MINIMAL PATH-DISCONNECTED GRAPHS

by

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Abstract

It is conjectured ([Tho80] [Cyp80]) that if a graph $G$ is $k$-edge-connected with $k$ odd then between any $k$ pairs of vertices $(s_1,t_1), ..., (s_k,t_k)$ there exist $k$ edge-disjoint paths $p_1, \ldots, p_k$ ($p_i$ is a $s_i-t_i$ path). In this paper it is shown that if the conjecture is false then there exists a counterexample $G$ which is a regular graph with at most $8k$ vertices.
1. Introduction.

Let $G = G(V,E)$ be a graph and $s,t \in V(G)$ an $s-t$ path $\pi$, is a path in $G$ between the vertices $s$ and $t$. A graph $G$ is $\kappa_{E} = \kappa_{E}(G)$ edge-connected if between any two vertices $s,t \in V(G)$ there are $\kappa_{E}$ edge-disjoint paths. This definition is equivalent to one in which $s,t$ are replaced by two sets $S,T \subseteq V(G)$ (Mengers Theorem). Similarly for two vertices (and sets) in $V(G)$ $\kappa_{E}(s,t;G)$ is the maximum number of edge-disjoint paths between $s$ and $t$ in $G$.

These definitions do not impose any relation between the endpoints of the paths. In order to relate the endpoints of those paths we introduce another definition which appeared in [IZB3]. Let $P = ((s_{1},t_{1}), \ldots, (s_{t},t_{t}))$ be a sequence of vertices in a graph $G$ then $P$ is path-connected iff there are $t$ edge-disjoint paths $\pi_{1}, \ldots, \pi_{t}$ where $\pi_{i}$ is a $s_{i}-t_{i}$ path. Denote by $|P| = t$.

A graph $G$ is $k$-path-connected iff every sequence $P$ of $k$ pairs of vertices in $V(G)$ is path-connected. $\kappa_{P}(G)$ is the maximal path connectivity of $G$. In [Tho80] and [Cyp80] the following conjecture appears:

Conjecture A.

For any graph if $\kappa_{E}(G)$ is odd then $\kappa_{P}(G) = \kappa_{E}(G)$.

It is known ([Cyp80] [Tho80] [IZB3]) that the conjecture is false when the edge-connectivity is an even integer. In [Oka83] it is proved that: $\kappa_{E}(G) \geq 2m-3$ entails $\kappa_{P}(G) \geq m$.

Also a weaker result $\kappa_{E}(G) \geq 5$ implies $\kappa_{P}(G) \geq 4$ appeared in [Eno83]. Furthermore, in [IZB3] it is proved that the conjecture holds for sequences $P$ which consist only of two distinct vertices.

In this paper the characteristics of a possible counterexample are analyzed. In section 2 we show that if the conjecture is false then there exists a
A $k$-regular graph contradicting the conjecture. The rest of the paper is devoted to analyzing the structure and size of a minimal regular counterexample $G$.

An initial attempt to prove the conjecture is by induction on the number of vertices. Choose a minimal edge-separating set $S$ and obtain the graph $G'$ by contracting all the vertices on one side of $S$ into a single vertex. From the paths in $G'$ one can try to construct $\kappa_{G'}(G)$ paths in $G$. One of the problems in this proof is that $S$ might separate only a single vertex such that $|V(G)|=|V(G')|$. Thus we are led to consider separating set $S$ in every side of which there are more than one vertex. It is proved in this paper that if such an edge-separating set exists in a minimal regular counterexample $(G,P)$ then it separates the vertices of $P$.

In section 3 we analyze the structure of some connected components of $G \setminus S$. Their number is bounded in section 4. This bound and the bound on the number of such minimal separating sets found in section 5 is used in section 6 to obtain a bound on the size of a minimal regular counterexample.

2. Regularity conditions.

Let $G(V,E)$ be a graph, $v \in V(G)$ a vertex and $e_1=(v,v'), e_2=(v,v'') \in E(G)$ two edges incident at $v$ then the graph $G^{e_1e_2}_v=G(V(G),(E(G) \setminus \{e_1,e_2\}) \cup \{(v',v'')\})$ is called a lifting of $G$ at $v$. The following results are from [Ma76]:

**Theorem 2.1**

Let $G(V,E)$ be a graph and $v \in G(V,E)$ then there exists a lifting $G^{e_1e_2}_v$ in $G$ at $v$ such that for any two vertices $s,t \neq v \kappa_{G}(s,t;G)=\kappa_{G(v^{e_1e_2}_v)}(s,t;G)$.

**Theorem 2.2**

Let $v \in V(G)$. If for any two vertices $s,t \in V(G)$ $s,t \neq v \kappa_{G}(s,t;G)\geq k$ and $\deg(v)\geq k$ then $\kappa_{G}(G)\geq k$. 
Let
\[ \Delta(G) = \max\{\deg(v) | v \in V(G)\} \]
\[ \delta(G) = \min\{\deg(v) | v \in V(G)\}. \]
A graph \( G \) is \( k \)-semi-regular iff \( k = \delta(G) \leq \Delta(G) \leq k + 1 \). Using Theorem 2.1 and Theorem 2.2 the following result is obtained.

**Lemma 2.1**

If Conjecture A is valid for semi-regular graphs then it is valid for all graphs.

**Proof**

Let \( G \) be a graph which is not semi-regular and for which Conjecture A does not hold and assume \( |E(G)| \) is minimal. Let \( v \in V(G) \) \( \deg(v) > k + 1 \). By Theorem 2.1 there exists a lifting \( G_v^{e_1 e_2} \) which preserves the edge-connectivity between all pairs of vertices excluding \( v \). As \( \deg(v; G_v^{e_1 e_2}) = \deg(v; G) - 2 \geq k \). By Theorem 2.2 \( \kappa_p(G_v^{e_1 e_2}) = \kappa_p(G) \). Now \( \kappa_p(G_v^{e_1 e_2}) \leq \kappa_p(\bar{G}) \) thus \( G_v^{e_1 e_2} \) is also a counterexample with fewer edges than \( G \). *

Let \( \pi = (x_1, x_2, \ldots, x_r) \) be a path then denote by \( \pi[x_i, x_j] \) the subpath of \( \pi \) between the vertices \( x_i \) and \( x_j \). Let \( \sigma = (x_r, y_1, \ldots, y_t) \) be another path then the **concatenation** of the paths \( \pi \) and \( \sigma \) is the path \( \pi\sigma = (x_1, \ldots, x_r, y_1, \ldots, y_t) \).

**Lemma 2.2**

Let \( G \) be a \( k \) edge-connected graph \( (k > 1) \) \( v_1, v_2 \in V(G) \) and \( w_1 \in \Gamma(v_1) \) (a neighbor of \( v_1 \)). Then there are \( k \) edge-disjoint \( v_1 - v_2 \) paths \( \pi_1, \ldots, \pi_k \), at least one of which passes through the edge \( (v_1, w_1) \).

**Proof**

Assume no path of \( \pi_1, \ldots, \pi_k \) passes through \( (v_1, w_1) \).
Let \( \tau \) be a \( w_1 - v_2 \) path in \( G \setminus (v_1,w_1) \). Let \( z \) be the first vertex in which \( \tau \) intersects \( \bigcup_{j=1}^{k} \pi_j (z \neq v_2) \) and assume this intersection occurs on \( \pi_i \). Replace \( \pi_i \) by \( \pi'_i = (v_1,w_1)\tau[w_1,z]\pi_i[z,v_2] \). \( \pi'_i \) is edge-disjoint of \( \pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_k \) and passes through \( (v_1,w_1) \).

**Theorem 2.3**

If Conjecture A holds for regular graphs then it holds for all graphs.

**Proof**

Assume Conjecture A is false. By Lemma 2.1 there exists a semi-regular graph \( G \) which is a counterexample. Furthermore, assume that \( G \) is not regular and \( V(G) \) is minimal. Let \( \kappa_r(G)=k \) (odd) \( \Delta(G)=k+1 \). Let \( p \in V(G) \ \deg(p)=k+1 \) and \( \Gamma_p = \{p,v_1,\ldots,p,v_{k+1}\} \). Denote by \( K \) a \( k+1 \) clique and \( \bar{M} \) a \( \frac{k+1}{2} \) matching on the same set of vertices \( V(K)=\{p_1,\ldots,p_{k+1}\} \). \( G_1=(V(K),E(K) \setminus E(\bar{M})) \). Replace \( p \) in \( G \) by \( G_1 \) and replace each edge \( (p,u_1) \) by \( (p_1,u_1) \).

i.e.

\[
G' = G((V(G) \cup V(G_1)) \setminus \{p\},(E(G) \setminus \Gamma_p) \cup E(G_1) \cup \{(p_1,u_1),\ldots,(p_{k+1},v_{k+1})\})
\]
Obviously \( G = G' \cap G_1 \) (the contraction of \( G_1 \) in \( G \)). Therefore \( \kappa_E(G') \leq \kappa_E(G) = k \).

**Claim** \( \kappa_E(G') = \kappa_E(G) \).

**Proof of claim**

The claim is true if and only if for any two vertices \( q_1, q_2 \in V(G') \)
\( \kappa_E(q_1, q_2; G') \geq \kappa_E(G) \).

**Case 1 :** \( q_1, q_2 \notin V(G_1) \)

Contracting \( G_1 \) yields \( k \) \( q_1 \)-\( q_2 \) paths in \( G = G' \cap G_1 \) at most \( \left| \frac{k}{2} \right| \) of these paths pass through \( p = G' \cap G_1 \) and as \( \kappa_E(G_1) = k - 1 \) these paths can be extended also in \( G' \) (by [Oka] \( G_1 \) is \( \left| \frac{k}{2} \right| \) path-connected).

**Case 2 :** \( q_1 = p_1 \in V(G_1) \)

Let \( p, v_1 \in V(G) \) and \( (p_1, v_1) \in E(G') \) by Lemma 2.2 there are \( k \) edge-disjoint \( p - q_2 \) paths in \( G \; \pi_1, \ldots, \pi_k \) one of which, \( \pi_1 \), passes through \( (p, v_1) \). This path can be extended to \( p \) by adding the edge \( (p, v_1) \) and as \( \kappa_E(G_1) = k - 1 \) all the other \( k - 1 \) paths can also be continued to \( p_1 \).

**Case 3 :** \( q_1 = p_1 \) \( q_2 = p_2 \) are vertices of \( G_1 \).

As \( \kappa_E(G_1) = k - 1 \) there are \( k - 1 \) \( q_1 \)-\( q_2 \) paths in \( G_1 \; \pi_1, \ldots, \pi_{k-1} \). Let \( w \in V(G') \setminus V(G_1) \). By Lemma 2.2 there exists a \( p - w \) path \( \tau_1 \) passing through \( v_1 \) \( ((p_1, v_1) \in E(G')) \) and a \( p - w \) path \( \tau_2 \) passing through \( v_2 \) \( ((p_2, v_2) \in E(G')) \).

Let \( z \) be the first intersection of \( \tau_1 \) and \( \tau_2 \) (they intersect at \( w \)). Then \( \tau = \tau_1[p_1, z] \tau_2[z, p_2] \) is \( p_1 \)\( p_2 \) path edge-disjoint of \( \pi_1, \ldots, \pi_{k-1} \). Therefore \( \kappa_E(G') = \kappa_E(G) \). But \( \kappa_p(G') \leq \kappa_p(G) \) and the number of vertices of degree \( k + 1 \) in \( G' \) is less than in \( G \).
3. Subgraphs in a minimal regular counterexample.

A minimal edge-separating set $S \subseteq E(G)$ in a graph $G$ is a set of edges in $E(G)$ such that $|S| = \kappa_E(G)$ and $G \setminus S = G_1 \cup G_2$ is disconnected ($G_1$, $G_2$ are its two components).

**Lemma 3.1**

Let $S$ be a minimal edge-separating set in a graph $G$ and $G_1$ a component of $G \setminus S$ then

$$\kappa_E(G_1) \geq \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor.$$

**Proof**

Let $p_1, p_2 \in V(G)$. In $G$ there are $\kappa_E(G)$ edge-disjoint $p_1 - p_2$ paths. Each $p_1 - p_2$ path which contains edges from $S$ must contain at least 2 edges of $S$. Therefore at most $\left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor$ $p_1 - p_2$ paths pass through $S$. All other paths lie entirely in $G_1$.

The following two lemmas will also be useful.

**Lemma 3.2**

Let $G$ be a graph $|E(G)| \leq \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor$, $p_1, p_2 \in V(G)$ then

$$|E(p_1, p_2)| \leq \left\lfloor \frac{\kappa_E(G)}{2} \right\rfloor.$$

**Proof**

Otherwise $|E(p_1, p_2), G \setminus \{p_1, p_2\}| < \kappa_E(G)$.

**Lemma 3.3**

Let $S$ be a minimal edge-separating set in a graph $G$, $G_1$ a component of
$G \setminus S$ such that $|V(G_1)| > 1 \ p \in V(G_1)$ then

$$|\Gamma_E(p) \cap S| \leq \lceil \frac{1}{2} \kappa_E(G) \rceil.$$  

Proof

Assume the contrary and let $q \in V(G_1)$ and $q \neq p$. Then the number of $p-q$ paths contained entirely in $G_1$ is at most $\kappa_E(G) - \frac{1}{2} \kappa_E(G)$. Assume the number of edges of $q$ in $G_1$ is at least $\frac{1}{2} \kappa_E(G)$ then the number of $p-q$ paths passing outside $G_1$ is at most $\frac{1}{2} |S|$. Therefore the connectivity of $p$ and $q$ is at most $(\kappa_E(G) - \frac{1}{2} \kappa_E(G)) + \frac{1}{2} |S| < \kappa_E(G)$. 

Let $k$ be an odd integer, $G$ a $k$-regular graph and $P$ a sequence of $k$ pairs of vertices in $G$. $(G,P)$ is a counterexample if conjecture A does not hold for $(G,P)$ i.e. $\kappa_E(G)=k$ and $P$ is not connected.

$(G,P)$ is a minimal regular counterexample if

1. $(G,P)$ is a counterexample.
2. $k=\kappa_E(G)$ is the minimal odd integer for which there exists a counterexample.
3. $G$ is the smallest regular graph satisfying (1) and (2).

An edge $e=(w_1,w_2)$ is reducible if there exists two vertices $v_1,v_2$ distinct of $w_1,w_2$ such that $\kappa_E(v_1,v_2;G) > \kappa_E(v_1,v_2;G \setminus e)$.

A minimal edge separating set $S$ in a graph $G$ is reducible if it contains a reducible edge, otherwise $S$ is irreducible.
Lemma 3.4

Every irreducible edge in a minimal regular counterexample \((G,P)\) is incident to a vertex of \(P\).

Proof.

Assume \(e=(v_1,v_2)\) is irreducible and \(v_1,v_2 \notin V(P)\). Then for any two vertices \(P_1,P_2\) such that \(\{P_1,P_2\} \cap \{v_1,v_2\} = \phi\) \(\kappa_G(P_1,P_2;G \setminus e)=\kappa_G(G)\). As \(\deg(v_1;G \setminus e) = \deg(v_2;G \setminus e)\) is even Theorem 2.2 can be applied repeatedly until \(v_1, v_2\) remain isolated. At this point remove \(v_1, v_2\) and by Theorem 2.1 the connectivity of the resulting graph \(G'\) is \(\kappa_G(G)\). As \(P\) remains disconnected in \(G'\) \((G,P)\) is not a minimal regular counterexample. 

A lump in a graph \(G\) is an induced subgraph \(L\) such that \(E(L,G \setminus L)\) is a reducible minimal edge-separating set and all edges of \(L\) are irreducible. Let \(G_1 \subset G\) then \(\mu_P(G_1)\) is the number of pairs \((v_1,v_2)\) of \(P\) such that either \(v_1\) or \(v_2\) belongs to \(G_1\) (if both \(v_1\) and \(v_2\) belong to single pair of \(P\), the pair is counted twice). \(\mu_P(G_1)\) is called the \(P\)-measure of \(G_1\) \((P\) will be omitted when understood). Next we relate \(\mu(L)\) (for a lump \(L\)) to \(|V(L)|\).

Theorem 3.1

For any lump \(L\) in a minimal regular counterexample \((G,P)\)

\[
|V(L)| \leq 2\mu(L)+1.
\]

Proof

\[
|E(L)| = \frac{k|V(L)| - |E(L,G \setminus L)|}{2} = \frac{k(|V(L)|-1)}{2} \quad (3.1.1)
\]

where \(k = \kappa_G(G)\). By Lemma 3.4 every irreducible edge has an endpoint in \(P\)

\[
|E(L)| \leq k \cdot \mu(L). \quad (3.1.2)
\]

Combining (3.1.2) with (3.1.1) yields \(|V(L)| \leq 2\mu+1\). 

The next result is a bound for $\mu(L)$ (L is a lump).

**Theorem 3.2**

For any lump $L$ in a minimal regular counterexample $G$, $\mu(L) \geq 3$.

**Proof**

Let $\kappa_G(G) = k = 2m + 1$. From its definition $L$ contains an irreducible edge. Therefore by Lemma 3.4 $\mu(L) \geq 1$.

We first show that $\mu(L) \geq 2$. Let $s \in V(L) \cap V(P)$ and $q \in V(L) \setminus \{s\}$. As $G$ is $k$ connected, by Lemma 3.3 $|\Gamma_E(q) \cap E(L, G \setminus L)| \leq m$. Therefore, $\deg(q; L) = k - |\Gamma_E(q) \cap E(L, G \setminus L)| \geq m + 1$. Also by Lemma 3.2 $|E(s, q)| \leq m$. Therefore $|\Gamma_E(q; L) \setminus E(s, q)| = \deg(q; L) - |E(s, q)| \geq 1$. Thus $E(L)$ contains an edge $e$ not incident to $s$. From the definition of a lump, $e$ is irreducible and by Lemma 3.4 $e$ has an endpoint in $V(P)$. As $s \not\in V(e)$, $\mu(L) \geq 2$.

Assume the theorem is false then $\mu(L) = 2$.

**Case 1 : $\kappa_p(L) \geq m + 1$.**

Let $G/L$ be the contraction of $L$ in $G$ (i.e. the graph resulting by replacing all vertices of $L$ by a single vertex $L/L$). As $|E(L, G \setminus L)| = \kappa_G(G) = k$ $G/L$ remains regular. $P/L$ is obtained from $P$ by replacing every occurrence of a vertex of $L$ by $L/L$. Since $G$ is a minimal regular counterexample and $|V(G/L)| < |V(G)|$ $P/L$ is path-connected in $G/L$. Each path in $G/L$ not having $L/L$ as an endpoint does not pass through $L/L$ or passes through at least two edges incident to $L/L$.

**Case 1.1 : $V(P) \cap V(L)$ consists of a single pair.**

In $G/L$ at most $m$ paths pass through $L/L$. As $\kappa_p(L) \geq m + 1$ it is possible to construct these paths and an additional path entirely in $L$.
Case 1.2: $V(P) \cap V(L)$ consists of vertices from two different pairs of $P$

In $G/L$ two paths have $L/L$ as endpoints. Through the remaining $2m-1$ edges there may pass at most $m-1$ additional paths. Thus at most $m+1$ paths need be constructed in $L$. This is always possible since $\kappa_p(L) \geq m+1$.

Case 2: $\kappa_p(L) < m+1$.

By Theorem 3.1 $2 \leq |V(L)| \leq 2\mu(L)+1=5$. As $|E(L)|$ is an odd integer (by 3.1.1)

$$|V(L)|=3 \text{ V } |V(L)|=5.$$  \hspace{1cm} (3.2.1)

Also $\kappa_E(G) > \kappa_E(L) \geq m+1$ (by Lemma 3.1). Since $G$ is a minimal regular counterexample $\kappa_p(L) \geq m$. Therefore

$$\kappa_E(L)=m+1, \kappa_p(L)=m.$$  \hspace{1cm} (3.2.2)

Also $m$ must be odd, otherwise $m+1$ is odd and $\kappa_p(L)=m+1$.

Case 2.1: $|V(L)|=3$.

Let $v,w \in V(L)$. As $|\Gamma_E(v) \cap E(L,G \setminus L)| \leq m$ (Lemma 3.3) and $|E(v,w)| \leq m$ (Lemma 3.2) $v$ is connected by an edge to all other vertices of $L$ and is at most at distance 1 from any edge of $E(L,G \setminus L)$ (see Figure 2).
Construct again the paths of $P/L$ in $G/L$. One of these paths (having $L/L$ as an endpoint) can be extended in $G$ by adding at most one edge $e$ of $E(L)$. But then $\kappa_G(L \setminus \{e\})=m$ is odd and (by 3.2.3) $\kappa_p(G \setminus \{e\})=m$. Therefore as in case 1, $m$ additional paths in $G/L$ passing through $L/L$ can be extended to paths in $G$.

**Case 2.3:** $|V(L)|=5$.

$$|E(L)| = \frac{k|V(L)| - |E(L,G \setminus L)|}{2} = \frac{5k-k}{2} = 2k.$$ Since every edge of $E(L)$ is incident to $A=V(L) \cap V(P)=\{a_1,a_2\}$, $E(L) \subseteq \Gamma_G(A)$. And since $|A|=2$, $2k=|E(L)| \leq |\Gamma_G(A)| \leq 2k$ we have $E(L)=\Gamma_G(A)$. Moreover $E(A)=\emptyset$. Therefore $L$ is bipartite $L=(A,B)$ with $A=V(L) \cap V(P)$ (see Figure 3).
FIGURE 3

Also $a_4$ are at distance 1 from $E(L,G \setminus L)$.

Case 2.2.1: $a_1, a_2$ are vertices of two different pairs of $P$.

A path of $P/L$ in $G/L$ (having $L/L$ as endpoint) can be extended in $G$ by using one edge $e$ of $L$, and as $\kappa_p(L \setminus \{e\})=m$ is odd $\kappa_p(L \setminus \{e\})=m$ (by the minimality of $(G,P)$). Therefore the additional $m$ paths in $G/L$ passing through $L/L$ can be extended in $G$.

Case 2.2.2: $a_1, a_2$ belong to a single pair in $P$. 
Let \( b \in B \) be such that \(|\Gamma_\delta(b;L)| > m+2\) and let \( I_\delta^{i_1i_2} \) be a lifting at \( b \in L \) such that \( \kappa_\delta(L_\delta^{i_1i_2}) = m+1 \) (which exists by Theorem 2.1 and Theorem 2.2). Then in \( L_\delta^{i_1i_2} \) \( a_1 \) and \( a_2 \) are connected by an edge \( e \) and the path \( a_1 - a_2 \) in \( G \) can be constructed by using \( e = [e_1, e_2] \). Now \( \kappa_p(L \setminus \{e_1, e_2\}) \geq \kappa_p(L_\delta^{i_1i_2} \setminus \{e\}) = m \).

Therefore after constructing the \( a_1 - a_2 \) path in \( G \) all other \( m \) paths of \( P/L \) in \( G/L \) passing through \( L/L \) can be extended to \( G \). Therefore \( G \) is not a counterexample.

**Corollary 3.2.1**

If \((G,P)\) is a minimal regular counterexample and \( G \) contains a reducible edge separating set \( S \) then \( S \) separates \( P \).

**Proof**

By Theorem 3.2 each component of \( G \setminus S \) contains vertices of \( P \) (because each component contains a lump).

Using similar methods and considering more special cases we can prove

**Theorem 3.3**

For any lump \( L \) in a minimal regular counterexample \( G \) \( \mu(L) \geq 5 \).

Let \( M \) be a subgraph of \( G \) all the edges of which are irreducible. If \( E(M,G \setminus M) \subseteq S_1 \cup S_2 \) where \( S_1 \), \( S_2 \) are two reducible minimal edge-separating sets in \( G \) and \( M \) does not contain a lump then \( M \) is a layer of \( G \).

**Lemma 3.5**

If \( \kappa_\delta(G) = 2m+1 \) then every layer in \( G \) is connected.

**Proof**

Let \( S_1, S_2 \) be the two minimal edge-separating sets which separate \( M \). Let \( G_1, G_2 \) be the two components of \( G \setminus S_1 \). If \( S_2 \) separates both \( G_1 \) and \( G_2 \) then by Lemma 3.1 \( |S_2 \cap E(G_1)| \geq m+1 \) and \( |S_2 \cap E(G_2)| \geq m+1 \) which yields \( |S_2| \geq 2m+2 \).
Therefore $S_2$ separates only one of the components of $G \setminus S_1$ and if $M$ is not connected then it contains a lump.

**Lemma 3.6**

For any layer $M$ in a minimal regular counterexample $\mu(M) \geq 1$.

**Proof**

If $|V(M)| = 1$, let $V(M) = \{p\}$ then by Lemma 3.3 $|S_1 \cap P(p) \subseteq m$, thus $\deg(p) \leq 2m < k$. Therefore $|V(M)| > 1$ and $M$ is connected by Lemma 3.5, thus $E(M)$ contains at least one edge, which by definition must be irreducible and by Lemma 3.4 is incident to a vertex of $P$.

Also a result similar to Theorem 3.1 can be obtained:

**Theorem 3.4**

For any layer $M$ in a minimal regular counterexample $|V(M)| \leq 2\mu(M) + 2$.

Next a theorem similar to Theorem 3.3 is proved.

**Theorem 3.5**

If $M$ is a layer in a minimal regular counterexample $G$ and $|V(M)| > 2$ then $\mu(M) \geq 2$.

**Proof**

Assume $\mu(M) = 1$ and $|V(M)| > 2$ then $2 < |V(M)| \leq 2\mu(M) + 2 = 4$. Therefore $|V(M)| = 4$ or $|V(M)| = 3$.

**Case 1 : $|V(M)| = 3$**

$|E(M)| = \frac{1}{2}(3k - |E(M,G \setminus M)|)$ is an integer, therefore $|E(M,G \setminus M)|$ is odd.

By Lemma 3.5 $M$ is connected and therefore the edges of $S_1 \cap S_2$ are exactly those edges of $S_1 \cup S_2$ not incident to $M$. But $S_1 \cup S_2$ includes the set which separates $M$ from $G \setminus M$. Thus $S_1 \cap S_2 = (S_1 \cup S_2) \setminus E(M,G \setminus M)$ and $2k =$
\[ |S_1| + |S_2| = |S_1 \cap E(M, G \setminus M)| + |S_2 \cap E(M, G \setminus M)| + 2 |S_1 \cap S_2| \] is even. Therefore \[ |S_1 \cap E(M, G \setminus M)|, \ |S_2 \cap E(M, G \setminus M)| \] must have the same parity. On the other hand \[ |E(M, G \setminus M)| = |S_1 \cap E(M, G \setminus M)| + |S_2 \cap E(M, G \setminus M)| \] and \[ |S_1 \cap E(M, G \setminus M)|, \ |S_2 \cap E(M, G \setminus M)| \] cannot have the same parity.

Case 2 : \(|V(M)| = 4\).

Since all edges of \(E(M)\) are irreducible, by Lemma 3.4 \(M\) is bipartite \(M = \{\{p\}, V(M) \setminus \{p\}\}\) and \(E(M) \subseteq \Gamma_{p}(p)\), thus \(|E(M)| \leq k\). As \(|E(M, G \setminus M)| \leq 2k\) \(|E(M)| = 2k - |E(M, G \setminus M)|\) we have \(|E(M)| \geq k\), therefore \(E(M) = \Gamma_{p}(p)\) (see FIG 4).

Let \(V(M) = \{p, v_1, v_2, v_3\}, e_i \in E(p, v_i) (i = 1, 2, 3)\) and \(G' = G \setminus \{e_1, e_2, e_3\}\).

Claim: If \(s, t \in V(G) \setminus V(M)\) then \(x_{G}(s, t; G') = x_{G}(s, t; G)\).

Proof of claim

Let \(S\) be a minimal \(s-t\) edge separating set. We show that \(|S| \geq x_{G}(G) (= 2m + 1)\).

Case 2.1 : \(S\) separates both \(G_1\) and \(G_2\).

As \(x_{G}(G_1), x_{G}(G_2) \geq m + 1\) (Lemma 2.1), \(|S| \geq 2m + 2 > x_{G}(G)\).

Case 2.2 : \(S\) separates \(G_1\) but does not separate \(M\) or \(G_2\).

In this case \(S\) is also an edge-separating set in \(G\) thus \(|S| \geq x_{G}(G)\).

Case 2.3 : \(S\) separates both \(G_1\) and \(M\) but does not separate \(G_2\).

In this case \(S\) separates a vertex \(u_i\) from \(p\) and \(S\) does not separate \(u_i\) from \(G_2\).

By Lemma 3.3 \(|E(u_i, G_2)| \leq m\) and \(S\) contains all other edges incident to \(u_i\).

Therefore in \(G'\) \(|\Gamma_{p}(u_i) \cap S| \geq m\). Furthermore \(|S \cap E(G_1)| \geq m + 1\) (Lemma 3.1)
and $E(G_1) \cap \Gamma_S(u_i) = \emptyset$ therefore $|S| \geq 2m + 1$.

If $u_i$ is not separated from $G_1$ then $S$ contains all the edges incident to $u_i$ which are not in $E(v_i, G_1)$ and the previous argument applies also in this case.

**Case 2.4 $S$ separates only $M$.**

Assume $S$ separates $v_1$ from $p$, therefore $E(v_1, p) \subseteq S$. Also $S$ separates $v_1$ from $G_1$ or $G_2$. Therefore $|S \cap \Gamma_S(v_1)| \geq m$. If $S$ separates another vertex $v_2$ then again $|S \cap \Gamma_S(v_2)| \geq m$, and as $S$ must contain an edge of $\Gamma_S(v_2)$ (there is a $G_1 - G_2$ path passing through $v_2$) we have $|S| \geq 2m + 1$. If $S$ does not separate another vertex of $M$ from $p$ then $S_1 \setminus \Gamma_S(v_1) \subseteq S$ or $S_2 \setminus \Gamma_S(v_1) \subseteq S$. In both cases $|S| \geq 2m + 1$.

This proves the claim.

**FIGURE 4**

- Remove $e_1, e_2, e_3$ then $p, v_1, v_2, v_3$ are of even degree and can be eliminated by a
sequence of liftings yielding a graph $G_0$ which is $2m+1$ connected (Theorem 2.1). Replace $v'$ in $P$ by any vertex $q' \in V(G_2)$ denote the new sequence by $P_0$. As $|V(G_0)| < |V(G)|$ and $G$ was a minimal regular counterexample $P_0$ is connected. Restoring $v_1,v_2,v_3,p$ leaves $P_0$ connected and the $q-q'$ paths can be extended into a $q-p$ path by using one of the edges $e_1,e_2,e_3$.

Therefore $\mu(G) > 1$.

4. Bounds on the number of lumps and layers.

Denote by:

- $l$: the number of lumps in $G$.
- $m_1$: the number of layers with 2 vertices in $G$.
- $m_2$: then number of layers with more than two vertices in $G$.

Then as $|V(P)| \leq 2 \cdot \kappa(G)$ for a minimal regular counterexample Theorem 3.3, Lemma 3.2 and Theorem 3.5 yield:

**Lemma 4.1**

$$5l + m_1 + 2m_2 \leq 2 \cdot \kappa(G).$$

Following is the main result of this section.

**Theorem 4.1**

If $u$ is the number of reducible edge-separating sets in a minimal regular counterexample $(G,P)$, $l$ the number of lumps in $G$ and $m$ the number of layers in $G$ then $m \geq u - 2l + 3$ (for $u > 0$).

**Proof**

The theorem is proved by induction on $u$. 
Base $u=1$ \( l=2 \) \( m=0 \).

Induction step.

Assume the theorem is true for $u$. Let $u' = u + 1$ and $G$ a graph with $u$ reducible edge separating sets. Let $L$ be a lump in $G$ (if $u \geq 1$ $G$ contains at least one). Let $G' = G / L$ be the contraction of $L$ in $G$. Let $u', m', l'$ defined as before for $G'$ then by the induction hypothesis ($u' \geq 1$)

\[
m' \geq u' - 2l' + 3
\]  

(4.1.1)

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5}
\caption{Figure 5}
\end{figure}

Now the following relations hold between $u', m', l'$ and $u, m, l$

$u = u' + 1$, $m \geq m' - 1$, $l = l' + 1$. Substituting in (4.1.1) yield $m \geq u - 2l + 3$.

5. Bounds on reducible minimal edge-separating sets.

As each irreducible edge in a minimal regular counterexample has a vertex from $V(P)$, a minimal regular counterexample contains at most $2 \cdot k^2$ irreducible edges. Therefore the number of reducible edges is at least $k \cdot |V(G)| - 2 \cdot k^2$. Let $g$ be the number of irreducible edges in layer of 2 vertices $M$. If $M$ is such a layer (see FIG 6)
then \( E(\hat{M}, G \setminus M) \subseteq S_1 \cup S_2 \). (\( S_i \) are minimal edge-separating sets). A short computation yields \(|S_1 \cup S_2 \setminus E(\hat{M}, G \setminus M)| = |E(M)|\) and if \( e \in S_1 \cup S_2 \setminus E(\hat{M}, G \setminus M)\) then \( e \in S_1 \cap S_2 \). The number of irreducible edges not in the two vertex layers is at most \( k \cdot (2k - m_1)\). Therefore the number of reducible edge separating sets is at least

\[
\frac{\frac{1}{2}k \cdot |V(G)| - k(2k - m_1)}{k}
\]

or

**Theorem 5.1**

If \( G \) is a minimal regular counterexample, \( u \) the number of reducible edge separating sets in \( G \) and \( m_1 \) the number of two vertex layers then

\[ u \geq \frac{1}{2}k \cdot |V(G)| - 2 \kappa_k(G) + m_1.\]

**6. A bound on minimal regular counterexample.**

Let \( l, m_1, m_2 \) and \( u \) be as in the previous section. Summing the results up to this point gives:

\[5l + m_1 + 2m_2 \leq 2 \kappa_k(G) \quad (6.1)\]

\[3l + m_1 + 2m_2 \leq 2 \kappa_k(G) \quad (6.1a)\]

(If only Theorem 3.2 is used instead of Theorem 3.3).
\[ u \geq \frac{1}{2} |V(G)| - 2 \kappa_E(G) + m_1 \]  
(from Theorem 5.1).

\[ m_1 + m_2 \geq u - 2 \cdot l + 3 \]  
(from Theorem 4.1).

Solving these inequalities for \(|V(G)|\) yields:

**Theorem 6.1**

If \(G\) is a minimal regular counterexample then \(|V(G)| \leq \frac{20}{3} \kappa_E(G) - 6.\)

**Proof**

In order to complete the proof the case \(u = 0\) has to be considered because in this case Theorem 5.1 cannot be used. But if \(u = 0\) all the edges of the graph are irreducible and by Lemma 3.4 \(E(G) \subseteq \Gamma_E(V(P))\). As \(|V(P)| \leq 2 \cdot k\) \(|E(G)| \leq 2 \cdot k^2\) and \(|V(G)| = \frac{2 \cdot |E(G)|}{k} \leq 4 \cdot k.\)

Using Theorem 3.3 yields (after rephrasing)

**Theorem 6.2**

If Conjecture A is true for regular graphs \(G\) such that \(|V(G)| \leq 8 \cdot \kappa_E(G) - 6\) then Conjecture A is true for all graphs.

A similar result can be proved for eulerian graphs.

**Theorem 6.3**

If Conjecture A is true for regular eulerian graphs \(G\) such that \(|V(G)| \leq 8 \cdot \kappa_E(G)\) then Conjecture A is true for all eulerian graphs.

**Proof**

In this case all vertices not in \(V(P)\) can be eliminated and \(|V(G)| = |V(P)| \leq 2 \cdot k.\)
7. References

[Cyp80]

[ES83]


[Ma78]

[Oka83]

[PS78]

[Sey80]

[Tho80]