A DUALITY PROPERTY FOR THE SET OF ALL FEASIBLE SOLUTIONS TO AN INTEGER PROGRAM

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ABSTRACT

It is shown how to transform the set of all feasible solutions to an Integer program represented by a system of linear diophantine inequalities into an "equivalent" set represented by a system of linear diophantine equations and congruences. A similar transformation is given working in the opposite direction (i.e. from a system of equations to a system of inequalities).
1. INTRODUCTION

Many papers in the literature are concerned with the problem of reducing integer programs from one form to another equivalent form, see e.g. [1, 2, 3, 5]. Such reductions are often useful as some forms are easier to solve than others. In this paper we will prove (algorithmically) a duality principle showing a correspondence between sets of constraints (to integer programs) expressed in the form of linear equalities and linear congruences and such sets of constraints expressed in the form of linear inequalities. Specifically we will show that under very general conditions the following transformations are possible:

(1) Given a convex set in the positive orthant (i.e. \( x_i \geq 0 \)) of the \( n \)-dimensional Euclidean space represented by \( m > n \) linear inequalities with integral coefficients:

   An \( I \)-equivalent convex set (i.e. there exists an explicit \( 1 \)-1 correspondence between the (integral) points in the first set and the (integral) points in the second) can be constructed in the \( m \)-dimensional Euclidean space, represented as the set of nonnegative solutions of a system consisting of \( m - n \) diophantine linear equations and up to \( n \) linear congruences with integral coefficients.

(2) Given a convex set in the \( n \)-dimensional Euclidean space represented as the set of nonnegative solutions of a system of \( m < n \) diophantine linear equations. An \( I \)-equivalent convex set in the \( n - m \) dimensional Euclidean space can be constructed, represented as the intersection of the positive orthant with a set defined by \( n \) linear inequalities.
2. PRELIMINARIES

Given a set of \( m \) inequalities in \( n \) variables

\[
\frac{1}{j=n} a_{ij}x_j + b_i \geq 0; \quad 1 \leq i \leq m
\]  

(1)

We represent the above set in the following matrix form

\[
(1,x_n,\ldots,x_1) A \geq 0
\]  

(2)

Then \( A \) is \( (n+1) \times m \) (with the \( b \)'s in its first row).

We shall refer to the convex set \( K \) defined below

\[
K = \{(x_n,\ldots,x_1) : (1,x_n,\ldots,x_1) A \geq 0\}
\]  

(3)

and we shall make the following assumptions:

(a) \( m \geq n+1 \)

(b) The entries of \( A \) are all integers

(c) Every subset of \( n+1 \) columns of \( A \) are linearly independent (otherwise some inequalities in (1) are superfluous and can be discarded).

(d) All the vertices of \( K \) (provided that \( K \) is not empty) are located in the positive orthant of the \( n \)-dimensional Euclidean space

(a vertex is a point satisfying \( n \) of the \( m \) inequalities when changed into equalities and verifying all the other inequalities).

If condition (d) above is not satisfied then one can translate \( K \) using a translation of the form \( x_n + x_1 + k_1 \) \( x_1 \) an integer to derive a new convex set \( K' \) satisfying the condition. Notice that a translation as above preserves the natural lattice.

The following facts are implied by the above assumptions.

Fact 1. The matrix \( A \) has a generalized inverse \( A^{-1} \) (by property c) whose entries are rational numbers (by property b) and is \( m \times (n+1) \).

Fact 2. The manifold \( (1,\alpha_n,\ldots,\alpha_1)A = (y_m,\ldots,y_1) \) where the \( \alpha \)'s are parameters is equivalent to a system of \( (m-n) \) simultaneous equations.
in the variables \( y_m \ldots y_1 \) which can be found easily: Choose \( n \) columns in \( A \) together with the corresponding \( \gamma \)'s; Solve the corresponding equations for the \( a \)'s, expressing the \( a \)'s as functions of the \( y \)'s; Substitute the value of the \( a \)'s in the remaining \( m-n \) equations.

Denote the set of equations derived as above by

\[
\sum_{j=m}^{1} c_{ij} y_j = d_i, \quad 1 \leq i \leq m-n
\]  

**Fact 3.** Let \((\hat{y}_m, \ldots, \hat{y}_1)\) be a vector satisfying (4). Then

\[(\hat{y}_m, \ldots, \hat{y}_1) A^{-1} = (\hat{y}_m, \ldots, \hat{y}_1)\].

This follows from Fact 2 above: any vector satisfying (4) can be expressed in the form \((1, \hat{a}_n, \ldots, \hat{a}_1)A\) for some parameter values \(\hat{a}_n, \ldots, \hat{a}_1\). Thus \((\hat{y}_m, \ldots, \hat{y}_1)A^{-1} = (1, \hat{a}_n, \ldots, \hat{a}_1)AA^{-1} = (\hat{y}_m, \ldots, \hat{y}_1)\).

**Fact 4.** Let \((\hat{y}_m, \ldots, \hat{y}_1)\) be a vector satisfying (4). Then

\[(\hat{y}_m, \ldots, \hat{y}_1)A^{-1} = (1, \hat{a}_n, \ldots, \hat{a}_1)\] for some values \(\hat{a}_n, \ldots, \hat{a}_1\).

This follows from Fact 2. As shown in Fact 3, \((\hat{y}_m, \ldots, \hat{y}_1)A^{-1} = (1, \hat{a}_n, \ldots, \hat{a}_1)AA^{-1} = (\hat{y}_m, \ldots, \hat{y}_1)\).

**Fact 5.** Let \((\hat{y}_{i1}, \ldots, \hat{y}_{im})\) be the \( i \)-th column of \( A^{-1} \), and let \( g_i \) be the l.c.m. (least common multiple) of the denominators of the entries in that column of \( A^{-1} \), \( 1 \leq i \leq n+1 \). Then, all the coefficients in the system of equations below are integers

\[
\sum_{j=m}^{1} (\gamma_{i,j}g_i) y_j \equiv 0 \pmod{g_i}; \quad 2 \leq i \leq n+1
\]  

\( g_i \) exists as follows from Fact 1.

Moreover, if \((\hat{y}_m, \ldots, \hat{y}_1)\) is a vector of integers satisfying (4) and (5) then the vector \((\hat{x}_{n+1}, \ldots, \hat{y}_1) = (\hat{y}_m, \ldots, \hat{y}_1)A^{-1}\) is a vector of integers. To prove this notice that \( \hat{x}_{n+1} = 1 \) by Fact 4 and
for the other entries, we have that

\[ \frac{1}{g_i} \sum_{j=m}^{n} \gamma_{ij} \hat{y}_j = \frac{1}{g_i} \sum_{j=m}^{n} (\gamma_{ij} g_i) \hat{y}_j = \frac{1}{g_i} (\tau g_1) = t \]

where \( t \) is an integer by (5).

Notice that if \( g_i = 1 \) for \( 2 \leq i \leq n+1 \) (i.e., if the corresponding columns of \( A^{-1} \) are columns of integers) then any vector of integers satisfies the modular equations (5).

3. FIRST REDUCTION THEOREM

We are now ready to prove the following:

**Theorem 1:** Let \( K \) be a convex set as defined in (3) and satisfying the conditions (a) to (d).

Let \( K_1 \) be a convex set as defined below

\[ K_1 = \{ (y_m, \ldots, y_1) : (y_m, \ldots, y_1) \text{ satisfies (4) and (5) and} \]

\[ \hat{y}_1 > 0 \} \quad (6) \]

There exists a 1-1 linear mapping from \( K \) onto \( K_1 \) such that

\[ \tau(x_n, \ldots, x_1) = (y_m, \ldots, y_1) \text{ has integral coordinates iff} \]

\[ (x_n, \ldots, x_1) = \tau^{-1}(y_m, \ldots, y_1) \text{ has integral coordinates.} \]

**Proof** Let \( (x_n, \ldots, x_1) \) be a vector of integers satisfying (2). Then, by property (d) all the \( x_i \) are nonnegative. Set

\[ (\hat{y}_m, \ldots, \hat{y}_1) = (1, x_n, \ldots, x_1)A \]

then \( \hat{y}_m, \ldots, \hat{y}_1 \) is a vector of integers (property (b)); the \( \hat{y}_i \)'s are nonnegative (the vector \( (x_n, \ldots, x_1) \) satisfies (2)); the vector \( (\hat{y}_n, \ldots, \hat{y}_1) \) satisfies (4) by Fact 2, and it satisfies (5) too which can be shown as follows (for \( 2 \leq i \leq n+1 \)):

\[ \frac{1}{g_i} \sum_{j=m}^{n} (\gamma_{ij} g_i) \hat{y}_j = g_i \sum_{j=m}^{n} \gamma_{ij} \hat{y}_j = g_i \hat{y}_i \equiv 0 \text{ (mod. } g_i) \]

\( (\hat{y}_i \) was assumed to be an integer).
Conversely, let \( \hat{y}_n \ldots \hat{y}_1 \) be a vector of nonnegative integers satisfying (4) and (5). Define \( \{\hat{x}_n \ldots \hat{x}_1\} \) by the relation

\[
(\hat{x}_{n+1} \hat{x}_n \ldots \hat{x}_1) = (\hat{y}_n \ldots \hat{y}_1) A^{-1}.

Then, by Fact 5 \( (\hat{x}_n \ldots \hat{x}_1) \) is a vector of integers and

\[ \hat{x}_{m+1} = 1. \]

By Fact 3, we have also that \( (1, \hat{x}_n \ldots \hat{x}_1) A = (\hat{y}_n \ldots \hat{y}_1) A^{-1} A = (\hat{y}_m \ldots \hat{y}_1) \) so that the \( \hat{x} \) vector satisfies (2) (the \( \hat{y}_1 \) are assumed to be nonnegative). This implies, by property (d) that the \( \hat{x}_1 \) are nonnegative. The proof is now complete. Q.E.D.

Remark: As mentioned in Fact 5 some of the modular equations (5) can be discarded in the definition of \( K \), specifically, those equations corresponding to columns of \( A^{-1} \) whose entries are integers.

4. AN EXAMPLE

Given the set of inequalities:

\[
\begin{align*}
&x_2 + 110x_1 \geq 5172 \\
&6x_2 + 663x_1 \leq 31171 \\
&5x_2 + 552x_1 \geq 25952 \\
&x_1 \geq 0
\end{align*}
\]

The corresponding convex set \( K \) is defined by

\[
\begin{bmatrix}
-5172 & 31171 & -25952 & 0 \\
1 & -6 & 5 & 0 \\
110 & -663 & 552 & 1
\end{bmatrix} \geq 0.
\]

Denote the above \( 3 \times 4 \) matrix by \( A \).

Conditions (a), (b) and (c) are satisfied.

The vertices of \( K \) are: \( \left( \frac{31171}{6}, 0 \right), \left( \frac{25952}{5}, 0 \right), \left( \frac{226}{3}, \frac{139}{3} \right) \) and \( (112, 46) \) showing that condition (d) is also satisfied.
We construct now the equations (4)

Out of \((1, 2, 3, 4) = (y_4, y_3, y_2, y_1)\) we find \(a_1 = y_1\) from the equation corresponding to the fourth column of \(A\); then \(-5172 + a_2 + 110y_1 = y_4\) from the equation corresponding to the first column of \(A\).

Thus \(a_1 = y_1\) and \(a_2 = 5172 - 110y_1 + y_4\).

Substituting in the other two equations we get

\[
y_3^2 = 31171 - 6(5172 - 110y_1 + y_4).
\]

or

\[
6y_4 + y_3 + .3y_1 = 139.
\]

Similarly, the remaining equation is found to be \(5y_4 - y_2 + 2y_1 = 92\).

We compute now the matrix \(A^{-1}\). One possible solution is:

\[
A^{-1} = \begin{bmatrix}
\frac{1}{47} & 251 & 0 \\
\frac{1}{47} & 20 & 0 \\
\frac{1}{47} & -26 & 0 \\
\frac{1}{47} & 2 & 1
\end{bmatrix}
\]

The last two columns of \(A^{-1}\) have integral entries so that the modular equations (5) are superfluous for this example. The resulting convex set \(K_1\) is therefore defined by the following constraints

\[
6y_4 + y_3 + .3y_1 = 139
\]

\[
5y_4 - y_2 + 2y_1 = 92
\]

\[
y_i \geq 0; \quad 1 \leq i \leq 4.
\]

Given any vector \((\hat{x}_2, \hat{x}_1)\) of integers in \(K\) the vector \((1, \hat{x}_2, \hat{x}_1)A = (\hat{y}_4, \hat{y}_3, \hat{y}_2, \hat{y}_1)\) is a vector of integers in \(K_1\). Given any vector of integers \((\hat{y}_4, \hat{y}_3, \hat{y}_2, \hat{y}_1)\) in \(K_1\), the vector \((\hat{x}_2, \hat{x}_1)\) defined by

\[
(\hat{y}_4, \hat{y}_3, \hat{y}_2, \hat{y}_1)A^{-1} = (1, \hat{x}_2, \hat{x}_1)
\]

is a vector of integers in \(K\). Thus, e.g., the vector \((20, 10, 14, 3)\) is in the set \(K_1\). The corresponding vector

\[
(20, 10, 14, 3)A^{-1} = (1, 4862, 3)
\]

reduces to the vector \((4862, 3)\) which is in the set \(K\).
Given two convex sets $K_1$ and $K_2$, $K_1$ will be called $I$-equivalent to $K_2$ if there exists a 1-1 mapping from $K_1$ onto $K_2$ such that points with integral coordinates in the first set are mapped to points with integral coordinates in the second set, and vice versa.

In the previous section we have shown how to reduce a convex set defined by a system of inequalities to an $I$-equivalent convex set defined by a system of equations over an expanded space. In this section we consider the dual problem, i.e., given a convex set defined by a system of equations, find an $I$-equivalent convex set defined by a system of inequalities over a reduced space.

5. A USEFUL PROCEDURE

For the sake of completeness we reproduce here a procedure, needed for our purpose, which was introduced and proved by this author in a previous paper [14].

Procedure 1. Given a diophantine linear equation

$$\sum_{i=1}^{n} a_i x_i = M$$

(6)

Construct a matrix $A$ with integral entries whose determinant is equal to $\pm 1$ and whose first column is equal to the vector $(a_n \ldots a_1)$.

The construction is as follows

1. Set $f_n = a_n$

2. Find $(f_i, s_i)$, $n-1 \geq i \geq 1$, satisfying

$$t_i f_{i+1} - s_i a_1 = \gcd(f_{i+1}, a_1) = f_i$$
3. Construct $A$ as follows:

3.1. The first column of $A$ is $(a_n \ldots a_1)$.

3.2. The second column of $A$ is $(s_{n-1}, t_{n-1}, 0 \ldots 0)$.

3.3. For the $i$-th column $i > 2$ set

\[ a_{ij} = a_{j-1} s_{n-j+1}/f_{n-j+2} \quad \text{for } j < i \]
\[ a_{jj} = t_{n-j+1} \]
\[ a_{ji} = 0 \quad \text{for } j > i \]

E.g. for $n = 4$

\[
A = \begin{bmatrix}
  a_4 & s_3 & a_4 s_2/f_3 & a_4 s_1/f_2 \\
  a_3 & t_3 & a_3 s_2/f_3 & a_3 s_1/f_2 \\
  a_2 & 0 & t_2 & a_2 s_1/f_2 \\
  a_1 & 0 & 0 & t_1
\end{bmatrix}
\] (7)

It was proved in [ ] that $A$ has the required properties.

Notice that if for some $i > 1$, $f_i = 1$ then $t_j = 1$ and $s_j = 0$ for all $j > i$.

Step 2 in this procedure is done via the Euclidean algorithm and requires at most $n-2+5$ length $(a_1)$ iterations assuming that $a_1$ is the smallest entry among the $a_i$'s.

Due to the fact that the determinant of $A$ is $\pm 1$ we have that the determinant of the inverse matrix $A^{-1}$ is also equal to $\pm 1$ and all the entries of $A^{-1}$ are integers.

In fact $A^{-1}$ can be defined symbolically based on the values $(t_i, s_i)$ computed in Step 2 of the above procedure.
E.g., for \( n = 5 \)

\[
A^{-1} = \begin{bmatrix}
t_4t_3t_2t_1 & -s_4t_3t_2t_1 & -s_3t_2t_1 & -s_2t_1 & -s_1 \\
-\frac{a_4}{f_4} & \frac{a_5}{f_4} & 0 & 0 & 0 \\
-\frac{t_4a_3}{f_3} & \frac{s_4a_3}{f_3} & f_4/t_3 & 0 & 0 \\
-\frac{t_4a_2}{f_2} & \frac{s_4a_2}{f_2} & s_3a_2/f_2 & f_3/f_2 & 0 \\
-\frac{t_4a_1}{f_2} & \frac{s_4a_1}{f_2} & s_3a_1/f_2 & s_2a_1 & f_2 \\
\end{bmatrix}
\]

(8)

6. SECOND REDUCTION THEOREM

We are able now to prove the following

**Theorem:** Given a convex set \( K \) represented by a system as defined below

\[
\sum_{j=n}^{1} u_{ij}x_j + b_i \geq 0 \quad 1 \leq i \leq m \quad (9.1)
\]

\[
\sum_{j=n}^{1} u_{ij}x_j + b_i = 0 \quad m < i \leq m_1 \text{ with } m_1 > m. \quad (9.2)
\]

An \( I \)-equivalent convex set \( K_1 \) can be found such that \( K_1 \) is represented by a system having the following form

\[
\sum_{j=n-1}^{1} c_{ij}y_j + d_i \geq 0 \quad 1 \leq i \leq m \quad (10.1)
\]

\[
\sum_{j=n-1}^{1} c_{ij}y_j + d_i = 0 \quad m < i \leq m_1-1 \quad (10.2)
\]

where all the \( a's, b's, c's \) and \( d's \) are integers and the set (10.2) of equations in the representation of \( K_1 \) is void if \( m_1 = m+1 \).

**Proof** We show first how to construct \( K_1 \) and then prove that \( K_1 \) is \( I \)-equivalent to \( K \).

1. Delete the last equation from (9.2) and represent the set (9.1) and the remaining set of equations in (9.2) in the form
\[(1, x_n, \ldots, x_1)U \geq 0, \quad U \text{ is } (n+1) \times m \quad (11.1)\]
\[(1, x_n, \ldots, x_1)Y = 0, \quad Y \text{ is } (n+1) \times (m_1-m-1) \quad (11.2)\]

2. Apply procedure 1 described above to the last equation in the set (9.2)
\[\sum m_{j} x_j + b_{m_1} = 0, \quad (M = \pm b_{m_1}) \quad (12)\]
resulting in the matrix \(A\) which is \(n \times n\) and has its first column equal to the coefficients of the equation in (12).

3. Find \(A^{-1}\) and multiply its first row by the value \(-b_{m_1}\) to get a new matrix \(\hat{A}\).

4. Set
\[U_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (13)\]

\(U_1\) is \(n \times m\) and \(V_1\) is \(n \times (m_1-m-1)\) and both matrices have integral entries.

5. The required \(I\)-equivalent convex set \(K_1\) is now defined by
\[(1, y_{n-1}, \ldots, y_1)U_1 \geq 0 \quad (14.1)\]
\[(1, y_{n-1}, \ldots, y_1)V_1 = 0 \quad (14.2)\]

and the transformation between the two sets is defined by:

given \((x_n, \ldots, x_1)\), define \((y_{n-1}, \ldots, y_1)\) by
\[(x_n, \ldots, x_1)A = (\pm b_{m_1} y_{n-1}, \ldots, y_1).\]

given \((y_{n-1}, \ldots, y_1)\), define \((x_n, \ldots, x_1)\) by
\[(x_n, \ldots, x_1) = (1, y_{n-1}, \ldots, y_1) \hat{A}.\]
To show that $K$ is $I$-equivalent to $K_1$ assume first that

$$(x_n \ldots x_1) = \tilde{x} \in K,$$

with the $x_i$'s integers. Then

$$\bar{x}A = (\sum_{j=1}^{n} m_j x_j y_{n-1}, \ldots, y_1) = (-b_{m_1}, y_{n-1}, \ldots, y_1) = (-b_{m_1}, \bar{y}),$$

where the $y$'s are integers. This follows from the construction of $A$, and from the fact that $\bar{x}$ satisfies (12). ($\bar{x} \in K$). Therefore

$$\bar{x} = (-b_{m_1}, \bar{y})A^{-1} = (1, \bar{y})\hat{A},$$

by the definition of $\hat{A}$.

$\bar{x} \in K$ also implies that

$$(1, \bar{x})U \geq 0$$

or

$$(1, \bar{x})V = 0$$

which implies that

$$(1, \bar{y})\begin{bmatrix} 1 & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \end{bmatrix} U = (1, \bar{y})U_1 \geq 0$$

and

$$(1, \bar{y})\begin{bmatrix} 1 & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots \end{bmatrix} V = (1, \bar{y})V_1 = 0$$

showing that $\bar{y} \in K_1$.

Assume now that $$(y_{n+1} \ldots y_1) = \bar{y} \in K_1.$$ Reversing the above set of implications we then get that

$$(1, (1, \bar{y})\hat{A})U \geq 0$$

and

$$(1, (1, \bar{y})\hat{A})V = 0.$$

or

$$(1, (-b_{m_1}, \bar{y})A^{-1})U \geq 0$$

and

$$(1, (-b_{m_1}, \bar{y})A^{-1})V = 0.$$

Setting $\tilde{x} = (-b_{m_1}, \bar{y})A^{-1} = (1, \bar{y})\hat{A}$ we get that

$$(1, \tilde{x})U \geq 0$$

and

$$(1, \tilde{x})V = 0.$$

(15)
We have also $\tilde{x}A = (-b_{m_1}, y)$. From the construction of $A$ we know that $\tilde{x}A = (\xi_u m_1 x_j, y)$. Thus $\xi_u m_1 x_j = -b_{m_1}$ which together with (15) implies that $\tilde{x} \in K$. If $y$ is a vector of integers, then by the definition of $\tilde{x}$ and by the construction of $A$, $\tilde{x}$ is a vector of integers. The proof is now complete.

Corollary. Given a convex set $K$ represented by a system as defined below with integral coefficients

$$\sum_{j=1}^{n} u_{ij} x_j + b_i \leq 0 \quad 1 \leq i \leq m \tag{16.1}$$

$$x_j \geq 0 \quad 1 \leq j \leq n, \ n > m \quad \tag{16.2}$$

An $I$-equivalent convex set $K_1$ can be found such that $K_1$ is represented by a system having the form

$$\sum_{j=1}^{n-m} c_{ij} y_j + d_i \geq 0 \quad 1 \leq i \leq \hat{n} \tag{17}$$

with integral coefficients.

Moreover, the vertices of $K_1$ are in the positive orthant of the $(n-m)$-dimensional Euclidean space.

Proof. Rewrite (16) to comply with the representation of $K$ in (9):

$$\sum_{j=1}^{n} u_{ij} x_j + b_i \geq 0 \quad 1 \leq i \leq n$$

with

$$u_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases} \quad \tag{18.1}$$
and

\[ b_1 = 0 \]

\[
\sum_{j=n+1}^{m+1} u_{i,j} x_j + b_1 = 0 \quad n+1 \leq i \leq n+m
\]  \hspace{1cm} (18.2)

Apply now the construction given in the proof of the theorem to the above system successively until all the equations (18.2) are removed.

The matrices \( A \) reducing the convex set \( K \) to the convex set \( K_1 \) have all their entries nonnegative, thus preserving the positive orthant, and the given set (18) is in the positive orthant by definition. This proves the second statement of the corollary.

An Example. Given the convex set defined by the following equations

\[
\begin{align*}
5x_4 + x_3 + x_2 &- 7 = 0 \\
2x_4 + 5x_3 + 3x_2 + x_1 - 21 &= 0 \\
3x_4 + 2x_3 + x_2 + x_1 - 18 &= 0 \\
x_1 &\geq 0
\end{align*}
\]  \hspace{1cm} (19)

**First reduction:** The matrix \( \hat{A} \) with regard to the third equation is found to be:

\[
\hat{A}_1 = \begin{bmatrix}
18 & -18 & 0 & 0 \\
-2 & 3 & 0 & 0 \\
-1 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}
\]

The set (19) is therefore \( I \)-equivalent to the set

\[
[1, y_3, y_2, x_1] \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow (1, y_3, y_2, y_1) \hat{A}_1 \succeq 0.
\]  \hspace{1cm} (20.1)
together with

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & A_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
5 \\
1
\end{bmatrix}
= \begin{bmatrix}
-7 & -21 \\
5 & 2 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
65 \\
-7 \\
-3
\end{bmatrix}
= 0
\]

(20.2)

Second reduction: The matrix \( \hat{A} \) with regard to the second equation in (20.2) is found to be

\[
\hat{A}_2 = \begin{bmatrix}
-75 & 150 & 0 \\
-6 & 11 & 0 \\
4 & -8 & 1
\end{bmatrix}
\]

The set (20) is therefore I-equivalent to set

\[
(1, z_2, z_1) \hat{A}_1 = (1, z_2, z_1) \begin{bmatrix}
18 & -95 & 150 & 0 \\
1 & -7 & 11 & 0 \\
-1 & 8 & 8 & 1
\end{bmatrix} \succ 0
\]

(21.1)

together with

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \hat{A}_2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
65 \\
-7 \\
-3
\end{bmatrix}
= \begin{bmatrix}
140 \\
9 \\
8
\end{bmatrix}
= 0
\]

(21.2)

Third reduction: The matrix \( \hat{A} \) with regard to the remaining equation is found to be

\[
\hat{A}_3 = \begin{bmatrix}
-140 & -140 \\
8 & 9
\end{bmatrix}
\]
The set (21) is therefore I-equivalent to the set,

\[
(1,w) \begin{bmatrix}
1 \\
A_3 \\
0
\end{bmatrix}
\begin{bmatrix}
18 & -93 & 1500 \\
-7 & 110 \\
5 & 81
\end{bmatrix}
= (1,w) \begin{bmatrix}
18 & 187 & -270 & -140 \\
-1 & -11 & 16 & 9
\end{bmatrix} \geq 0 \quad (22)
\]

For any solution \( \hat{w} \) satisfying (22) the corresponding solution to the original set of equations is found by multiplying \((1,\hat{w})\) by the matrix

\[
\begin{bmatrix}
18 & 187 & -270 & -140 \\
-1 & -11 & 16 & 9
\end{bmatrix}
\]

computed above.

One finds easily that the only solution satisfying (22) is \( w = 17 \) so that the original set (19) has the corresponding solution

\[
x_4 = 18-17 = 1, \quad x_3 = 187-17*11 = 0
\]

\[
x_2 = -270+17*11 = 2, \quad x_1 = -140+17*9 = 13
\]

and this is the only solution of (19).
BIBLIOGRAPHY


