The Golden Ratio Policy Strikes Again

OR

On Distributed Protocols
for a Multiple-Access Channel
with Average Delay Criterion

by

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ABSTRACT
Consider \( n \) transmission stations sharing a single communication channel. Packets arrive at the stations according to independent renewal processes. The stations are assumed to be able to store an unlimited number of packets in their buffers. The stations transmit packets at time slots allocated to them according to a given conflict-free distributed protocol. The cost criterion according to which protocols are evaluated is the long-run weighted average buffer occupancies. (The average waiting time is a special case of such a weighting.) A lower bound to the cost criterion under Time Division Multiplexing (TDM) protocols is given and the costs of two protocols are analyzed. The first protocol is the \textit{Random-Control} policy and the second is the \textit{Golden-Ratio} policy which is shown to achieve a cost close to the lower bound for realistic parameters.

\textbf{Keywords:} Communication protocols, Golden ratio, Fibonacci numbers, Multiple-access channel, Optimal control, Distributed control.

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1. Introduction

1-1 Consider $n$ transmission stations sharing a single communication channel. The channel is assumed to be slotted, i.e. the time axis is divided into equal segments called slots. Messages are correspondingly "packetized", and the transmission time of a packet is one slot. Each station has an independent arrival process of packets and is capable of storing an unlimited number of packets in its buffer. At each slot several stations may be granted permission to transmit according to some predetermined distributed protocol. When granted permission, a station transmits a packet within the slot, if its buffer is not empty. As a result of those transmissions an "empty", "successful", or "conflict" event may occur depending on whether zero, one or more than one packet were transmitted. The transmission result becomes known to all the stations at the end of the slot.

1-2 Several aspects of this multiple access channels were studied:

(i) The maximal throughput which can be obtained under stability restrictions on the system. An upper bound was first found by Pippenger [P] and gradually improved by several studies till the lowest known so far found by Mikhailov and Tsybakov [MT].

(ii) Performance analysis of given protocols. For a general survey the reader is referred to Massey [Ma]. The most attractive protocol is probably the tree algorithm suggested first by Capetanakis [C] and Tsybakov and Mikhailov [TM1]. The throughput and the delay of the tree algorithm with various sets of assumptions were studied by the proposers [C, TM1, TM2], as well as by Gallager [G], Fayolle and Hofri [FH], and others.

(iii) Optimization over a class of protocols. Optimal or nearly optimal protocols were formulated under various restrictions and cost criteria in Humblet and Mosely [HM], Berman [B], Varaiya and Walrand [VW], Rosberg [R1], Itai and Rosberg [IR] and Hluchyj [Hl].

1-3 In this report we discuss only conflict-free policies (i.e. each slot is allocated to at most one station). This conflict-free model is mainly applicable to data communication systems which use a satellite communication channel (see [Sc]), terrestrial loop circuit (see [KM]), or local area networks of computers (see [CFPW]). It appears that even when collision detection and resolution is cost effective and reliable, it is not worthwhile to allow conflicts when the arrival rates of messages are large.

Another restriction that we pose on the policies is that they only depend on the buffer occupancies, regardless of whether they are observable or not. In the second case one uses surrogate variables, which are the information which is available to the transmitters. It is not difficult to imagine situations where auxiliary information will lead to more efficient policies: e.g. consider an arrival process according to which the number of arrivals in a slot is zero or two. A policy that uses this information, and permits a station, once it produced a successful transmission one more slot (certain to succeed) and only then considered other stations will probably do better than one which only considered occupancy levels.
We believe that giving up this restriction will rarely pay, but the analysis is quite intractable otherwise.

1-4 This study is a continuation of Itai and Rosberg [IR], where the buffer capacity was set at one (i.e. no queues form) and the throughput of the channel was taken as the cost criterion. It was shown there that the Golden ratio Policy (which is defined below) achieves nearly optimal throughput. In this study we show that the same policy performs extremely well also under different ground rules, which are probably more realistic. When the buffers are unlimited the throughput of a stable system is necessarily equal to the input rate, and other criteria raise their head. Notably - the expected packet delay or equivalently the expected buffer occupancies, which are related by Little's theorem. We shall use a more general criterion: a linear combination of the expected buffer occupancies, with arbitrary weights:

1-5 In Section 2 we formulate a Markov Decision Process with incomplete information. In Section 3 we compute the buffer occupancy level in a station under an arbitrary loop policy which is used in Section 4 to provide a lower bound to the cost function under any TDM policy. In Sections 5 and 6 we analyze two specific policies: a random control policy and a deterministic one: the Golden Ratio loop-policy.
2. MARKOV DECISION PROCESS FORMULATION

2-1 Let \( n \) be the number of transmission stations and \( \{V^{(i)}(t)\}_{t=0}^\infty \) the arrival process of packets at station \( i \). We assume that the number of packets arriving at a station during successive time slots form a renewal process. We further assume that the arrival processes are mutually independent.

For every station \( i \) define the following non-negative integer-valued random variables:

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X^{(i)}(t) )</td>
<td>the number of packets in the buffer of station ( i ) at the beginning of slot number ( t ).</td>
</tr>
<tr>
<td>( V^{(i)}(t) )</td>
<td>the number of packets arriving at station ( i ) during slot number ( t ). The first two moments of ( V^{(i)}(t) ) are ( \lambda^{(i)} ) and ( \xi^{(i)} ), respectively, independent of ( t ).</td>
</tr>
<tr>
<td>( u^{(i)}(t) )</td>
<td>1 or 0 depending on whether station ( i ) had or had not permission to transmit at slot number ( t ).</td>
</tr>
<tr>
<td>( w^{(i)}(t) )</td>
<td>1 or 0 depending on whether station ( i ) did or did not transmit a packet during slot ( t ). Note that</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\lambda^{(i)}(t) &= \mu^{(i)}(t)(X^{(i)}(t) > 0), \\
\mu^{(i)}(t) &= \sum_{i=1}^{\infty} u^{(i)}(t) = I(A) \\
\end{align*}
\]

where \( I(A) \) is the indicator of the set \( A \).

For any set of \( n \) variables \( y^{(i)} \), \( 1 \leq i \leq n \) we shall use the vector notation \( y = (y^{(1)}, \ldots, y^{(n)}) \).

Recall that only conflict-free policies are considered, i.e., at any slot \( t \) at most one station has permission to transmit. Thus \( \sum_{i=1}^{\infty} u^{(i)}(t) \) equals 0 or 1.

Call the action which gives rise to \( \sum_{i=1}^{\infty} u^{(i)}(t) = 0 \) the empty action. For our choice of a cost criterion (expected buffer occupancies), it does not appear to serve any useful purpose; for every policy that has an empty action there is a policy with no empty actions with no more queued packets: by simply transforming each such action into an arbitrary permission. Henceforth, we assume \( \sum_{i=1}^{\infty} u^{(i)}(t) = 1 \) \( \forall \ t \geq 1 \).

From the definitions

\[
X^{(i)}(t+1) = X^{(i)}(t) + V^{(i)}(t) - I(X^{(i)}(t) > 0)u^{(i)}(t), \quad 1 \leq i \leq n, \tag{2.2}
\]

or equivalently

\[
X^{(i)}(t+1) = X^{(i)}(t) - w^{(i)}(t) + V^{(i)}(t)
\]

where \( X^{(i)}(0) \) are arbitrarily distributed. We assume that all packets arriving during a slot join the buffer at its end.

2-2 Let \( C^{(i)} \) be the cost per unit time of keeping a packet in station \( i \).

Further let \( V_T(\pi) \) be the total expected cost until time \( T \) of using policy \( \pi \), and define the long-run average cost

\[
\bar{V}(\pi) = \lim_{T \to \infty} \sup \frac{V_T(\pi)}{T}. \tag{2.3}
\]

Finally, let
\[ \overline{V} = \nu \overline{f}(\pi). \]

**\( \overline{V} \)** is the value function. A transmission policy \( \pi^* \) is optimal if \( \overline{V}(\pi^*) = \overline{V} \).

From the definition

\[ V_{\pi}(x) = \sum_{t=1}^{T} (C, x) E_{\pi}(X(t) = x), \]

where \( (C, x) = \sum_{t=1}^{T} g(t) x(t) \).

The sum in (2.4) is taken over all possible \( x \) and the expectation is with respect to the probability measure induced by policy \( \pi \) and the arrival process. From (2.4) it follows that under stationary conditions

\[ \overline{V}(\pi) = \sum_{t=1}^{T} C(t) \mu(t), \]

where \( \mu(t) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E_{\pi}(X(t)) \) is the expected number of packets at station \( i \) at beginnings of slots under the stationary distribution of the system states.

**Remark 2.1** For \( C(t) = \frac{1}{\lambda(t)} \), Little's lemma provides that \( \overline{V}(\pi) \) is the long run average waiting time.

2-3 Under any given policy \( \pi \), \( \{X(t)\}_{t=1} \) is a Markov Process. We are interested in decentralized policies - policies which can be implemented at each station without exchanging information beyond that which is obtained through the three-level channel feedback defined above. Note that a policy \( \pi \) takes action at the end of each time slot.

If \( X(t) \) were known by all the stations it could serve as a state descriptor in the decision process. However, under the decentralization requirement \( X(t) \) can only be considered a random variable with a distribution that depends on the information available to all stations: the history of the control actions and the transmission results.

As in [IR] we have to make do with sufficient statistics for \( X(t) \). The results of this section rely heavily on Striebel's monograph [ST] on this topic, especially Chapters 2 and 3 there.

2-4 Let

\[ u^i = (u(1), u(2), \ldots, u(t)), \quad w^i = (w(1), w(2), \ldots, w(t)). \]

Denote by \( H_t \) the \( \sigma \)-field generated by \( \{u^i, w^i; 1 \leq t \leq T\} \).

An element in \( H_{t-1} \) will be denoted by \( h_{t-1} \). It is the (observable) history of the process up to time \( t \) Clearly \( h_{t-1} \) could be used as a sufficient statistic. We shall use however an equally efficient and more compact statistic.

A variable which is convenient to use is \( \lambda(t) \), the number of slots since the last
unused permission given to station $i$, up to time $t$: $\bar{X}^{(i)}(0) = 0$. Thus

$$\bar{X}^{(i)}(t+1) = 1 + \bar{X}^{(i)}(t)(1-u^{(i)}(t))(1-w^{(i)}(t)).$$

(2.6)

Joining to it the last $\bar{X}^{(i)}$ values of $w^{(i)}$ yields the sufficient statistic at time $t$

$$s(t) = \{\bar{X}^{(i)}(t), w(t-r); 1 \leq r \leq \bar{X}(t)\}.$$

Note that $\bar{X}(t)$ can be computed from $u^{-1}$ and $w^{-1}$.

Since the $X^{(i)}(t)$'s, $1 \leq i \leq n$, $t=0,1,2,\ldots$, are independent, collisions are avoided and $X^{(i)}(t)$ depends on $s(t)$ and $X^{(i)}(\cdot)$ only, we have from (2.6) the following:

**Lemma 2.1**

(i) Given $s(t)$, $\{X^{(1)}(t), X^{(2)}(t), \ldots, X^{(n)}(t)\}$ are mutually independent r.v.'s with their distributions completely defined by $s(t)$.

(ii) \( E(X^{(i)}(t) | s(t)) = \lambda^{(i)}(t) \bar{X}^{(i)}(t) - \sum_{j=1}^{\bar{X}^{(i)}(t)} w^{(i)}(j). \)... (2.7)

2-6 From Lemma 2.1 it follows that we can use $\{s(t)\}$ as a sufficient statistic for $\{X(t)\}$, $t=1,2,3,\ldots$ and have the following Markov Decision Process (MDP) model for decentralized transmission protocols.

The state space $S$ is a denumerable space, consisting of all possible values $s(t)$. The decisions are taken at discrete epochs of time (beginnings of slots) and the action space, at state $s \in S$ is

$$A_s = \{u = (u^{(1)}, u^{(2)}, \ldots, u^{(n)}); u^{(i)} = 0,1; \sum_{i=1}^{n} u^{(i)} = 1\}.$$  

(Recall that for MDP formulation an admissible policy may use randomization on elements of $A_s$, and may also depend on the entire history of the process states and control actions).

The transition probabilities from state $s$ to state $s'$ given an action $u$ are completely defined from (2.1), (2.6) and Lemma 2.1(ii).

The immediate cost function is the conditional cost function

$$E[C|X(t); s(t)],$$

which again is well defined by Lemma 2.1. The cost criterion is the long-run cost and is then clearly the same as defined in (2.4).

The MDP defined above introduces the set of all decentralized control policies which can be formulated as an MDP. Henceforth we shall restrict our attention to a subset of these policies which will be defined as follows:

Let $k^{(i)}(t)$ be the number of slots since station $i$ was last given permission to transmit. Define $k^{(i)}(0) = 1$. From the protocol definition

$$k^{(i)}(t+1) = 1 + k^{(i)}(t)(1-u^{(i)}(t)).$$

(2.8)

and we shall proceed to only consider policies $\pi$ for which $u^{(i)}(t) = u^{(i)}(k^{(i)}(t))$, $1 \leq i \leq n$. 

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Since $k(t)$ gives merely the elapsed times since last permissions, we may regard it as the "first order history", and policies based on it as "first order approximation policies".

Note that (2.6) implies that any "first order approximation policy" corresponds to a predetermined allocation of the time axis among the $n$ stations. Let $t^{(i)}_j < t^{(i+1)}_j < \cdots$ be the slots allocated thus to station $i$, $1 \leq i \leq n$.

$$d^{(i)}_j = t^{(i)}_{j+1} - t^{(i)}_j, \quad j = 0, 1, \ldots$$

and

$$d^{(i)} = \liminf_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} d^{(i)}_j \quad (2.9)$$

These sequences define Time Division Multiplexing (TDM) policies, and the optimization problem over this restricted set reduces to dividing (allocating) the time slots in a satisfactory manner.

An important consideration is that we may safely limit our attention to these policies under which $\{X(t)\}_{t=1}^{\infty}$ is ergodic (i.e. positive recurrent), since otherwise, for a non-ergodic policy $\pi$, $\bar{D}(\pi)$ simply diverges.

2-6 In the following theorem we give a necessary and sufficient condition for a policy $\pi$ to be ergodic (i.e. $X(t)$ is ergodic under $\pi$).

From Lemma 2.1 and the fact that under a given $\pi$ the allocations are predetermined, $\{X^{(i)}(t)\}_{t=1}^{\infty}$, $1 \leq i \leq n$, are independent processes. Therefore, it is sufficient to specify conditions under which each of the process $\{X^{(i)}(t)\}_{t=1}^{\infty}$ is ergodic.

Let $\pi$ be a given policy. Define the following quantities

$$\gamma^{(i)}(x) = E_\pi [X^{(i)}(1) - X^{(i)}(0) \mid X^{(i)}(0) = x]$$

where $\gamma^{(i)}(x)$ is the average drift of $\{X^{(i)}(t)\}$ under $\pi$, $P^t = (P^t_{ij})$ is the $t$-step transition probability matrix of $\{X^{(i)}(t)\}_{t=1}^{\infty}$ and $N^{(i)}(t)$ the number of slots allocated to station $i$ until slot $t$.

Obviously, for every slot $t$ and $z \neq 0$

$$\gamma^{(i)}(x) = \lambda^{(i)} - u^{(i)}(t). \quad (2.11)$$

Hence:

**Theorem 2.1** A stationary policy $\pi$ is ergodic iff

$$\liminf_{t \to \infty} (N^{(i)}(t) - \lambda^{(i)} t) > 0. \quad (2.12)$$

**Proof.** The proof is based on Theorem 2 in Rosberg [R1] which gives necessary and sufficient conditions for the ergodicity of Markov chains. From the theorem there it is sufficient to show that (2.12) is equivalent to
\[ \lim_{T \to \infty} \inf \frac{1}{T} \sum_{i=1}^{T} \sum_{x \neq 0} P_{0i}^{(i)}(x) < 0. \]

From (2.11), the properties of irreducible aperiodic Markov chains and the definition of \( N_{0i}^{(i)}(t) \)

\[ \lim_{T \to \infty} \inf \frac{1}{T} \sum_{i=1}^{T} \sum_{x \neq 0} P_{0i}^{(i)}(x) = \lim_{t \to \infty} (1 - P_{0i}^{(i)}) \lambda^{(i)} - \lim_{t \to \infty} \sup \frac{1}{T} \sum_{t \in A_T} (1 - P_{0i}^{(i)}). \] (2.13)

where \( A_T = \{ t \leq T | u^{(i)}(t) = 1 \ \text{under} \ \pi \} \).

Since \( \lim_{t \to \infty} (1 - P_{0i}^{(i)}) \) exists, (2.13) becomes

\[ \lim_{t \to \infty} \left( 1 - P_{0i}^{(i)} \right) \lambda^{(i)} - \lim_{t \to \infty} \left( 1 - P_{0i}^{(i)} \right) \lambda^{(i)} - \lim_{t \to \infty} \sup \frac{N_{0i}^{(i)}(t)}{t}. \] (2.14)

Since \( \lim_{t \to \infty} (1 - P_{0i}^{(i)}) > 0 \), the last expression is negative iff (2.12) is satisfied.

Note that under an ergodic policy \( \pi \) the limits in (2.9) and (2.12) exist. Furthermore, the stationary probability that the buffer of station \( i \) is not empty under an arbitrary allocation to the station is

\[ \lim_{t \to \infty} \frac{\lambda^{(i)}t}{N_{0i}^{(i)}(t)}. \]

From the definition of \( d^{(i)} \) in (2.9) we also have

\[ \lim_{t \to \infty} \frac{t}{N_{0i}^{(i)}(t)} = d^{(i)}. \] (2.15)

In the following sections we shall find a lower bound to the cost function \( \bar{V}(\pi) \) for any deterministic TDM policy, which is the same as for any "first order approximation policy". Then we analyze two specific, realizable control policies. While it is not evident that the lower bound thus computed does indeed bound \( \bar{V} \) under any circumstance, we conjecture that possibly excepting "pathological" cases, it does. Hence its value, together with the lower of the two of the realizable policy values we can compute\(^8\), bracket \( \bar{V} \). In Section 6 we note that there are cases where it is obvious from symmetry that the Round Robin policy is optimal; thus we have at least in these special cases a direct comparison.

In Section 3 we proceed to compute the expected buffer occupancy under stationary conditions for TDM policies with a special structure (loop policies).

\(^8\) This value will practically always be the one produced by the Golden Ratio policy.
3. The Buffer Occupancy Problem Under a Loop Policy

3-1 In this Section we derive the results necessary to compute the distribution of buffer occupancies ("queue lengths") at each of the $n$ stations at the beginning of an arbitrary slot under any ergodic loop policy which is defined as follows:

**Definition.** A policy $\pi$ is a loop (periodic) policy if there exists an integer $N$ (period) such that for all $t$, the station which is allocated slot $t$ is also allocated slot $t + N$.

Note that under a loop policy the sequence of inter-permission times $d_j^{(i)}$, $j = 0, 1, 2, \ldots$ is periodic for every station $i$; that is, there are periods $N^{(i)}$ such that $d_{j+N}^{(i)} = d_j^{(i)}$ for every $t$.

The following analysis as a queueing-theoretical flavour. While the problem is significant in its own right, it is rather a technical digression from the main thrust of the paper. The reader may find the salient results in the Theorems and the Corollary.

Since the queues at distinct stations do not interact under such a policy, they are independent and we may focus on a generic station $i$. To simplify notation we omit in this section the station index.

Let $N$ be the loop size and $L$ the number of slots in the loop which are allocated to a given station. Further, let $d_j$, $0 \leq j < L$ be distances between two successive allocations. Clearly, $\sum_{j=0}^{L-1} d_j = N$. During the time from the beginning of the $i$-th transmission slot in the loop to the beginning of the $(i+1)$-st the station is said to be in its $i$-th phase, $0 \leq i < L$.

During each slot a random number $V$ of packets arrive at the station. Let $\lambda$ (respectively $\xi$) be its first (respectively second) moment, and $\alpha(z)$ its probability generating function (pgf).

Let $\mathcal{V}$ be the $d_i$-fold convolution of $V$ and $\alpha(z) = \mathcal{V}(z)$ the pgf of $\mathcal{V}$.

Since the computation is limited to ergodic policies, we may assume that the queue length is under stationary conditions (i.e., the distribution of the queue length does not depend on the time phase 0 in the cycle started).

3-2 Denote by $Y_j$ the queue length at the beginning of phase $j$ and by $G_j(z)$ its pgf.

The queue length evolution is governed by

$$Y_{j+1} = Y_j - I(Y_j > 0) + \mathcal{V},$$

(3.1)

where $[j+1]$ is $(j+1)$ modulo $L$.

Thus, $G_{j+1}(z)$ is related to $G_j(z)$ through the operator $T_j$,

$$G_{j+1}(z) = T_j[G_j(z)] = \alpha_j(z)[G_j(0) + \frac{1}{\lambda} (G_j(z) - G_j(0))].$$

(3.2)

By iterating (3.2) $L$ times we have

$$G_j(z)(z^L - \beta(z)) = \sum_{k=j}^{L-1} (-1)^{k-j} \left( \sum_{m=0}^{k-1} \frac{1}{\lambda^m} \alpha_m(z) \right),$$

(3.3)

where
\[
\beta(z) = \prod_{k=0}^{L-1} \alpha_k(z) \quad \rho_k = G_k(0),
\]

and the summation and product indices wrap-around from \( L-1 \) to 0 where needed.

Hence, all the queue length distributions are determined up to the \( \rho_k, k = 0, \ldots, L-1 \), which are given in Theorem 3.1 below.

3.3 Let \( q_j(k) = \Pr(V^j = k) \).

\[
H(k) = \begin{bmatrix}
    0 & q_0(k) & 0 & \cdots & 0 \\
    0 & 0 & q_1(k) & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & \cdots & \cdots & 0 \\
    0 & \cdots & \cdots & q_{L-1}(k)
\end{bmatrix}, \quad k \geq 0
\]

(3.4)

and \( \gamma \) be the matrix which satisfies

\[
\gamma = \sum_{k=0}^{L-1} H(k) \gamma^k.
\]

(3.5)

It is easy to verify that \( \gamma \) is stochastic. Let \( \eta = (\eta_0, \ldots, \eta_{L-1}) \) be the stationary probability vector of \( \gamma \), that is

\[
\eta = \eta \gamma, \quad \sum_{i=0}^{L-1} \eta_i = 1.
\]

Theorem 3.1

\[
\hat{p}_j = G_j(0) = (L-\lambda N)\eta_j, \quad j = 0, 1, \ldots, L-1.
\]

Proof. Define a random matrix \( G_j \), \( 0 \leq i, j \leq L-1 \), of the down-level-crossing-periods (dlcp).

\[
\Pr(C_{ij} = r) = \Pr(\text{the queue length reaches } k-1 \text{ for the first time after } r \text{ allocations to the station and this happens at the beginning of phase } i \text{ given that the system is at the beginning of phase } i \text{ and there are } k \text{ packets in the queue}).
\]

We first evaluate the distribution of the \( G_j \)'s and then show how they are related to the boundary probabilities \( \hat{p}_j \).

(1) The distribution of \( G_j \).

From the definitions

\[
\Pr(C_{ij} = 1) = \delta_{i, j} \delta_{j, i+1},
\]

(3.6)

\[
\Pr(C_{ij} = r) = \sum_{k=1}^{r-1} \delta_{k} q_k(k) \Pr(C_{ik+1} = r-1), \quad r > 1,
\]

where \( \delta_{i,j} \) is the Kronecker delta and \( C^\ast k \) is the \( k \)-fold convolution of \( C \).

Let

\[
\gamma_k(z) = \sum_{r=1}^{\infty} \Pr(C_{ij} = r)z^r, \quad \gamma(z) = (\gamma_k(z)).
\]
From (3.6),

\[ \gamma_i(z) = z \alpha_i(0) \delta_{i+1} + \sum_{k=1} q_i(k)z \gamma_i^{k+1,j}(z), \]

(3.7)

where \( \gamma_i^{k,j}(z) \) are the elements of \( \gamma^k(z) \).

In matrix form (3.7) becomes

\[ \gamma(z) = z \sum_{k=0} H(k) \gamma^k(z). \]

(3.8)

At \( z=1 \) equation (3.8) reduces to (3.5), if we identify \( \gamma \) with \( \gamma(1) \).

(2) The boundary probabilities

Consider the process that assumes the value of the phase number at those instances where the queue is empty at beginnings of phases. Clearly, this process is a first order Markov chain.

Let \( A = (a_{ij}) \) be its transition probability matrix, that is

\[ a_{ij} = \text{Prob}(\text{given that } Y_i = 0, \text{ the next time the queue is empty happens at the beginning of phase } j). \]

(3.9)

By conditioning on the value of \( Y^i \) we have

\[ a_{ij} = \alpha_i(0) \delta_{i+1} + \sum_{k=1} q_i(k) \text{Prob}(\text{given that } Y_{i+1} = k, \text{ the queue will first assume the value } 0 \text{ at phase } j). \]

(3.10)

Thus, \( \eta_j \) (the j-th element of \( \eta \), \( \gamma \)'s stationary probability vector), is the stationary probability of being at the beginning of phase \( j \), given that the queue is empty.

\[ p_j = \text{Prob}(\text{The queue is empty} \mid \text{The system is at the beginning of phase } j) = \text{Prob}(\text{The system is at the beginning of phase } j \mid \text{The queue is empty}) \times \frac{1}{\text{Prob}(\text{The system is at phase } j)}. \]

(3.11)

Taking the expectation of (3.1) with respect to the stationary distribution and then iterating it \( L \) times gives

\[ \text{Prob}(\text{The queue is empty at phase beginning}) = 1 - \lambda N/L. \]

(3.12)

Since the station "visits" all phases with equal frequency,

\[ \text{Prob}(\text{The system is at phase } j) = \frac{1}{L}. \]

(3.13)

Now the theorem follows from (3.10) - (3.13).
Remark 3.1 The matrix $\gamma$ can be computed by the following successive approximations

$$\gamma_k = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 1 \\ 1 & 0 & \ldots & \ldots & 0 \end{bmatrix}$$

for $k \geq 0$.

and

$$\gamma_{k+1} = \sum_{k=0}^{\infty} H(k) \gamma_k$$

Since $\gamma$ is stochastic and $H(k)$ a probability function, $\gamma_j \to \gamma$. The convergence is geometric, at a rate that depends on the largest eigenvalue of $\gamma$ that is less than 1.

3.4 Now $\bar{Y}_j$, the first moment of $Y_j$, can be obtained by differentiating (3.3) at $z = 1$, and also directly from (3.1).

By taking the expectation of (3.1) and iterating the resulting equation $L$ times it follows from the definitions that

$$\sum_{k=0}^{L-1} (1-p_k) = \lambda N.$$  

(3.14)

By squaring (3.1), then taking the expectation with respect to the stationary distribution, iterating it $L$ times and using (3.1) and (3.14) we obtain

$$\bar{Y}_j = \frac{(1+\lambda-\lambda^2)N + \sum_{j=0}^{L-1} \lambda d_j (\lambda d_j - 2(1-p_j))} {2(L-\lambda N)}.$$  

(3.15)

where the empty sum vanishes, and

$$\bar{Y}_j = \bar{Y}_0 + \sum_{k=0}^{L-1} (\lambda d_k - (1-p_k)).$$

(3.16)

Clearly, $Y_j(d_j) = Y_{j+1}$.

Since the loop contains $N$ slots,

$$\bar{Y} = \frac{1}{N} \sum_{j=0}^{d_j-1} \sum_{k=0}^{d_j-1} \bar{Y}_j(k) = \frac{1}{N} \sum_{j=0}^{d_j-1} d_j [\bar{Y}_j - (1-p_j) + \frac{1}{2} \lambda (d_j - 1)].$$  

(3.17)

From (3.15) and (3.17) we have the following theorem.
Theorem 3.2
\[
\bar{Y} = \frac{(\xi - \lambda^2 + \lambda)N + \sum_{j=0}^{L-1} \lambda d_j (\lambda d_j - 2(1 - p_j)) - 2 \sum_{j=0}^{L-1} (1 - \lambda d_j) \sum_{k=0}^{d_j-1} (\lambda d_k - (1 - p_k))}{2(L - \lambda N)} \tag{3.16}
\]

For the ensemble of \( n \) stations denote by \( Y^{(i)} \) the expected queue length at station \( i \) at a random slot.

For a given loop policy, let \( d_j^{(i)}, 0 \leq j < N^{(i)} \) be the distances between successive allocations to station \( i \). Replacing \( \lambda, \xi, L, d_j, p_k \) in Theorem 3.2 by \( \lambda^{(i)}, \xi^{(i)}, N^{(i)}, d_j^{(i)}, p_k^{(i)} \), we obtain an expression for \( \bar{Y}^{(i)} \).

**Corollary 3.1** Under every loop policy \( \pi \)
\[
\bar{V}(\pi) = \sum_{i=1}^{n} C^{(i)} \bar{Y}^{(i)},
\]
where \( \bar{Y}^{(i)} \) is given in Theorem 3.2 with the above substitutions.

In the next section we shall use these results to develop a lower bound to the value of \( \bar{V}(\pi) \), for all loop policies.
4. A Lower Bound on $\mathcal{V}(\pi)$.

4-1 In this Section we bound $\mathcal{V}(\pi)$ from below, for every loop policy $\pi$. We also show that in the special case where the packet arrival processes are generated by independent Poisson point processes, the same bound bounds from below every TDM policy (not necessarily periodic).

Let $\pi$ be a loop policy. From Corollary 3.1 follows $\mathcal{V}(\pi) = \sum_{i=1}^{n} C^{(i)} \overline{\mathcal{V}}^{(i)}$, where the $\overline{\mathcal{V}}^{(i)}$ are given in (3.18) (and more fully below, (4.2)).

We proceed as follows to find the bound:

(i) For each $\overline{\mathcal{V}}^{(i)}$ we independently obtain a lower bound, based on minimizing it under the constraint that the average inter-allocation distance of the station (its mean phase length, in terms of Section 3) is equal to a specified value $d^{(i)}$. These minima are denoted by $\mathcal{E}^{(i)}$.

(ii) Then we consider the ensemble of $n$ stations and solve the minimization problem

$$\min_{\{d^{(i)}\}_i} \sum_{i=1}^{n} C^{(i)} \mathcal{E}^{(i)}$$

s.t.

$$\sum_{i=1}^{n} \frac{1}{d^{(i)}} = 1$$

4-2 Let $i$ be a given station and $d^{(i)}$ be a given positive number. Now let $\pi$ be a loop policy which gives $N^{(i)}$ permissions in a loop of $N$ slots, giving rise to phase lengths $d^{(i)}_{j}, 0 \leq j < N^{(i)}$. Viewing the phase lengths as real numbers, rather than integers, the following optimization will clearly produce a lower bound to any achievable value function, with these parameters:

$$\min_{\{d^{(i)}_{j}\}} \overline{\mathcal{V}}^{(i)}$$

s.t.

$$\sum_{j=0}^{N^{(i)}-1} d^{(i)}_{j} = d^{(i)} N^{(i)} = N$$

**Theorem 4.1 (local optimum)** $\overline{\mathcal{V}}^{(i)}$ is minimized when

$$d^{(i)}_{j} = d^{(i)} , \quad 0 \leq j < N^{(i)}$$

**Proof.** From equation (3.18) we have

$$\overline{\mathcal{V}}^{(i)} = \frac{1}{2(N^{(i)}-\lambda^{(i)} N)} \left\{ (\varepsilon^{(i)}-(\lambda^{(i)})^2+\lambda^{(i)}) N + \sum_{j=0}^{N^{(i)}-1} \lambda^{(i)} d^{(i)}_{j} (\lambda^{(i)} d^{(i)}_{j} - 2(1-p^{(i)})) \right\}$$

$$+ \frac{1}{N} \sum_{j=1}^{N^{(i)}-1} d^{(i)}_{j} \left( \sum_{k=0}^{j-1} (\lambda^{(i)} d^{(i)}_{j} - 1-p^{(i)}) \right) \left( \lambda^{(i)} d^{(i)}_{j} - 1-p^{(i)} \right) \right\}$$

Using (3.14) which here reads
Consider now \( \mathcal{Y}(i) \) as a function of the \( 2N(i) \) variables \( \mathbf{d}^{(i)} = (d_0^{(i)}, d_1^{(i)}, \ldots, d_{N(i)-1}^{(i)}) \) and \( \mathbf{p}^{(i)} = (p_0^{(i)}, p_1^{(i)}, \ldots, p_{N(i)-1}^{(i)}) \), temporarily suppressing the functional dependence between \( \mathbf{d}^{(i)} \) and \( \mathbf{p}^{(i)} \) which is given by Theorem 3.1.

We pose the programming problem

\[
\min_{d^{(i)}, p^{(i)}, \nu_1, \nu_2} g(d^{(i)}, p^{(i)}) = \mathcal{Y}(i) \quad \text{s.t.} \quad \sum_{j=0}^{N(i)-1} d_j^{(i)} = N, \quad \sum_{j=0}^{N(i)-1} (1-p_j^{(i)}) = \lambda^{(i)} N
\]

and show that the functional relationship between \( d^{(i)} \) and \( p^{(i)} \) is satisfied at the extremum point obtained. Thus this is indeed an optimum to the original problem. To solve (4.3) we form the Lagrangian

\[
F(d^{(i)}, p^{(i)}, \nu_1, \nu_2) = g(d^{(i)}, p^{(i)}) - \nu_1 \left( \sum_{j=0}^{N(i)-1} d_j^{(i)} - N \right) - \nu_2 \left( \sum_{j=0}^{N(i)-1} (1-p_j^{(i)}) - \lambda^{(i)} N \right)
\]

Differentiating \( F \) with respect to the components of \( d^{(i)} \) and \( p^{(i)} \) respectively, we obtain the following two nonsingular sets of linear equations:

\[
\nu_1 = \frac{\xi^{(i)} + (\lambda^{(i)})^2 (2N-1) - 2\lambda^{(i)} \xi^{(i)} - \frac{1}{2} \lambda^{(i)} (1 - \frac{1}{2} \sum_{j=0}^{N(i)-1} (1-p_j^{(i)}) - \lambda^{(i)} N)}{2(N(i)-\lambda^{(i)} N)} + \lambda^{(i)} (1 - \frac{1}{2} \sum_{j=0}^{N(i)-1} (1-p_j^{(i)} - \lambda^{(i)} N)
\]

and

\[
\nu_2 = \frac{(N(i)-1-k)}{2(N(i)-\lambda^{(i)} N)} - \frac{1}{N} \sum_{j=k}^{N(i)-1} d_j^{(i)}
\]

Adding the constraints in (4.3) to (4.5) and (4.6) produces the solutions

\[
d_j^{(i)} = \frac{N}{N(i)} - \frac{\lambda^{(i)} N}{N(i)} \quad \text{and} \quad p_j^{(i)} = \frac{N(i) - \lambda(i) N}{N(i)} \quad 0 \leq j < N(i)
\]

By symmetry it is obvious that the optimal values in (4.7) satisfy the functional relationship.

Now let \( \pi \) be any TDM policy (not necessarily periodic), for which

\[
\lim_{k \to k} \frac{1}{k} \sum_{j=0}^{k-1} d_j^{(i)} = d^{(i)}
\]

Also, let \( \mu^{(i)} \) be the expected buffer occupany level at station \( i \) under stationary conditions. Using a slight variation of the methods used by Hajek recently [Ha, section 6] we obtain
Theorem 4.2 If packet arrival epochs at the stations form independent Poisson point processes than $\mu^{(i)}$ is minimized when $d_j^{(i)} = d^{(i)}$.

Remark 4.1 When $d^{(i)}$ is not integral, an achievable minimum in Theorems 4.1 and 4.2 is obtained by "nearly uniform" determination of $\{d_j^{(i)}\}$, as is shown by Hajek [Ha]. This allocation scheme is as follows:

Slot $t$ is allocated to station $i$ iff \[ \lfloor (t+1)/d^{(i)} \rfloor - \lfloor t/d^{(i)} \rfloor = 1, \] where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to $x$.

Corollary 4.1

(a) Under every loop policy $\pi$, which satisfies (4.8)

\[ \mu^{(i)} \geq \frac{d^{(i)} \text{Var}(y^{(i)})}{2(1-\rho_i)} - \frac{\lambda^{(i)}}{2}, \tag{4.9} \]

where $\rho_i = \lambda^{(i)}d^{(i)}$ and $\text{Var}(y^{(i)}) = \epsilon^{(i)}(\lambda^{(i)})^2$.

(b) If the packet arrival processes form independent Poisson point processes than (4.9) is valid for every TDM policy.

Proof. The corollary is immediate by replacing $d_j^{(i)}$ with $d^{(i)}$ in (4.2) and using Theorems 4.1 and 4.2.

4.5 Now consider the ensemble of $n$ stations and let $\pi$ be an ergodic policy with the average distance between the allocations it grants station $i$ equalling $d^{(i)}$, $1 \leq i \leq n$. From equation (2.5) and Corollary 4.1 we have the following Theorem:

Theorem 4.3

(a) For every periodic policy $\pi(d)$, which satisfies (4.8) for $1 \leq i \leq n$

\[ \bar{V}(\pi(d)) \geq \sum_{i=1}^{n} C^{(i)} \left[ \frac{d^{(i)} \text{Var}(y^{(i)})}{2(1-\rho_i)} - \frac{\lambda^{(i)}}{2} \right] \tag{4.10} \]

(b) If the arrivals are generated by Poisson point processes then (4.10) is satisfied for every TDM policy.

4.6 The Theorem introduced in this subsection gives a further characterization of the lower bound to $V(\pi)$. Let

\[ z^{(i)} = \frac{1}{d^{(i)}} \tag{4.11} \]

$z^{(i)}$ is the long-run fraction of slots allocated to station $i$. Thus from (4.11) and Theorem 4.3 the optimal solution to the

\[ \underline{\mu}^{(i)} \] as when $V^{(i)}$ was minimized, this provides a realizable minimum when $d^{(i)}$ is an integer; otherwise it is not realizable, but one can compute in principle a lower bound by inserting the nonintegral values $d_j^{(i)} = d^{(i)}$ at the formula for $V^{(i)}$. 

\[ \Box \]
following optimization problem would be a lower bound to the values $\bar{V}(\pi(d))$ over all $d$ that produce stable systems:

$$\min_{\{s^{(i)}\}} \left\{ \sum_{i=1}^{n} C^{(i)} \text{Var}(V^{(i)}) / 2(1-\lambda^{(i)}) / x^{(i)} - \frac{1}{2} \sum_{i=1}^{n} C^{(i)} \lambda^{(i)} \right\},$$

s.t. $^4$ $\sum_{i=1}^{n} x^{(i)} = 1, \quad x^{(i)} > \lambda^{(i)}$

4-7 The solution of (4.12) provides the following

**Theorem 4.4** For every $\pi$,

$$\bar{V}(\pi) \geq \frac{1}{2(1-\lambda)} \left( \sum_{i=1}^{n} C^{(i)} \text{Var}(V^{(i)}) \right)^2 - \frac{1}{2} \sum_{i=1}^{n} C^{(i)} \lambda^{(i)}, \quad \text{where} \quad \lambda = \frac{1}{n} \sum \lambda^{(i)}.$$

Proof. We show that a solution which satisfies $x^{(i)} > \lambda^{(i)}, \ i = 1, \ldots, n,$ exists for the following problem:

$$\min_{\{s^{(i)}\}} \sum_{i=1}^{n} C^{(i)} \text{Var}(V^{(i)}) / 2(x^{(i)} - \lambda^{(i)})$$

s.t. $\sum_{i=1}^{n} x^{(i)} = 1$

where the last term in (4.12) that does not depend on $x^{(i)}$ was scuttled. To solve (4.13) define a Lagrangian with the multiplier $\nu$ for the one constraint:

$$F(x, \nu) = \sum_{i=1}^{n} C^{(i)} \text{Var}(V^{(i)}) / 2(x^{(i)} - \lambda^{(i)}) - \nu(1-\sum_{i=1}^{n} x^{(i)})$$

Since the problem and the constraint are convex, a necessary and sufficient condition for the minimum is

$$\frac{\partial F}{\partial x^{(i)}} = \nu - \frac{C^{(i)} \text{Var}(V^{(i)})}{2(x^{(i)} - \lambda^{(i)})} = 0,$$

and

$$\sum_{i=1}^{n} x^{(i)} = 1$$

The requirement of ergodic policies provides $x^{(i)} > \lambda^{(i)}$ (see Theorem 2.1). Thus the only acceptable solution to (4.14) is

$$x^{(i)} = \lambda^{(i)} + \left[ \frac{C^{(i)} \text{Var}(V^{(i)})}{2\nu} \right]^{\frac{1}{2}},$$

with

$$\nu = \frac{1}{(1-\lambda)^2} \left[ \sum_{i=1}^{n} (C^{(i)} \text{Var}(V^{(i)})) \right]^{\frac{1}{2}}$$

$^4$ The form of the constraint assures the adoption only of allocation patterns that make the buffer occupancies ergodic.
and the Theorem follows from (4.12), (4.13) and (4.15). Substituting \( v \) in the solution yields the explicit optimal allocation fractions:

\[
x^{(i)} = \lambda^{(i)} + (1-\lambda) \frac{(C^{(i)} \text{Var}(V^{(i)}))^i}{\sum_{j=1}^{n} (C^{(j)} \text{Var}(V^{(j)}))^i}
\]

(4.16)

**Corollary 4.2** If \( \lambda^{(i)} = \lambda^{(1)} \) and \( C^{(i)} \text{Var}(V^{(i)}) = C^{(1)} \text{Var}(V^{(1)}) \), \( 2 \leq i \leq n \), then the **Round Robin** (RR) policy is optimal. This policy assigns to station \( i \) slots \( i \pmod{n} \) – 1.

**Proof.** From (4.16) we have \( x^{(i)} = \frac{1}{n} \). Since under the RR policy the fractional allocations are \( 1/n \), and the allocations to each station are equidistant with \( d = n \) then by Corollary 4.1 the lower bound is obtained.

Consider the \( x^{(i)} \)'s in (4.16) and let \( d^{(i)} = \frac{1}{x^{(i)}} \). An equally spaced loop policy (as in Corollary 4.2) for all the stations is almost never feasible. The infeasibility springs from two sources:

1. Usually the \( x^{(i)} \)'s are irrational, thus a finite loop is precluded.
2. Even when the \( x^{(i)} \)'s are rational, the implied phase lengths for different stations will usually clash (see example 4.1 below).

Therefore we try to approximate the optimal solution by policies which permit station \( i \) to transmit \( N^{(i)} \) times, at equally spaced slots in a loop of size \( N \). Thus, we first calculate the optimal \( x^{(i)} \)'s according to (4.16) and then consider policies that give station \( i \) \( N^{(i)} \) permissions. These policies will be close to the optimal only if the permissions to each station are nearly regularly spaced.

Thus we are confronted with the following placement problem:

Given \( N^{(1)}, N^{(2)}, \ldots, N^{(n)} \), \( \sum_{i=1}^{n} N^{(i)} = N \),

place the permissions of each station thus that

\[
\left| \frac{N}{N^{(i)}} \right| \leq g^{(i)} \leq \bar{d}^{(i)} \leq \left| \frac{N}{N^{(i)}} \right|
\]

where \( g^{(i)} \) and \( d^{(i)} \) are the minimum and the maximum distances between successive permissions to station \( i \).

This problem does not always have a solution.

**Example 4.1:** When \( n = 3, N = 6 \), \( N^{(1)} = 1, N^{(2)} = 2, N^{(3)} = 3 \), then ideally station 1 should be given permission once in the loop, station 2 every half loop and station 3 every other slot. This is infeasible.

Note that this incompatibility does not spring from the irrationality of the \( x^{(i)} \)'s; it would exist even if the optimal allocation fractions calculated from (4.11) for the above example were \( \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \).

To bypass this intractable placement problem we introduce and analyze in the next two sections two realizable control policies. The first one is a random control policy and the second is a deterministic TDM policy, the Golden Ratio policy.
5. A Random Control Policy

5-1 The purpose of this short section is to demonstrate the merit of a deterministic control policy compared with a random one. We use a specific example to quantify this merit.

5-2 Consider the channel capacity allocation as given in (4.16) with all $C^{(i)}$ equal to 1:

$$z^{(i)} = \lambda^{(i)} + (1-\lambda) \frac{\sqrt{\text{Var}(\hat{V}^{(i)})}}{\sum_{j=1}^{n} \sqrt{\text{Var}(\hat{V}^{(j)})}}.$$  

(5.1)

**Definition 5.1** Let $\pi_R$ be the conflict-free policy which at every slot $t$, $t=1,2,\cdots$ permits station $i$ to transmit with probability $z^{(i)}$, $i=1,2,\cdots,n$.

This policy can be implemented distributively if all the stations are synchronized with respect to the time origin and are using identical random number generators. In order to evaluate $\pi_R$ we compute the mean stationary buffer occupancy at station $i$, $i=1,2,\cdots,n$ and compare it to the lower bound obtained in Corollary 4.1.

$$LB^{(i)} = \frac{\text{Var}(\hat{V}^{(i)})}{2(z^{(i)}-\lambda^{(i)})} - \frac{\lambda^{(i)}}{2}.$$  

(5.2)

where $d^{(i)}$ was replaced by $1/z^{(i)}$. Under $\pi_R$ station $i$ is a GI/Geom/1 queueing system and through an equation analogous to (3.1) one obtains, the mean stationary buffer occupancy at station $i$, under $\pi_R$ as

$$\overline{\rho}^{(i)}(\pi_R) = \frac{\text{Var}(\hat{V}^{(i)}) + \lambda^{(i)} - (\lambda^{(i)})^2}{2(z^{(i)} - \lambda^{(i)})}.$$  

(5.3)

Since (5.1) and (5.2) imply $\lambda^{(i)} < z^{(i)} < \lambda^{(i)} + 1 - \lambda$, it follows from (5.2) and (5.3)

$$1 + \frac{\lambda^{(i)}(1-\lambda^{(i)})}{\text{Var}(\hat{V}^{(i)})} \leq \frac{\overline{\rho}^{(i)}(\pi_R)}{LB^{(i)}} \leq \frac{\text{Var}(\hat{V}^{(i)}) + \lambda^{(i)}(1-\lambda^{(i)})}{\text{Var}(\hat{V}^{(i)})} - \frac{\lambda^{(i)}(1-\lambda^{(i)})}{LB^{(i)}}.$$  

For illustration, if the number of arrivals per slot has the Poisson distribution, then $\text{Var}(\hat{V}^{(i)})$ is $\lambda^{(i)}$ and

$$2 - \lambda^{(i)} \leq \frac{\overline{\rho}^{(i)}(\pi_R)}{LB^{(i)}} \leq \frac{2 - \lambda^{(i)}}{\lambda},$$  

(5.4)

whereby one also gets an idea about the tightness of these bounds, remembering that $\lambda$ is bounded above by 1, and $\lambda^{(i)}$ is a fraction thereof, typically $\lambda/n$. 

...
The Golden Ratio Control Policy

6-1 Let \( x(i) > 0 \), \( i = 1, 2, \ldots, n \), \( \sum_{i=1}^{n} x(i) = 1 \) be the desirable fractions of permissions to each of the stations, as given by equation (4.16). Also, let \( F_k \) be the \( k \)-th Fibonacci number and \( N_k(i) \), \( i = 1, 2, \ldots, n \) be integers such that

\[
[x(i)F_k] \leq N_k(i) \leq [x(i)F_k], \quad \sum_{i=1}^{n} N_k(i) = F_k. \tag{6.1}
\]

Thus

\[
\lim_{k \to \infty} \frac{N_k(i)}{F_k} = x(i). \tag{6.2}
\]

For each \( k \), the Golden Ratio policy assigns \( N_k(i) \) slots to station \( i \) and attempts to distribute the permissions uniformly over a loop of size \( F_k \). (The analysis of Section 4 implies that it is optimal to distribute the permissions uniformly.)

6-2 Open address hashing confronts a similar problem: To distribute keys uniformly over a hash table. The uniformity of the distribution depends on the hash function. It has been shown that multiplicative hashing with the Golden Ratio multiplicand, \( \varphi^{-1} = (\sqrt{5} - 1)/2 \approx 0.6180339887 \), distributes the keys most uniformly (Knuth [Kn Vol. 1]). The Golden Ratio policy applies some of these results. Fibonacci numbers are related to the golden ratio \( \varphi^{-1} \) by the equation: \( F_k = \varphi^k - (1 - \varphi)^k \).

Let \( \text{frac}(y) = y - \lfloor y \rfloor \), \( a_j = \text{frac}(j \varphi^{-1}) \) and \( A_N = \{a_j \mid j = 0, \ldots, N-1\} \). The \( t \)-th smallest point of \( A_N \) is associated with the \( t \)-th slot of the loop.

**Definition 6.1** The Golden Ratio policy, \( \pi_{\text{GR}}(k) \), is the policy which assigns to station \( i \) the slots corresponding to the points

\[
\{a_j \mid \sum_{m=1}^{j-1} N_k(m) \leq j < \sum_{m=1}^{j} N_k(m)\}.
\]

It will be convenient to identify the points 0 and 1, and thus the points \( a_j \) are distributed over a unit circle.

**Example 6.1:** Suppose \( n = 3 \), \( x(1) = \frac{1}{2} + \epsilon_1 \), \( x(2) = \frac{3}{8} + \epsilon_2 \), \( x(3) = \frac{1}{8} + \epsilon_3 \), where \( \epsilon_i > 0 \) are arbitrarily small and \( x(1) + x(2) + x(3) = 1 \). Taking \( F_8 = 8 \), \( N_8(1) = 4 \), \( N_8(2) = 3 \) and \( N_8(3) = 1 \), \( \pi_{\text{GR}}(8) \) assigns to station 1 the slots corresponding to 0, \( \varphi^{-1} \), \( \text{frac}(2\varphi^{-1}) \) and \( \text{frac}(3\varphi^{-1}) \); to station 2 the slots corresponding to \( \text{frac}(4\varphi^{-1}) \), \( \text{frac}(5\varphi^{-1}) \) and \( \text{frac}(6\varphi^{-1}) \); and to station 3 the point corresponding to \( \text{frac}(7\varphi^{-1}) \). Thus the loop policy keeps giving permission to the stations in the following cyclic order:

"1,2,1,3,2,1,2,1".

6-3 The policy \( \pi_{\text{GR}(k)} \) defines the distances \( d_j(k), j = 0, 1, \ldots, N(j)-1, 1 \leq i \leq n \). Now \( \overline{d}(k) \) and \( \overline{V}(\pi_{\text{GR}(k)}) \) can be computed using Theorem 3.2 and corollary 3.1. In the following two theorems which are proved in [IR, Section 5] we give the uniformization characteristics of \( \pi_{\text{GR}(k)} \).
**Theorem 6.1** For each station $i$ (with $N_i^{(i)} = F_{k_i} + s_i^{(i)}, 0 \leq s_i^{(i)} < F_{k_i-1}$ where $k_i$ satisfies (6.1)), there are distances of at most three values between consecutive allocations:

- $s_i^{(i)}$ occurrences of distance $F_{k_i-k_i}$
- $F_{k_i-2} + s_i^{(i)}$ occurrences of distance $F_{k_i-k_i+1}$
- $F_{k_i-1} - s_i^{(i)}$ occurrences of distance $F_{k_i-k_i+2}$

**Remark 6.1** It is also shown in [IR] that the different distances are uniformly mixed.

**Remark 6.2** If a loop of a length which is not a Fibonacci number is used for the above allocation, the results are similar in uniformity but more values of distances are generated.

Let $j_i$ satisfy $\varphi^{-j_i} \leq x(i) < \varphi^{-j_i+1}$.

**Theorem 6.2** For each station $i$, with $N_i^{(i)} = F_{k_i} + s_i^{(i)}$ ($0 \leq s_i^{(i)} < F_{k_i-1}$) allocations in a loop of size $N=F_k$, and $k$ being sufficiently large, the following proportions of distances are generated:

$$1 - \frac{\varphi^{-j_i}}{x(i)} \text{ of distance } F_{k_i};$$

$$\varphi^{-j_i-2} + \frac{\varphi^{-j_i}}{x(i)} -1 \text{ of distance } F_{k_i+1};$$

$$1 - \frac{\varphi^{-j_i-2}}{x(i)} \text{ of distance } F_{k_i+2}.$$ 

---

**To evaluate the quality of $\pi_{GR}$ we need to compute** $E[\bar{V}/\bar{V}(\pi_{GR})]$, as a function of the number of transmitting stations and the size of the loop used by $\pi_{GR}$, where the expectation is taken in some suitable sense over the range of admissible arrival processes. This we cannot do because firstly, $\bar{V}$ is not known, and secondly, there is no satisfactory measure over the set of arrival processes.

**To obviate the first difficulty we use instead of $\bar{V}$ the bound provided by Theorem 4.4.** The second one is circumvented in a sense by estimating the ratio at an arbitrarily selected set of points: only Poisson arrival processes were considered, and two load levels were chosen, $\lambda=0.3, 0.9$. To obtain the values of $\lambda^{(i)}$ we sampled values from a gamma distribution, with parameters computed to produce coefficient of variation (denoted by $\gamma$) at the six values 0.1, 0.25, 0.5, 1, 2 and 10. This plan was adopted on the hypothesis that variation among the $\lambda^{(i)}$ is likely to be an important determinant of the performance of the channel (under any policy). Support for this hypothesis may be found in the expression for the bound on $\bar{V}$, in Theorem 4.4; specializing for Poisson arrival processes, where $\text{Var}(Y^{(i)}))=\lambda^{(i)}$, then $\sum(\lambda^{(i)})^2$ increases (in expectation) as the variation of the population from which the $\lambda^{(i)}$ are drawn decreases. Computations were performed for 5, 15 and 30 transmitting stations, and loop lengths were fixed at 89, 233 and 610 slots (for 5 stations 55 slots were used rather than 610; note that these are all Fibonacci numbers). The cost coefficients $C^{(i)}$ were taken all equal to 1.
Finally, we repeat that, $V(n_{GR})$ fails to achieve $V$ for two distinct reasons:

(i) It uses a finite loop, hence the optimal capacities as given by (4.16) cannot usually be assigned.

(ii) The policy $n_{GR}$ does not normally produce for a given loop and $N^{(t)}$ the best possible placement. Indeed, in the special case where the $\lambda^{(t)}$ are close enough and $n$ divides $N$, $N^{(t)}=N/n$ is clearly the optimal capacity assignment and $n_{GR}$, the Round-Robin policy is optimal; still $n_{GR}$ would use inter-assignment distances of (usually) three distinct values.

To estimate the contribution of the first factor, we used the bound of Theorem 4.4 both with the optimal allocation, from (4.16) and the actually achieved capacities$^5$. We checked at a few points the difference between $n_{GR}$ and $n_{GR}$, where the latter was indeed optimal, and the values appear to be in line with the others, which rather strengthens the evidence of the following results.

Denote the bound with the optimal capacities by $V_1$, and the one with the actually achieved capacities by $V_2$. Let $\tau_i = V_i / V(n_{GR}), i=1,2$.

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</table>

Table 1 - Performance ratios for decreasing arrival rate variation.

In Table 1 appear the values computed for $\tau_i$. In each case the upper value is $\tau_1$ and the lower one is $\tau_2$. Each point is averaged from 10 samples of $\lambda^{(t)}$. The following conclusions may be drawn:

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$^5$ The latter were obtained by first rounding $z^{(t)}N$ and if these did not sum to $N$ - by adding or deleting slots from allocations where the resulting difference $|z^{(t)}N-N^{(t)}|$ is then the smallest. Again, this is not necessarily optimal, but appears to have caused a loss that is vastly dominated by the others.
(1) \( r_{ij} \) is bounded from below by 0.9, i.e. the placement produced by \( \pi_{GR} \) "costs" less then 10%, and often much less.

(2) The achieved "rational" allocations harm the performance very little unless "difficult conditions" transpire (- some of the following was gleaned from more detailed printouts):

(a) \( \lambda \) is high: stations with very low \( \lambda^{(i)} \), still get one slot (even if their optimal allocation is much less than \( 1/N \)). This "pushes" other stations to values of \( M \lambda^{(i)} / N^{(i)} \) very close to 1. Actually, the zeroes in the table are all for \( \lambda=0.9 \), and they correspond with one exception to cases were in all the samples some of the stations were thus reduced to instability. (The exception, for \( n=15 \) and \( N=610 \) arose because in all samples some stations were assigned too many slots for the computation to take reasonable time. We allowed up to 120 slots per station.) As \( N \) increases this effect diminishes.

(b) The ratio \( N/n \) is too low for \( x^{(i)}N \) to be well approximated. This is more harmful the higher is \( \lambda \) (again, because over-allocation for one station would bring others to too high utilization levels); it happens more often when the coefficient of variation of \( \lambda^{(i)} \) is large (the first phenomenon is then also more pronounced).

(3) The extent of the variation among \( \lambda^{(i)} \) did not impact the placement algorithm at all, but as noted above occasionally impaired the capacity assignment mechanism.

The computations were robust, and no precision problems were apparent. They required however substantial computing times, most of it in the iteration outlined in Remark 3.1. The use of an arrival process of smaller support for the distribution of \( V \) (such as the Bernoulli distribution) would reduce it considerably: for the rates and loops we used, by an order of magnitude.
REFERENCES


[Ha] B. Hajek: Extremal Splittings of Point Processes, Presented in the ConferenCe Dedicated to the Memory of Jack Kiefer and Jacob Wolfowitz, Cornell University, July 1983.


