) a_{k+1} a_{k+2} \ldots a_{R-1} = (a_{R-k} \ldots a_{R-1}) \alpha_{k+1} \alpha_{k+2} \ldots \alpha_{R-1}
\]

Since \( |\alpha_i| \geq 2 \) for \( i = k+1, \ldots, R-1 \), it follows that \( |\alpha_{k+1} \ldots \alpha_{R-1}| \geq 2 \cdot (R-1-k) \).

In order to insure that \( w \) will produce in G a word with a length not smaller than \( R-1 \) it is necessary that

\[
2 \cdot (R-1-k) \geq R-1
\]

Hence, for \( R \geq 2(k+1) + 1 \) this condition is fulfilled.

Proof of Theorem 2

Suppose that G is \( R-B < k, \ell < L \) system.

Let \( w = a_1 a_2 \ldots a_{2R-k-\ell-2} \) be a subword (with a length \( 2R-k-\ell-2 \)) of a word in \( g^L(G)g^L \).

Note that

\[
\text{SUF}_{R-1}(w) = a_{R-k-\ell} \ldots a_{2R-k-\ell-2}
\]

\[
\text{PRE}_k(\text{SUF}_{R-1}(w)) = a_{R-k-\ell} \ldots a_{R-1}
\]

\[
\text{SUF}_\ell(w) = a_{2R-k-2\ell-1} \ldots a_{2R-k-\ell-2}
\]
Hence there are words \( \alpha_{k+1} \cdots \alpha_{k+2} \cdots \alpha_{2R-k-2} \) such that

\[
(a_1 \cdots a_k) \alpha_{k+1} \cdots \alpha_{R-\ell} \alpha_{R-\ell-1} \alpha_{R-\ell-2} \cdots \alpha_{2R-k-2} \alpha_{2R-k-2} \cdots \alpha_{2R-k-2}
\]

Since \( G \) is \( R \)-bounded

\[
|\alpha_{k+1} \cdots \alpha_{R-\ell-1}| \geq R-1, \quad |\alpha_{R-\ell} \cdots \alpha_{2R-k-2}| \geq R-1
\]

Hence

\[
|\alpha_{k+1} \cdots \alpha_{2R-k-2}| \geq 2R-2 > 2R-k-2
\]

Hence \( G \) is \( R' \)-bounded for \( R' = 2R-k-1 \).

Remark: For propagating systems (PIL-systems without erasing) the theorem holds for \( R'^* = R+1 \).

Lemma 1: Suppose that \( x \Rightarrow^* y \) for some \( k, \ell > 1 \) system \( G \), \( k+\ell > 0 \), and that any subword with length \( R-1 \) in \( g^k \times g^\ell \) produces a subword of \( y \) with a length not smaller than \( R-1 \). Then any subword of \( y \) with a length \( R \) is a subword of a subword in \( y \) produced by a subword of \( g^k \times g^\ell \) not greater than \( R \).

Proof: Let \( \beta_1 \beta_2 \) be any subword of \( y \) such that

1. \( |\beta_1 \beta_2| = R \)
2. \( |\beta_1| = 1 \), \( |\beta_2| \geq 1 \)

and choose \( \beta_1, \beta_2 \) so that using the production rules for \( x \Rightarrow^* y \) we have

\[
(\delta_1) a_1 a_2 (\delta_2) \Rightarrow \gamma_1 \beta_1 \beta_2 \gamma_2, \quad a_1, a_2 \in \Sigma, \quad \gamma_1, \gamma_2 \in \Sigma^*
\]

where \( w = \delta_1 a_1 a_2 \delta_2 \) is the subword of \( x \) producing the subword of \( y \)

\( \gamma_1 \beta_1 \beta_2 \gamma_2 \) (a specified occurrence of this subword). Clearly, \( w \) is the smallest subword of \( x \) producing a subword of \( y \) containing this occurrence of \( \beta_1 \beta_2 \) as a subword. (Using the same production rules of \( G \) for the same letters of \( w \) and taking a proper subword of \( w \) we get produced a subword of \( \gamma_1 \beta_1 \beta \) or of \( \beta_2 \gamma_2 \).)
Suppose that $|w| > R$ or equivalently $|\alpha| \geq R-k-\delta-1$ or equivalently

$$|\text{SUF}_k(\delta_1 \delta_{\alpha}) \alpha \text{PRE}_2 (a_2 \delta_2)| \geq R-1$$

Hence by the assumption of the lemma

$$|\beta| \geq R-1$$

implying by using (2) that $|\beta_1 \beta_2| \geq R+1$ which contradicts (1). Hence $|w| \leq R$.

---

Proof of Theorem 3

Let $G$ be a $<k,\ell>l$ system, $G = \langle \Sigma, P, \omega \rangle$.

1. Build the set $A_R$ as follows:

   If $|g^k \omega g^\ell| > R$ then $A_R \neq \{\omega\}$. 
If not let $A_R$ be the set of words $w$, such that $|g^k w g^l| > R$, and such that there is a sequence of direct derivations $\bar{w} = w_0 \Rightarrow w_1 \Rightarrow w_2 \cdots \Rightarrow w_n = w$ with $|g^k w_i g^l| < R$ for $i = 0, 1, \ldots, n-1$.

$A_R$ can be built by the following algorithm using a stack and additional set $B_R$ both empty at the beginning.

```
begin
  add $w$ to $B_R$;
  push $w$ onto STACK;
  while STACK is not empty do begin
    pop $x$ from STACK;
    for each $y$ such that $x + y \Rightarrow G$ do
      if $|g^k y g^l| > R$ then add $y$ to $A_R$
      else if $y$ is not in $B_R$ then begin
        add $y$ to $B_R$;
        push $y$ onto STACK
      end
  end

(2) Suppose that $|\Sigma| = m$.

Produce all words derived from $A_R$ in $(m+1)^R$ direct derivations or less.
Examine all generalized rules with the following structure taking part in these derivations:

$\alpha \beta \gamma \Rightarrow \delta$ where $|\alpha| = k$, $|\beta| = R-k-\lambda-1$, $|\gamma| = \lambda$ ($|\alpha \beta \gamma| = R-1$).

If one of them fails to fulfill $|\delta| > R-1$ then $G$ is not $R$-bounded by definition.

If for all these generalized rules $|\delta| > R-1$, $G$ is $R$-bounded by the following argument:
Let $F(n)$ be the set of different subwords with length $R$ of words $g^k w g^l$ such that $w$ is derived from a word in $A_R$ in not more than $n$ direct derivations, $n = 0, 1, 2, \ldots$.

Obviously

$$F(0) \subseteq F(1) \subseteq F(2) \subseteq \ldots$$

It follows from lemma 1 and from the condition $|\delta| > R-1$ above that all words in $F(n)$ (or subwords received by erasing all occurrences of $g$ in these words) are produced from words in $F(n-1)$, $n = 1, 2, \ldots (m+1)^R$.

Hence, if for some specific $i$

$$F(i) = F(i+1), \quad 0 \leq i \leq (m+1)^R$$

then

$$F(i) = F(i+j) \text{ for any } j > 0.$$ 

As the number of different subwords of length $R$ is at most $(m+1)^R$ ($m+1$ letters in $\Sigma \cup \{g\}$), (*) should hold for some $i$.

Hence $F(i)$ includes all active subwords with length $R$ of $G$, and only rules participating in the generalized rules found above will be active.

**Proof of Theorem 4**

The proof is by a reduction to the halting problem for Turing machines which start with an empty tape.

1) For any given Turing machine $TR$ which starts with an empty tape build a $D<1,1> L$ system $G'$ with an axiom of length 1 simulating $TR$. (For details see [Herman, Rozenberg (1975b)]).

2) Define $G'' = \langle \{a\}, \{(g)a(g) \rightarrow a a, (g)a(a) \rightarrow a a, (a)a(g) \rightarrow a a, (a)a(a) \rightarrow a a, \ldots \rangle$.

   By theorem 1 $G''$ is a $R$-BD $<1,1> L$ system for every $R \geq 5$.

3) Build the $D<1,1> L$ system $G$ as follows:

   $G$ simulates $G'$ until $G'$ indicates that $TR$ has stopped by a special letter appearing in the current word. This letter triggers an erasing of the word, letter by letter, until one letter is left. Then a special production rule triggers a development identical to that of $G''$. 

4) If TR is not halting the growth of the words of G is at most linear since a word's length at every step is identical to the length of TR's tape. Hence, in this case G is not R-bounded for any R as an R-bounded system has an exponential growth. (It follows from its definition.) If TR halts then the number of words produced by G before starting to simulate G" is finite. Hence the lengths of these words are bounded by (the smallest bound) Q. Let \( R = \text{Max}(5, Q+2) \). As G" is R-bounded it follows that G is R-bounded, because all words of \( g \in L(G) \) are produced while simulating G".

Hence, G is R-bounded for some R if and only if TR halts.

Lemma 2: Let \( \beta_1, \ldots, \beta_n \in \Sigma^* \) such that

\[
| \prod_{j=i}^{i+p-1} \beta_j | \geq R-1, \quad i = 2, 3, \ldots, n-p.
\]

Then

\[
(*) \quad S_R(\prod_{i=1}^{n} \beta_i) = S_R(\prod_{j=1}^{p} \beta_j)^{p+1} \prod_{i=2}^{n-p} S_R(S_{R-1}(\prod_{j=1}^{i+p-1} \beta_j)^{p+1})
\]

\[
(\prod_{i=1}^{n} \beta_i \text{ stands for } \beta_1 \beta_2 \ldots \beta_n).
\]

Proof: Let us first prove

(1) \[ S_R(\gamma_1 \gamma_2 \gamma_3) = S_R(\gamma_1 \gamma_2) \cdot S_R(S_{R-1}(\gamma_2) \gamma_3), \]

for \( \gamma_i \in \Sigma^* \) ; \( i = 1, 2, 3 \), \( |\gamma_2| \geq R-1 \).

Suppose \( \gamma_1 = a_1 \cdots a_r \), \( r \geq 0 \)

\( \gamma_2 = b_1 \cdots b_s \), \( s \geq R-1 \)

\( \gamma_3 = c_1 \cdots c_t \), \( t \geq 0 \)

\( a_i, b_i, c_i \in \Sigma \).
\[ S_R(\gamma_1 \gamma_2) = [a_1 \ldots a_r] [a_{r+R+1} \ldots a_r] [a_{r+R+2} \ldots a_r b_1] \ldots [a_{r+R+2} \ldots a_r b_1] [b_1 \ldots b_R] \ldots [b_{S-R+1} \ldots b_S] \]

\[ S_R(SUF_{R-1}(\gamma_2) \gamma_3) = [b_{S-R+2} \ldots b_s c_1] \ldots [b_s c_1 \ldots c_{R-1}] [c_1 \ldots c_R] \ldots [c_{t-R+1} \ldots c_t] \]

Hence (1) holds.

The formula (*) in the lemma is proved by induction on \( n \).

Basis: for \( n = p+2 \), (*) will be

\[ (2) \quad S_R(\prod_{i=1}^{p+2} \beta_i) = S_R(\prod_{j=1}^{p} \beta_j \beta_{p+1}) S_R(SUF_{R-1}(\prod_{j=2}^{p+1} \beta_j) \beta_{p+2}) \]

and is proved by substituting in (1)

\[ \gamma_1 = \beta_1, \quad \gamma_2 = \prod_{j=2}^{p+1} \beta_j, \quad \gamma_3 = \beta_{p+2} \]

Now, assuming that (*) holds for \( n \), we shall prove it for \( n+1 \).

\[ S_R(\prod_{i=1}^{p} \beta_i \beta_{p+1}) \prod_{i=2}^{n-p} (\prod_{j=1}^{i+p-1} \beta_j) = S_R(SUF_{R-1}(\prod_{j=1}^{i+p-1} \beta_j) \beta_{p+i}) \]

= \[ [S_R(\prod_{j=1}^{p} \beta_j \beta_{p+1}) \prod_{i=2}^{n-p} S_R(SUF_{R-1}(\prod_{j=1}^{i+p-1} \beta_j) \beta_{p+i})] S_R(SUF_{R-1}(\prod_{j=1}^{n} \beta_j) \beta_{n+1}) \]

= \[ S_R(\prod_{i=1}^{p} \beta_i) S_R(SUF_{R-1}(\prod_{j=n-p+1}^{n} \beta_j) \beta_{n+1}) \]

= \[ S_R(\prod_{i=1}^{n-p} \beta_i) (\prod_{i=n-p+1}^{n} \beta_i) \beta_{n+1} = S_R(\prod_{i=1}^{n} \beta_i) \]

and is proved by (*).
Proof of Theorem 6

Since $G'$ is deterministic (Theorem 5) it is sufficient to prove

(1) \[ w_1 \xrightarrow{G} w_2 \implies S_R(g^{k}w_1g^l) \xrightarrow{G} S_R(g^{k}w_2g^l). \]

Let $g^{k}w_1g^l = a_1a_2\ldots a_m$.

(We can assume $m \geq R$. See remarks for definition 11.)

$w_1 \xrightarrow{G} w_2$ iff there exist words $\alpha_i \in \Sigma^*$, $i = k+1, k+2, \ldots, m-l$ such that

(2) \[ w_2 = \alpha_{k+1}^{l} \alpha_{k+2}^{l} \ldots \alpha_{m-l}^{l}, \quad \text{and} \]

(3) \[ (a_i, a_{i+1}, \ldots, a_{i+k-1})a_{i+k}^{l} \ldots a_{i+m-l}^{l}(a_{i+m-l+1}^{l} \ldots a_{i+R-1}) \xrightarrow{G} \alpha_{i+k}^{l} \ldots \alpha_{i+m-l}^{l} \]

By definition 11 if (3) holds the following are production rules in $G'$.

(4) \[
\begin{align*}
[a_1a_2\ldots a_R] & \xrightarrow{S_R} S_R(g^{k}a_{k+1}^{l} \ldots a_{R-1}^{l}g^l) & \text{if } m = R, \\
[a_1a_2\ldots a_R] & \xrightarrow{S_R} S_R(g^{k}a_{k+1}^{l} \ldots a_{R-1}^{l}) & \\
& \vdots & \\
[a_1a_1+1\ldots a_{i+R-1}] & \xrightarrow{S_R} S_R(SUF_{R-1}(a_{i+1}^{l} \ldots a_{i+R-2}^{l})a_{i+R-1}^{l}) & \text{if } m > R, \\
& \vdots & \\
& \vdots & \\
[a_{m-R+1}a_{m-R}^{l} \ldots a_m] & \xrightarrow{S_R} S_R(SUF_{R-1}(a_{m-R+1}^{l} \ldots a_{m-1}^{l}g^l)) & \\
\end{align*}
\]
Using (4) if \( m=R \) then (1) holds.

If \( m>R \) then (4) holds iff

\[
(5) \quad S_R(a_1 \ldots a_m) \succeq S_R\left(g^{k}\alpha_{k+1} \ldots \alpha_{R-1}\right) \prod_{i=2}^{m-R} S_R\left(S^{U\{F_1}\left(\alpha_{i+k} \ldots \alpha_{i+R-2}\right)\alpha_{i+R-1}\right) \left(\alpha_{m-R+1} \ldots \alpha_{m-\xi-1}\right)S_{\xi-1}^{R-1}\alpha_m g^\xi
\]

and by lemma 2, substituting \( n = m-k-\xi \),

\[
p = R-k-\xi-1
\]

\[
\beta_1 = g^{k}\alpha_{k+1}
\]

\[
\vdots
\]

\[
\beta_i = \alpha_{k+i}, \quad i = 2 \ldots n-1
\]

\[
\beta_n = \alpha_{m-\xi} g^\xi
\]

Hence (1) holds.

Proof of Theorem 7

By Theorem 5,

\[
L(G') = \{S_R(g^{k}w^\xi) | w \in L(G)\}
\]

Suppose that \( w \in L(G) \)

\[
w = a_1 a_2 \ldots a_m, \quad a_i \in \Sigma, \quad i = 1 \ldots m
\]

\[
S_R(g^{k}w^\xi) = [PRE_R(g^{k}w^\xi)][PRE_R(g^{k-1}w^\xi)] \ldots [PRE_R(a_1 a_2 \ldots a_m g^\xi)] \ldots
\]

\[
[PRE_R(a_1 \ldots a_m g^\xi)] \ldots [a_{m+\xi-R+1} \ldots a_m g^\xi]
\]
Using the definition of $h$

$$h(\text{PRE}_R(g^k \omega g^\ell)) = a_1 a_2 \ldots a_{R-k-1}$$

$$h(\text{PRE}_R(g^{k-i} \omega g^\ell)) = a_{i+R-k-\ell} \quad i = 1, 2, \ldots, k-1$$

$$h(\text{PRE}_R(a_1 a_{i+1} \ldots a_m g^\ell)) = a_{i+R-k-\ell-1} \quad i = 1, 2, \ldots, m+\ell-R+1$$

Hence

$$h(S_R(g^k \omega g^\ell)) = a_1 a_2 \ldots a_m = w$$

Proof of Theorem 8

Let $G = \langle \Sigma, P, \omega \rangle$ be an $R$-BD$<k, \ell> \ L$ system and let $G' = \langle \Sigma', P', \omega' \rangle$ be the associated one. $G'' = \langle \Sigma'', P'', \omega'' \rangle$ will be constructed as follows:

1. $\Sigma'' = \Sigma' \cup \Sigma$

2. Let $\omega' = S_R(g^k \omega g^\ell) = w_1 w_2 \ldots w_p$, $w_i \in \Sigma'$ $i = 1, \ldots, p$

where $w_1 = [g^k b_1 b_2 \ldots b_{R-k-\ell-1} a \gamma]$ $a, b_i \in \Sigma$ $i = 1, \ldots, R-k-\ell-1$.

Then $\omega'' = b_1 b_2 \ldots b_{R-k-\ell-1} w_1 w_2 \ldots w_p$.

3. Production rules for $G''$:

(i) If $[w] \in \Sigma'$ and $\text{PRE}_k(w) \neq g^k$ then the rule

$[w] \rightarrow S_R(\delta_w) \in P'$ will be in $P''$ as well.

(ii) If $[w] \in \Sigma'$ and $\text{PRE}_k(w) = g^k$ then for each rule

$[w] \rightarrow S_R(\delta_w) \in P'$

where $S_R(\delta_w) = w_1 w_2 \ldots w_p$, $w_i = [g^k b_1 b_2 \ldots b_{R-k-\ell-1} a \gamma]$

(with the notations as above) there will be a rule

$[w] \rightarrow b_1 b_2 \ldots b_{R-k-\ell-1} w_1 w_2 \ldots w_p \in P''$. 
(iii) For each \( a \in \Sigma \) there will be the rule

\[ a \rightarrow e \in P^n \]

It is easy to verify that

\[ w_1 w_2 \ldots w_p \in L(G') \iff b_1 b_2 \ldots b_{R-k-l-1} w_1 w_2 \ldots w_p \in L(G'') \]

where \( w_1 = [g^k b_1 b_2 \ldots b_{R-k-\ell-1} \gamma] \) by the above notations.

Hence, if we define a coding \( C \) as follows:

\[ C(w) = a \quad \text{if} \quad w = [a\beta\gamma] \in \Sigma^*, |\alpha| = k, |\gamma| = \ell \]

\[ C(a) = a \quad \text{if} \quad a \in \Sigma \]

it is clear that

\[ C(L(G'')) = h(L(G')) \]

when \( h \) is the homomorphism defined in theorem 6.

Hence, using theorem 6:

\[ C(L(G'')) = L(G) \]

Proof of Theorem 9:

Let \( G_1, G_2 \) be two \( R \)-BD \( < k, \ell > L \) systems. Construct the respective associated (DOL) systems \( G_1', G_2' \) and then apply the DOL equivalence algorithm to \( G_1', G_2' \). If \( G_1' \) and \( G_2' \) are equivalent, then \( G_1, G_2 \) are since

\[ L(G_1') = L(G_2') \iff h(L(G_1')) = h(L(G_2')) \iff L(G_1) = L(G_2) \]

by Theorem 7.

Remark:

If \( L(G_1) \) or \( L(G_2) \) contain words smaller than \( R \) then an additional equivalence test should be done for the (finite) sets of these words (see Remark in Definition 11).
Proof of Theorem 10:

This theorem is a result of Theorems 5, 6, 7 and an application of the homomorphism defined in Theorem 7. Each letter $x = [\alpha \beta \gamma]$ but the first one in any $w^n \in L(G')$ is mapped by $h$ to a single letter $"a"$ in $w_n \in L(G)$. The first letter $x$ where $x = [g^k \beta \gamma]$, $|\gamma| = \lambda$, in $w^n$ is mapped to $\beta \alpha$ which is the $R-k-\lambda$ prefix of $w_n$.

1) Each component in the Parikh vector of $w_n$, say the component of $a \in \Sigma$ can be calculated by

$$[\pi_G(n)]_a = \sum_{x \in \Sigma^*} [\pi_{G'}(n)]_x B_{xa}$$

where $B_{xa}$ represents the number of "$a"$ letters in $w_n$ donated by a single "$x"$ letter in $w^n$ through the homomorphism $h$.

$$B_{xa} = \begin{cases} 1 & \text{if } x = [\alpha \beta \gamma], \ a \neq g^k \\ 0 & \text{if } x = [\alpha \beta \gamma], \ a \neq g^k, a' \neq \alpha \\ \text{number of occurrences of } a \text{ in } \beta a' & \text{if } x = [g^k \beta \gamma] \end{cases}$$

Hence, in matrix notations

$$\pi_G(n) = \pi_{G'}(n)B = \pi_{G'}(0)\lambda_G^n, B$$

2) If the first letter in $w_n$ is $x = [g^k \beta \gamma]$, $|\gamma| = \lambda$ then

$$f_G(n) = |w_n| = |w'_n| - 1 + |\beta \alpha| = f_{G'}(n) - 1 + (R-k-\lambda)$$

$$= f_{G'}(n) + R - (k+\lambda+1) = \pi_{G'}(0)\lambda_G^n, n + R - (k+\lambda+1).$$