MINIMAL PATH-DISCONNECTED GRAPHS

by

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Abstract

It is conjectured ([Tho80] [Cyp80]) that if a graph $G$ is $k$-edge-connected with $k$ odd then between any $k$ pairs of vertices $(s_1,t_1), \ldots, (s_k,t_k)$ there exist $k$ edge-disjoint paths $\pi_1, \ldots, \pi_k$ ($\pi_i$ is a $s_i-t_i$ path). In this paper it is shown that if the conjecture is false then there exists a counterexample $G$ which is a regular graph with at most $8k$ vertices.
1. Introduction.

Let $G = G(V,E)$ be a graph and $s,t \in V(G)$ an $s-t$ path $\pi$, is a path in $G$ between the vertices $s$ and $t$. A graph $G$ is $\kappa_E(G)$ edge-connected if between any two vertices $s,t \in V(G)$ there are $\kappa_E$ edge-disjoint paths. This definition is equivalent to one in which $s,t$ are replaced by two sets $S,T \subseteq V(G)$ (Menger's Theorem). Similarly for two vertices (and sets) in $V(G)$ $\kappa_E(s,t;G)$ is the maximum number of edge-disjoint paths between $s$ and $t$ in $G$.

These definitions do not impose any relation between the endpoints of the paths. In order to relate the endpoints of those paths we introduce another definition which appeared in [IZB3]. Let $P = (s_1,t_1), \ldots, (s_l,t_l)$ be a sequence of vertices in a graph $G$ then $P$ is path-connected iff there are $l$ edge-disjoint paths $\pi_1, \ldots, \pi_l$ where $\pi_i$ is a $s_i-t_i$ path. Denote by $|P| = l$.

A graph $G$ is $k$-path-connected iff every sequence $P$ of $k$ pairs of vertices in $V(G)$ is path-connected. $\kappa_P(G)$ is the maximal path connectivity of $G$. In [Tho80] and [Cyp80] the following conjecture appears:

**Conjecture A.**

For any graph if $\kappa_E(G)$ is odd then $\kappa_P(G) = \kappa_E(G)$.

It is known ([Cyp80] [Tho80] [IZB3]) that the conjecture is false when the edge-connectivity is an even integer. In [Oka83] it is proved that: $\kappa_E(G) \geq 2m - 3$ entails $\kappa_P(G) \geq m$.

Also a weaker result $\kappa_E(G) \geq 5$ implies $\kappa_P(G) \geq 4$ appeared in [Eno83]. Furthermore, in [IZB3] it is proved that the conjecture holds for sequences $P$ which consist only of two distinct vertices.

In this paper the characteristics of a possible counterexample are analyzed. In section 2 we show that if the conjecture is false then there exists a
A regular graph contradicting the conjecture. The rest of the paper is devoted to analyzing the structure and size of a minimal regular counterexample $G$.

An initial attempt to prove the conjecture is by induction on the number of vertices. Choose a minimal edge-separating set $S$ and obtain the graph $G'$ by contracting all the vertices on one side of $S$ into a single vertex. From the paths in $G'$ one can try to construct $\kappa_{G'}(G)$ paths in $G$. One of the problems in this proof is that $S$ might separate only a single vertex such that $|V(G)| = |V(G')|$. Thus we are led to consider separating set $S$ in every side of which there are more than one vertex. It is proved in this paper that if such an edge-separating set exists in a minimal regular counterexample $(G,P)$ then it separates the vertices of $P$.

In section 3 we analyze the structure of some connected components of $G \setminus S$. Their number is bounded in section 4. This bound and the bound on the number of such minimal separating sets found in section 5 is used in section 6 to obtain a bound on the size of a minimal regular counterexample.

2. Regularity conditions.

Let $G(V,E)$ be a graph, $v \in V(G)$ a vertex and $e_1 = (v,v')$, $e_2 = (v,v'') \in E(G)$ two edges incident at $v$ then the graph $G_{v,e_1,e_2} = G(V(G),(E(G) \setminus \{e_1,e_2\}) \cup \{(v',v'')\})$ is called a lifting of $G$ at $v$. The following results are from [Ma78]:

**Theorem 2.1**

Let $G(V,E)$ be a graph and $v \in G(V,E)$ then there exists a lifting $G_{v,e_1,e_2}$ in $G$ at $v$ such that for any two vertices $s,t \neq v$ $\kappa_G(s,t;G) = \kappa_{G_{v,e_1,e_2}}(s,t;G_{v,e_1,e_2})$.

**Theorem 2.2**

Let $v \in V(G)$. If for any two vertices $s,t \in V(G)$ $s,t \neq v$ $\kappa_G(s,t;G) \geq k$ and $\deg(v) \geq k$ then $\kappa_G(G) \geq k$. 

Let
\[ \Delta(G) = \max \{ \deg(v) | v \in V(G) \} \]
\[ \delta(G) = \min \{ \deg(v) | v \in V(G) \}. \]
A graph \( G \) is \( k \) - semi-regular iff \( k = \delta(G) \leq \Delta(G) \leq k + 1 \). Using Theorem 2.1 and Theorem 2.2 the following result is obtained.

**Lemma 2.1**

If Conjecture A is valid for semi-regular graphs then it is valid for all graphs.

**Proof**

Let \( G \) be a graph which is not semi-regular and for which Conjecture A does not hold and assume \( |E(G)| \) is minimal. Let \( v \in V(G) \) \( \deg(v) > k + 1 \). By Theorem 2.1 there exists a lifting \( G_v^{i,e} \) which preserves the edge-connectivity between all pairs of vertices excluding \( v \). As \( \deg(v; G_v^{i,e}) = \deg(v; G) - 2k \). By Theorem 2.2 \( \kappa_p(G_v^{i,e}) = \kappa_p(G) \). Thus \( G_v^{i,e} \) is also a counterexample with fewer edges than \( G \).

Let \( \pi = (x_1, x_2, \ldots, x_r) \) be a path then denote by \( \pi[x_i, x_j] \) the subpath of \( \pi \) between the vertices \( x_i \) and \( x_j \). Let \( \sigma = (x_r, y_1, \ldots, y_l) \) be another path then the concatenation of the paths \( \pi \) and \( \sigma \) is the path \( \pi \sigma = (x_1, \ldots, x_r, y_1, \ldots, y_l) \).

**Lemma 2.2**

Let \( G \) be a \( k \) edge-connected graph \( (k > 1) \) \( v_1, v_2 \in V(G) \) and \( w_1 \in \Gamma(v_1) \) (a neighbor of \( v_1 \)). Then there are \( k \) edge-disjoint \( v_1 - v_2 \) paths \( \pi_1, \ldots, \pi_k \), at least one of which passes through the edge \( (v_1, w_1) \).

**Proof**

Assume no path of \( \pi_1, \ldots, \pi_k \) passes through \( (v_1, w_1) \).
FIGURE 1

Let $\tau$ be a $w_1 - v_2$ path in $G \setminus (v_1, w_1)$. Let $z$ be the first vertex in which $\tau$ intersects $\bigcup_{j=1}^{k} \pi_j$ ($z \neq v_2$) and assume this intersection occurs on $\pi_i$. Replace $\pi_i$ by $\pi_i' = (v_1, w_1) \tau (w_1, z) \pi_i [z, v_2]$. $\pi_i'$ is edge-disjoint of $\pi_1, \ldots, \pi_i-1, \pi_i+1, \ldots, \pi_k$ and passes through $(v_1, w_1)$.

**Theorem 2.3**

If Conjecture A holds for regular graphs then it holds for all graphs.

**Proof**

Assume Conjecture A is false. By Lemma 2.1 there exists a semi-regular graph $G$ which is a counterexample. Furthermore, assume that $G$ is not regular and $V(G)$ is minimal. Let $\kappa_G(G) = k$ (odd) $\Delta(G) = k + 1$. Let $p \in V(G)$ $\deg(p) = k + 1$ and $\Gamma_G(p) = \{p, v_1, \ldots, (p, v_{k+1})\}$. Denote by $K$ a $k + 1$ clique and $M$ a $\frac{k+1}{2}$ matching on the same set of vertices $(V(K) = \{p_1, \ldots, p_{k+1}\}, G_1 = (V(K), E(K) \setminus E(M))$. Replace $p$ in $G$ by $G_1$ and replace each edge $(p, v_i)$ by $(p_i, v_i)$.

I.e.

$$G' = G((V(G) \cup V(G_1)) \setminus \{p\}, (E(G) \setminus \Gamma_E(p)) \cup E(G_1) \cup \{(p_1, v_1), \ldots, (p_{k+1}, v_{k+1})\})$$
Obviously \( G = G' / G_1 \) (the contraction of \( G_1 \) in \( G \)). Therefore \( \kappa_E(G') \leq \kappa_E(G) = k \).

**Claim** \( \kappa_E(G') = \kappa_E(G) \).

**Proof of claim**

The claim is true if and only if for any two vertices \( q_1, q_2 \in V(G') \)
\[ \kappa_E(q_1, q_2; G') \geq \kappa_E(G) . \]

**Case 1 :** \( q_1, q_2 \notin V(G_1) \)

Contracting \( G_1 \) yields \( k \) \( q_1 - q_2 \) paths in \( G = G' / G_1 \) at most \( \left\lfloor \frac{k}{2} \right\rfloor \) of these paths pass through \( p = G_1 / G_1 \) and as \( \kappa_E(G_1) = k - 1 \) these paths can be extended also in \( G' \) (by [Oka] \( G_1 \) is \( \left\lfloor \frac{k}{2} \right\rfloor \) path-connected).

**Case 2 :** \( q_1 = p_1 \in V(G_1) \)

Let \( p, v_1 \in V(G) \) and \( (p_1, v_1) \in E(G') \) by Lemma 2.2 there are \( k \) edge-disjoint \( p - q_2 \) paths in \( G \) \( \pi_1, \ldots, \pi_k \) one of which, \( \pi_1 \), passes through \( (p, v_1) \). This path can be extended to \( p \) by adding the edge \( (p, v_1) \) and as \( \kappa_E(G_1) = k - 1 \) all the other \( k - 1 \) paths can also be continued to \( p_1 \).

**Case 3 :** \( q_1 = p_1, q_2 = p_2 \) are vertices of \( G_1 \).

As \( \kappa_E(G_1) = k - 1 \) there are \( k - 1 \) \( q_1 - q_2 \) paths in \( G_1 \) \( \pi_1, \ldots, \pi_{k-1} \). Let \( w \in V(G') \setminus V(G_1) \). By Lemma 2.2 there exists a \( p - w \) path \( \tau_1 \) passing through \( v_1 \) \((p_1, v_1) \in E(G')\) and a \( p - w \) path \( \tau_2 \) passing through \( v_2 \) \((p_2, v_2) \in E(G')\). Let \( z \) be the first intersection of \( \tau_1 \) and \( \tau_2 \) (they intersect at \( w \)). then \( \tau = \tau_1[p_1, z] \tau_2[z, p_2] \) is \( p_1 p_2 \) path edge-disjoint of \( \pi_1, \ldots, \pi_{k-1} \). Therefore \( \kappa_E(G') = \kappa_E(G) \). But \( \kappa_p(G') \leq \kappa_p(G) \) and the number of vertices of degree \( k + 1 \) in \( G' \) is less than in \( G \).
3. **Subgraphs in a minimal regular counterexample.**

A *minimal edge-separating set* \( S \subseteq E(G) \) in a graph \( G \) is a set of edges in \( E(G) \) such that \( |S| = \kappa_E(G) \) and \( G \setminus S = G_1 \cup G_2 \) is disconnected (\( G_1, G_2 \) are its two components).

**Lemma 3.1**

Let \( S \) be a minimal edge-separating set in a graph \( G \) and \( G_1 \) a component of \( G \setminus S \) then

\[
\kappa_E(G_1) \geq \left\lfloor \frac{\kappa_E(G) + 1}{2} \right\rfloor.
\]

**Proof**

Let \( p_1, p_2 \in V(G_1) \). In \( G \) there are \( \kappa_E(G) \) edge-disjoint \( p_1 - p_2 \) paths. Each \( p_1 - p_2 \) path which contains edges from \( S \) must contain at least 2 edges of \( S \). Therefore at most \( \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor \) \( p_1 - p_2 \) paths pass through \( S \). All other paths lie entirely in \( G_1 \).

The following two lemmas will also be useful.

**Lemma 3.2**

Let \( G \) be a graph \( |E(G)| \leq \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor \).\( p_1, p_2 \in V(G) \) then

\[
|E(p_1, p_2)| \leq \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor.
\]

**Proof**

Otherwise \( |E(p_1, p_2, G \setminus \{p_1, p_2\})| < \kappa_E(G) \).

**Lemma 3.3**

Let \( S \) be a minimal edge-separating set in a graph \( G \), \( G_1 \) a component of
\[ G \setminus S \text{ such that } |V(G_1)| > 1, p \in V(G_1) \text{ then} \]
\[ |\Gamma_G(p) \cap S| \leq \left[ \frac{1}{2} \kappa_G(G) \right]. \]

**Proof**

Assume the contrary and let \( q \in V(G_1) \) and \( q \neq p \). Then the number of \( p-q \) paths contained entirely in \( G_1 \) is at most \( \kappa_E(G) - \left[ \frac{1}{2} \kappa_E(G) \right] \). Assume the number of edges of \( q \) in \( G_1 \) is at least \( \frac{1}{2} \kappa_E(G) \) then the number of \( p-q \) paths passing outside \( G_1 \) is at most \( \frac{1}{2} |S| \). Therefore the connectivity of \( p \) and \( q \) is at most \( (\kappa_E(G) - \left[ \frac{1}{2} \kappa_E(G) \right]) + \frac{1}{2} |S| < \kappa_E(G) \).

Let \( k \) be an odd integer, \( G \) a \( k \)-regular graph and \( P \) a sequence of \( k \) pairs of vertices in \( G \). \( (G,P) \) is a counterexample if conjecture A does not hold for \( (G,P) \) i.e. \( \kappa_G(G) = k \) and \( P \) is not connected.

\( (G,P) \) is a **minimal regular counterexample** if

1. \( (G,P) \) is a counterexample.
2. \( k = \kappa_G(G) \) is the minimal odd integer for which there exists a counterexample,
3. \( G \) is the smallest regular graph satisfying (1) and (2).

An edge \( e = (w_1, w_2) \) is **reducible** if there exists two vertices \( v_1, v_2 \) distinct of \( w_1, w_2 \) such that \( \kappa_G(v_1, v_2; G) > \kappa_G(v_1, v_2; G \setminus e) \).

A minimal edge separating set \( S \) in a graph \( G \) is **reducible** if it contains a reducible edge, otherwise \( S \) is **irreducible**.
Lemma 3.4

Every irreducible edge in a minimal regular counterexample \((G,P)\) is incident to a vertex of \(P\).

Proof

Assume \(e=(v_1,v_2)\) is irreducible and \(v_1,v_2 \not\in V(P)\). Then for any two vertices \(p_1,p_2\) such that \(\{p_1,p_2\} \cap \{v_1,v_2\} = \emptyset\) \(\kappa_{\bar{e}}(p_1,p_2;G \setminus e) = \kappa_{\bar{e}}(G)\). As \(\deg(v_1;G \setminus e) = \deg(v_2;G \setminus e)\) is even Theorem 2.2 can be applied repeatedly until \(v_1, v_2\) remain isolated. At this point remove \(v_1, v_2\) and by Theorem 2.1 the connectivity of the resulting graph \(G'\) is \(\kappa_{\bar{e}}(G)\). As \(P\) remains disconnected in \(G'\) \((G,P)\) is not a minimal regular counterexample.

A lump \(L\) in a graph \(G\) is an induced subgraph \(L\) such that \(E(L,G \setminus L)\) is a reducible minimal edge-separating set and all edges of \(L\) are irreducible. Let \(G_1 \subset G\) then \(\mu_p(G_1)\) is the number of pairs \((v_1,v_2)\) of \(P\) such that either \(v_1\) or \(v_2\) belongs to \(G_1\) (if both \(v_1\) and \(v_2\) belong to single pair of \(P\), the pair is counted twice). \(\mu_p(G_1)\) is called the \(P\)-measure of \(G_1\) \((P\) will be omitted when understood). Next we relate \(\mu(L)\) \((\text{for a lump } L)\) to \(|V(L)|\).

Theorem 3.1

For any lump \(L\) in a minimal regular counterexample \((G,P)\)

\[ |V(L)| \leq 2\mu(L)+1. \]

Proof

\[ |E(L)| = \frac{k|V(L)| - |E(L,G \setminus L)|}{2} = \frac{k(|V(L)| - 1)}{2} \tag{3.1.1} \]

where \(k = \kappa_{\bar{e}}(G)\). By Lemma 3.4 every irreducible edge has an endpoint in \(P\)

\[ |E(L)| \leq k \cdot \mu(L). \tag{3.1.2} \]

Combining (3.1.2) with (3.1.1) yields \(|V(L)| \leq 2\mu+1\).
The next result is a bound for \( \mu(L) \) (\( L \) is a lump).

**Theorem 3.2**

For any lump \( L \) in a minimal regular counterexample \( G \), \( \mu(L) \geq 3 \).

**Proof**

Let \( \kappa_G(G) = k = 2m + 1 \). From its definition \( L \) contains an irreducible edge. Therefore by Lemma 3.4 \( \mu(L) \geq 1 \).

We first show that \( \mu(L) \geq 2 \). Let \( s \in V(L) \cap V(P) \) and \( q \in V(L) \setminus \{s\} \). As \( G \) is \( k \) connected, by Lemma 3.3 \( |\Gamma_G(q) \cap E(G \setminus L)| \leq m \). Therefore, \( \deg(q;L) = k - |\Gamma_G(q) \cap E(G \setminus L)| \geq m + 1 \). Also by Lemma 3.2 \( |E(s,q)| \leq m \).

Therefore \( |\Gamma_G(q;L) \cap E(s,q)| = \deg(q;L) - |E(s,q)| \geq 1 \). Thus \( E(L) \) contains an edge \( e \) not incident to \( s \). From the definition of a lump, \( e \) is irreducible and by Lemma 3.4 \( e \) has an endpoint in \( V(P) \). As \( s \not\in V(e) \), \( \mu(L) \geq 2 \).

Assume the theorem is false then \( \mu(L) = 2 \).

**Case 1 : \( \kappa_p(L) \geq m + 1 \).**

Let \( G/L \) be the contraction of \( L \) in \( G \) (i.e. the graph resulting by replacing all vertices of \( L \) by a single vertex \( L/L \)). As \( |E(G \setminus L)| = \kappa_G(G) = k \) \( G/L \) remains regular. \( P/L \) is obtained from \( P \) by replacing every occurrence of a vertex of \( L \) by \( L/L \). Since \( G \) is a minimal regular counterexample and \( |V(G/L)| < |V(G)| \) \( P/L \) is path-connected in \( G/L \). Each path in \( G/L \) not having \( L/L \) as an endpoint does not pass through \( L/L \) or passes through at least two edges incident to \( L/L \).

**Case 1.1 : \( V(P) \cap V(L) \) consists of a single pair.**

In \( G/L \) at most \( m \) paths pass through \( L/L \). As \( \kappa_p(L) \geq m + 1 \) it is possible to construct these paths and an additional path entirely in \( L \).
Case 1.2: $V(P) \cap V(L)$ consists of vertices from two different pairs of $P$

In $G/ L$ two paths have $L/ L$ as endpoints. Through the remaining $2m - 1$ edges there may pass at most $m - 1$ additional paths. Thus at most $m + 1$ paths need be constructed in $L$. This is always possible since $\kappa_p(L) \geq m + 1$.

Case 2: $\kappa_p(L) < m + 1$.

By Theorem 3.1 $2 \leq |V(L)| \leq 2 \mu(L) + 1 = 5$. As $|E(L)|$ is an odd integer (by 3.1.1)

$$|V(L)| = 3 \text{ V } |V(L)| = 5.$$  \hspace{1cm} (3.2.1)

Also $\kappa_G(G) > \kappa_G(L) \geq m + 1$ (by Lemma 3.1). Since $G$ is a minimal regular counterexample $\kappa_G(L) \geq m$, Therefore

$$\kappa_G(L) = m + 1, \text{ } \kappa_p(L) = m.$$  \hspace{1cm} (3.2.2)

Also $m$ must be odd, otherwise $m + 1$ is odd and $\kappa_p(L) = m + 1$.

Case 2.1: $|V(L)| = 3$.

Let $v, w \in V(L)$. As $|\Gamma_G(v) \cap E(L, G \setminus L)| \leq m$ (Lemma 3.3) and $|E(v, w)| \leq m$ (Lemma 3.2) $v$ is connected by an edge to all other vertices of $L$ and is at most at distance 1 from any edge of $E(L, G \setminus L)$ (see Figure 2).
Construct again the paths of $P/L$ in $G/L$. One of these paths (having $L/L$ as an endpoint) can be extended in $G$ by adding at most one edge $e$ of $E(L)$. But then $\kappa_p(L \setminus \{e\}) = m$ is odd and (by 3.2.2) $\kappa_p(G \setminus \{e\}) = m$. Therefore as in case 1, $m$ additional paths in $G/L$ passing through $L/L$ can be extended to paths in $G$.

Case 2.3: $|V(L)| = 5$.

$$|E(L)| = \frac{k|V(L)| - |E(L, G \setminus L)|}{2} = \frac{5k - k}{2} = 2k.$$ Since every edge of $E(L)$ is incident to $A = V(L) \cap V(P) = \{a_1, a_2\}$, $E(L) \subseteq \Gamma_L(A)$. And since $|A| = 2$, $2k = |E(L)| \leq |\Gamma_L(A)| \leq 2k$ we have $E(L) = \Gamma_L(A)$. Moreover $E(A) = \phi$. Therefore $L$ is bipartite $L = (A, B)$ with $A = V(L) \cap V(P)$ (see Figure 3).
Also \( a_4 \) are at distance 1 from \( E(L,G \setminus L) \).

**Case 2.2.1:** \( a_1, a_2 \) are vertices of two different pairs of \( P \).

A path of \( P/L \) in \( G/L \) (having \( L/L \) as endpoint) can be extended in \( G \) by using one edge \( e \) of \( L \), and as \( \kappa_p(L \setminus \{e\}) = m \) is odd \( \kappa_p(L \setminus \{e\}) = m \) (by the minimality of \( (G,P) \)). Therefore the additional \( m \) paths in \( G/L \) passing through \( L/L \) can be extended in \( G \).

**Case 2.2.2:** \( a_1, a_2 \) belong to a single pair in \( P \).
Let $b \in B$ be such that $|\Gamma_{\mathcal{G}}(b; L)| = m + 2$ and let $L_0^e \subseteq E$ be a lifting at $b \in L$ such that $\kappa_{\mathcal{G}}(L_0^e \setminus e) = m + 1$ (which exists by Theorem 2.1 and Theorem 2.2). Then in $L_0^e \subseteq E$ $a_1$ and $a_2$ are connected by an edge $e$ and the path $a_1 - a_2$ in $G$ can be constructed by using $e = [e_1, e_2]$. Now $\kappa_{\mathcal{G}}(L \setminus \{e_1, e_2\}) \geq \kappa_{\mathcal{G}}(L_0^e \setminus \{e\}) = m$.

Therefore after constructing the $a_1 - a_2$ path in $G$ all other $m$ paths of $P / L$ in $G / L$ passing through $L / L$ can be extended to $G$. Therefore $G$ is not a counterexample.

**Corollary 3.2.1**

If $(G, P)$ is a minimal regular counterexample and $G$ contains a reducible edge separating set $S$ then $S$ separates $P$.

**Proof**

By Theorem 3.2 each component of $G \setminus S$ contains vertices of $P$ (because each component contains a lump).

Using similar methods and considering more special cases we can prove

**Theorem 3.3**

For any lump $L$ in a minimal regular counterexample $G \mu(L) \geq 5$.

Let $M$ be a subgraph of $G$ all the edges of which are irreducible. If $E(M, G \setminus M) \subseteq S_1 \cup S_2$ where $S_1, S_2$ are two reducible minimal edge-separating sets in $G$ and $M$ does not contain a lump then $M$ is a layer of $G$.

**Lemma 3.5**

If $\kappa_{\mathcal{G}}(G) = 2m + 1$ then every layer in $G$ is connected.

**Proof**

Let $S_1, S_2$ be the two minimal edge-separating sets which separate $M$. Let $G_1, G_2$ be the two components of $G \setminus S_1$. If $S_2$ separates both $G_1$ and $G_2$ then by Lemma 3.1 $|S_2 \cap E(G_1)| \geq m + 1$ and $|S_2 \cap E(G_2)| \geq m + 1$ which yields $|S_2| \geq 2m + 2$.
Therefore $S_2$ separates only one of the components of $G \setminus S_1$ and if $M$ is not connected then it contains a lump.

**Lemma 3.6**

For any layer $M$ in a minimal regular counterexample $\mu(M) \geq 1$.

**Proof**

If $|V(M)| = 1$, let $V(M) = \{p\}$ then by Lemma 3.3 $|S_1 \cap \Gamma_S(p) | \leq m$ thus $\deg(p) \leq 2m < k$. Therefore $|V(M)| > 1$ and $M$ is connected by Lemma 3.5, thus $E(M)$ contains at least one edge, which by definition must be irreducible and by Lemma 3.4 is incident to a vertex of $P$.

Also a result similar to Theorem 3.1 can be obtained:

**Theorem 3.4**

For any layer $M$ in a minimal regular counterexample $|V(M)| \leq 2 \mu(M) + 2$.

Next a theorem similar to Theorem 3.3 is proved.

**Theorem 3.5**

If $M$ is a layer in a minimal regular counterexample $G$ and $|V(M)| > 2$ then $\mu(M) \geq 2$.

**Proof**

Assume $\mu(M) = 1$ and $|V(M)| > 2$ then $2 < |V(M)| \leq 2 \mu(M) + 2 = 4$. Therefore $|V(M)| = 4$ or $|V(M)| = 3$.

**Case 1**: $|V(M)| = 3$.

$|E(M)| = \frac{k(3-k - |E(M,G \setminus M)|)}{2}$ is an integer, therefore $|E(M,G \setminus M)|$ is odd. By Lemma 3.5 $M$ is connected and therefore the edges of $S_1 \cap S_2$ are exactly those edges of $S_1 \cup S_2$ not incident to $M$. But $S_1 \cup S_2$ includes the set which separates $M$ from $G \setminus M$. Thus $S_1 \cap S_2 = (S_1 \cup S_2) \setminus E(M,G \setminus M)$ and $2k =$
\[ |S_1| + |S_2| = |S_1 \cap E(M, G \setminus M)| + |S_2 \cap E(M, G \setminus M)| + 2|S_1 \cap S_2| \] is even. Therefore \(|S_1 \cap E(M, G \setminus M)| \text{ and } |S_2 \cap E(M, G \setminus M)| \) must have the same parity. On the other hand \(|E(M, G \setminus M)| = |S_1 \cap E(M, G \setminus M)| + |S_2 \cap E(M, G \setminus M)| \text{ and } \)

\[ |S_1 \cap E(M, G \setminus M)| \text{ and } |S_2 \cap E(M, G \setminus M)| \] cannot have the same parity.

**Case 2:** \(|V(M)| = 4\).

Since all edges of \(E(M)\) are irreducible, by Lemma 3.4 \(M\) is bipartite \(M = (\{p\}, V(M) \setminus \{p\})\) and \(E(M) \subseteq \Gamma_E(p)\). Thus \(|E(M)| \leq k\). As \(|E(M, G \setminus M)| \leq 2k\|E(M)\| = 2(4-k) - |E(M, G \setminus M)|\) we have \(|E(M)| \geq k\), therefore \(E(M) = \Gamma_E(p)\) (see FIG 4).

Let \(V(M) = \{p, v_1, v_2, v_3\}, e_i \in E(p, v_i)\) \((i = 1, 2, 3)\) and \(G' = G \setminus \{e_1, e_2, e_3\}\).

**Claim:** If \(s, t \in V(G) \setminus V(M)\) then \(\kappa_E(s, t; G') = \kappa_E(s, t; G)\).

**Proof of claim**

Let \(S\) be a minimal \(s \rightarrow t\) edge separating set. We show that \(|S| \geq \kappa_E(G) (=2m+1)\).

**Case 2.1:** \(S\) separates both \(G_1\) and \(G_2\).

As \(\kappa_E(G_1), \kappa_E(G_2) \geq m+1\) (Lemma 2.1). \(|S| \geq 2m+2 > \kappa_E(G)\).

**Case 2.2:** \(S\) separates \(G_1\) but does not separate \(M\) or \(G_2\).

In this case \(S\) is also an edge-separating set in \(G\) thus \(|S| \geq \kappa_E(G)\).

**Case 2.3:** \(S\) separates both \(G_1\) and \(M\) but does not separate \(G_2\).

In this case \(S\) separates a vertex \(v_i\) from \(p\) and \(S\) does not separate \(v_i\) from \(G_2\). By Lemma 3.3 \(|E(v_i, G_2)| \leq m\) and \(S\) contains all other edges incident to \(v_i\) and \(G'\). Therefore in \(G'\) \(|\Gamma_E(v_i) \cap S| \geq m\). Furthermore \(|S \cap E(G_1)| \geq m+1\) (Lemma 3.1).
and $E(G_1) \cap \Gamma_E(v_i) = \emptyset$ therefore $|S| \geq 2m+1$.

If $v_i$ is not separated from $G_1$ then $S$ contains all the edges incident to $v_i$ which are not in $E(v_i, G_1)$ and the previous argument applies also in this case.

**Case 2.4** $S$ separates only $M$.

Assume $S$ separates $v_1$ from $p$, therefore $E(v_1, p) \subseteq S$. Also $S$ separates $v_1$ from $G_1$ or $G_2$. Therefore $|S \cap \Gamma_E(v_1)| \geq m$. If $S$ separates another vertex $v_2$ then again $|S \cap \Gamma_E(v_2)| \geq m$, and as $S$ must contain an edge of $\Gamma_E(v_3)$ (there is a $G_1 - G_2$ path passing through $v_3$) we have $|S| \geq 2m+1$. If $S$ does not separate another vertex of $M$ from $p$ then $S_1 \setminus \Gamma_E(v_1) \subseteq S$ or $S_2 \setminus \Gamma_E(v_1) \subseteq S$. In both cases $|S| \geq 2m+1$.

This proves the claim.

![Diagram of graph with vertices and edges](FIGURE 4)

Remove $e_1, e_2, e_3$ then $p, v_1, v_2, v_3$ are of even degree and can be eliminated by a
sequence of liftings yielding a graph \( G_0 \) which is \( 2m+1 \)-connected (Theorem 2.1). Replace \( p' \) in \( P \) by any vertex \( q' \in V(G_2) \) denote the new sequence by \( P_0 \). As \( |V(G_2)| < |V(G)| \) and \( G \) was a minimal regular counterexample \( P_0 \) is connected. Restoring \( v_1,v_2,v_3,p \) leaves \( P_0 \) connected and the \( q-q' \) paths can be extended into a \( q-p \) path by using one of the edges \( e_1,e_2,e_3 \).

Therefore \( \mu(G) > 1 \).

**4. Bounds on the number of lumps and layers.**

Denote by:

- \( l \) - the number of lumps in \( G \).
- \( m_1 \) - the number of layers with 2 vertices in \( G \).
- \( m_2 \) - then number of layers with more than two vertices in \( G \).

Then as \( |V(P)| \leq 2 \cdot \kappa_E(G) \) for a minimal regular counterexample Theorem 3.3, Lemma 3.2 and Theorem 3.5 yield:

**Lemma 4.1**

\[
5l + m_1 + 2m_2 \leq 2 \cdot \kappa_E(G).
\]

Following is the main result of this section.

**Theorem 4.1**

If \( u \) is the number of reducible edge-separating sets in a minimal regular counterexample \( (G,P) \), \( l \) the number of lumps in \( G \) and \( m \) the number of layers in \( G \) then \( m \geq u - 2l + 3 \) (for \( u > 0 \)).

**Proof**

The theorem is proved by induction on \( u \).
Base \( u = 1 \) \( l = 2 \) \( m = 0 \).

**Induction step.**

Assume the theorem is true for \( u_0 \). Let \( u = u + 1 \) and \( G \) a graph with \( u \) reducible edge separating sets. Let \( L \) be a lump in \( G \) (if \( u \geq 1 \) \( G \) contains at least one). Let \( G' = G / L \) be the contraction of \( L \) in \( G \). Let \( u', m', l' \) defined as before for \( G' \) then by the induction hypothesis (\( u \geq 1 \))

\[
m' \geq u' - 2l' + 3
\]

(4.1.1)

\[ FIGURE 5 \]

Now the following relations hold between \( u', m', l' \) and \( u, m, l \)

\[
u = u' + 1, m = m' - 1, l = l' + 1.
\]

Substituting in (4.1.1) yield \( m \geq u - 2l + 3 \).

5. **Bounds on reducible minimal edge-separating sets.**

As each irreducible edge in a minimal regular counterexample has a vertex from \( V(P) \), a minimal regular counterexample contains at most \( 2 \cdot k^2 \) irreducible edges. Therefore the number of reducible edges is at least \( k \cdot |V(G)| - 2 \cdot k^2 \). Let \( g \) be the number of irreducible edges in layer of \( 2 \) vertices \( M \). If \( M \) is such a layer (see FIG 6)
then $E(M, G \setminus M) \subseteq S_1 \cup S_2$. ($S_i$ are minimal edge-separating sets). A short computation yields $|S_1 \cup S_2 \setminus E(M, G \setminus M)| = |E(M)|$ and if $e \in S_1 \cup S_2 \setminus E(M, G \setminus M)$ then $e \in S_1 \cap S_2$. The number of irreducible edges not in the two vertex layers is at most $k \cdot (2k - m_1)$. Therefore the number of reducible edge separating sets is at least

$$\frac{k(V(G)) - k(2k - m_1)}{k}$$

or

**Theorem 5.1**

If $G$ is a minimal regular counterexample, $u$ the number of reducible edge separating sets in $G$ and $m_1$ the number of two vertex layers then

$$u \geq \frac{k}{2} |V(G)| - 2 \kappa_2(G) + m_1.$$ 

6. A bound on minimal regular counterexample.

Let $l, m_1, m_2$ and $u$ be as in the previous section. Summing the results up to this point gives:

$$5l + m_1 + 2m_2 \geq 2 \kappa_2(G) \tag{6.1}$$
$$3l + m_1 + 2m_2 \geq 2 \kappa_2(G) \tag{6.1a}$$

(If only Theorem 3.2 is used instead of Theorem 3.3).
\[ u \geq \frac{1}{k} |V(G)| - 2 \kappa_{E}(G) + m_1 \]  
(6.2)

(from Theorem 5.1).

\[ m_1 + m_2 \geq u - 2l + 3 \]  
(6.3)

(from Theorem 4.1).

Solving these inequalities for \(|V(G)|\) yields:

**Theorem 6.1**

If \( G \) is a minimal regular counterexample then \(|V(G)| \leq \frac{20}{3} \kappa_{E}(G) - 6\).

**Proof**

In order to complete the proof the case \( u = 0 \) has to be considered because in this case Theorem 5.1 cannot be used. But if \( u = 0 \) all the edges of the graph are irreducible and by Lemma 3.4 \( E(G) \subseteq E(V(P)) \). As \(|V(P)| \leq 2k \) \(|E(G)| \leq 2k^2\)
and \(|V(G)| = \frac{2k}{k} |E(G)| \leq 4k. *\)

Using Theorem 3.3 yields (after rephrasing)

**Theorem 6.2**

If Conjecture A is true for regular graphs \( G \) such that \(|V(G)| \leq 8 \cdot \kappa_{E}(G) - 6\) then Conjecture A is true for all graphs.

A similar result can be proved for eulerian graphs.

**Theorem 6.3**

If Conjecture A is true for regular eulerian graphs \( G \) such that \(|V(G)| \leq 2 \cdot \kappa_{E}(G)\) then Conjecture A is true for all eulerian graphs.

**Proof**

In this case all vertices not in \( V(P) \) can be eliminated and \(|V(G)| = |V(P)| \leq 2k . *\)
7. References

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