COMPACTNESS, EMBEDDINGS AND DEFINABILITY

by

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CHAPTER 18:

COMPACTNESS, EMBEDDINGS and DEFINABILITY.

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Chapter dependency:
Prerequisite: Chapters 2,3
Material used also from chapters 4,8,9,17.

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INTRODUCTION.

This chapter presents an overview of the author's joint work with S. Shelah in abstract model theory, which had started as early as in 1972. It is mainly based on our papers [Makowsky-Shelah-Stavi 1978] and [Makowsky-Shelah 1979, 1981, 1983] and on an unpublished manuscript of S. Shelah, which he wrote while this chapter came into being. The present exposition, however, tries to give a more coherent picture by putting all our results into a single perspective together with results of M. Magidor; H. Manilla, D. Mundici and J. Stavi.

The main theme of this chapter is abstract model theory proper, especially the relationship between various compactness, embedding and definability properties which do not characterize first order logic. More precisely, we look at various classes of logics defined axiomatically, such as compact logics, logics satisfying certain model existence or definability properties. The classes of logics are sometimes further specified by set theoretic parameters, such as finitely generated, absolute, set presentable, bounds on the size function, or by set theoretic assumptions such as large cardinal axioms. Within such classes of logics we want to explore, what other properties of logics follow from the axiomatic description of the class. In chapter 3 first order logic was characterized in this way. In chapter 17 the class of absolute logics was studied. Most of the other chapters (with the exception of chapters 19 and 20) study families of logics which bear some inherent similarity which stems from the way they evolved, such as infinitary logics or logics based on cardinality quantifiers, and establish particular model theoretic results for those logics. In this chapter we want to clarify the conceptual and metalogical relationship between these model theoretic properties. Success in this program can be achieved in three ways: By establishing non-trivial connections between these properties; by applying the former to gain new insight about particular logics previously studied; and by using this insight to construct new examples of logics, and ultimately, by showing, that our list of examples is, in some reasonable sense, exhaustive.

The chapter consists of four sections, in each of which one aspect of abstract model theory is developed to a certain depth.

Section 1 is devoted to compactness properties and is almost self-contained. Its main results are the Abstract Compactness Theorem and the description of the Compactness Spectrum. Here a thorough understanding of various compactness phenomena is obtained and the theory is provided with new examples. Especially the examples described in section 1.6, play an important role in the successive sections as well.

Section 2 is devoted to the study of the dependence number. Its main result is the Finite Dependence Theorem, the proof of which is given completely on the basis of three lemmas, which are only stated. The complete proof may be found in [Makowsky-Shelah 1983]. The Finite Dependence Theorem clarifies how little compactness is needed to ensure that a logic is equivalent to a logic which has the Finite Dependence Property. In fact, assuming there are no uncountable measurable cardinals, [ω]-compactness suffices. Finally the dependence structure is introduced, a concept which appears here for the first time. It is the appropriate generalization of the
dependence number, as the examples and the Finite Dependence Structure Theorem show.

Section 3 is devoted to various aspects of embeddings, whose existence is implied by the compactness theorem; such as proper extensions, amalgamation and joint embeddings. Joint embeddings are also discussed in chapter 19 and amalgamations in chapter 20. The main result here is the connection between \([\omega]\)-compactness and proper extensions and the Abstract Amalgamation Theorem. Again, this section is rather self-contained. The Abstract Amalgamation Theorem lead also to the discovery that various logics with cardinality quantifiers do not satisfy the Amalgamation Property, which solved a problem which had been stated explicitly in [Malitz-Reinhard 1972].

Section 4, finally, is devoted to definability properties, as introduced already in chapter 2.7, and to preservation properties. Preservation properties for sum-like operations already played an important role in chapters 12 and 13. A common generalization of these two properties, the Uniform Reduction Property, was introduced in [Feferman 1974, FM]. The first two subsections are devoted to an exposition of these properties and their interrelations. The main results here are the equivalence of the Uniform Reduction Property \(UR\) with the Interpolation Property and the equivalence, for compact logics, of the Pair Preservation Property and the Uniform Reduction Property for Pairs. The Robinson Property and especially its weaker versions, the Finite Robinson Property and the Weak Finite Robinson Property are the topic of the next three subsections. In chapter 19 the Robinson Property is studied further.

Our main result here are: The Finite Robinson Property together with the Pair Preservation Property implies that a logic is ultimately compact, and therefore has the Finite Dependence Property, provided that there are no uncountable measurable cardinals. The Beth Property together with the Tree Preservation Property implies the Weak Finite Robinson Property and the Robinson Property together with the Pair Preservation Property implies the existence of models with arbitrarily large automorphism groups. The last subsection discusses more examples, in particular a compact logic which satisfies the Beth Property, the Pair Preservation Property, but not the Interpolation Property.

Measurable cardinals play an important role in our presentation. They are in some sense \(L\)-compact cardinals, which is to say, if such a cardinal \(\mu\) exists then every finitely generated logic is, stationary often, weakly compact below \(\mu\). The first cardinal for which a logic is \([\mu]\)-compact is always measurable (or \(\omega\)). But measurable cardinals, of which the first could conceivably be as big as the first strongly compact cardinal, appear also frequently in the hypotheses of various of our theorems. They also appear in various examples and counterexamples and sometimes there existence turns out to be equivalent to certain assumptions in abstract model theory.

In the same sense, it turns out, Vopenka's principle is a compactness axiom: It is equivalent to the statement that every finitely generated logic is ultimately compact or, alternatively, that every finitely generated logic has a global Hanf number. We have not centered our presentation around this theme, but the reader will easily
Finally, a word on future research. Some of the possible directions of future research in abstract model theory are outlined in chapters 19 and 20. The purpose there is to get away from the syntactic aspects of logic completely and to study classes of structures more in the spirit of universal algebra. If we want to stay in the framework of abstract model theory and logics I can see three directions to pursue further research:

The first one is to study, what we have rather neglected in this chapter, the impact of various axiomatizability and dependence properties of logics on their respective model theory. We know that axiomatizability implies recursive compactness. But we do not know, for instance, if there are any model theoretic properties distinguishing axiomatizable logics from logics axiomatizable by a finite set of axiom schemas. Only recently, in [Shelah-Steinhorn 1983], it is shown that the logic $L_{\omega_1\omega}(\text{Beth}_\omega)$ is an axiomatizable logic which cannot be axiomatized by schemas. This was the first example of its kind. Similarly, we know that $[\omega]$-compactness implies the Finite Dependence Property (assuming there are no uncountable measurable cardinals), but we have not investigated, if other model theoretic properties, such as Lowenheim or Hanf numbers, have similar effects. The same holds for the Finite Dependence Structure and dependence filters, as discussed in section 2.4.

The second one is the search for more model theoretic properties which fit into the abstract framework. In section 4.5, an attempt in this direction is presented: the existence of models with large automorphism groups. Incidentally, this also gives us a new proof for the case of first order logic. In [Shelah 1983 Manuscript] a host of new notions occur in his study of Beth closures of logics preserving compactness and preservation properties. There is a danger here of proving theorems which apply only to first order logic, such as compactness and chain properties imply the Robinson Property. Since it is open whether there are logics satisfying both the Robinson Property and the Pair Preservation Property, the results in 4.5 should be taken with a grain of salt.

The third direction consists in incorporating the theory of second order quantifiers, as presented in chapter 12, into the study of the model theoretic properties as presented in this chapter. What are the compact second order quantifiers, what are the second order quantifiers satisfying preservation and definability properties, etc. I am convinced that abstract model theory will remain a fruitful area of active research for many years to come.

We have not included detailed historical notes. Most of the results presented in this chapter are taken from my joint papers with S.Shelah and from his unpublished manuscript mentioned above. Some of the theorems and corollaries were stated here for the first time as a result of reflection upon the material presented. Results which appear here for the first time in print are marked with an asterisk. Whenever possible, we refer to the other chapters in the book rather than to original papers.
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1. COMPACT LOGICS

1.1 $[\kappa, \lambda]$-compactness.

In this section we study compactness properties of abstract logics. Traditionally one looks at a set $\Sigma$ of sentences of cardinality $\kappa$ such that every subset $\Sigma_0 \subset \Sigma$ of cardinality less than $\lambda$ has a model and concludes that $\Sigma$ has a model. This is called $(\kappa; \lambda)$-compactness. By abuse of notation we write $(\omega, \omega)$-compactness instead of $(<\omega, \omega)$-compactness. We call $(<\omega, \omega)$-compactness just compactness.

In contrast to this we look at two different situations:

(*) Given two sets of sentences $\Delta$ and $\Sigma$ with $\text{card}(\Sigma) = \kappa$, $\text{card}(\Delta)$ arbitrary and such that for every subset $\Sigma_0 \subset \Sigma$ of cardinality less than $\lambda$, $\Sigma_0 \cup \Delta$ has a model. Then $\Sigma \cup \Delta$ has a model.

(**) Given a family $\Gamma_\alpha (\alpha, < \kappa)$ of sets of sentences such that for every set $X \subset \kappa$ of cardinality less than $\lambda$, the union $\bigcup_{\alpha \in X} \Gamma_\alpha$ has a model. Then $\bigcup_{\alpha \in \kappa} \Gamma_\alpha$ has a model.

1.1.1. Proposition: For a regular logic $L$ properties (*) and (**) are equivalent.

Proof: $(*) \rightarrow (**)$: Let $P_\alpha (\alpha < \mu)$ be unary predicates not in $\bigcup_{\alpha \in \kappa} \Gamma_\alpha$ and let $\psi$ be the formula $\exists x P_\alpha (x)$. Now we put

$$\Delta = \{ \exists \alpha < \kappa. \forall \sigma \in \Gamma_\alpha \}$$

and

$$\Sigma = \{ \forall \alpha < \kappa. \psi \}$$

Clearly $\Delta \cup \Sigma_0$ has a model iff $\bigcup_{\alpha \in \kappa} \Gamma_\alpha$ has a model.

(**) $\rightarrow$ (*): Let $\{ \psi_\alpha : \alpha < \kappa \}$ be an enumeration of the formulas of $\Sigma$ and put

$$\Gamma_\alpha = \Delta \cup \{ \psi_\alpha \}$$

QED.

1.1.2. Remark:

(*) was first systematically studied in [Makowsky - Shelah, 1979], and in [Makowsky - Shelah, 1983]. (***) was introduced for topological spaces in [Alexandroff - Urysohn 1929], as was pointed out to us by H. Manilla. (*) was called first relative $(\kappa, \lambda)$-compact and then $(\kappa, \lambda)^{*}$-compact. (***) is called in the topological literature $(\kappa, \lambda)$-compact.

(II) The motivation behind (*) stems from working with elementary extensions and with diagrams: $\Delta$ usually plays the role of a `diagram,' and $\Sigma$ describes the properties the extension should have. A similar situation occurs in [Chang-Keisler 1973, exercise 4.3.22].

1.1.3. Definition: A regular logic $L$ with property (i), or (ii) is called $[\kappa, \lambda]$-compact. If
\( \kappa = \lambda \) we simply write \([\kappa] - \text{compact}\).

1.1.4. Examples:
(i) \( L(Q, \omega) \) is \((\omega, \omega) - \text{compact but not } [\omega] - \text{compact}\).
(ii) \( ([\text{Bell-Slimson: 1969, Theorem 2.2 \text{ p 263}}]) \) If \( \kappa \) is small for \( \lambda \), then \( L(Q, \kappa) \) is \([\kappa, \omega] - \text{compact}\). In particular \( \omega \) is small for \((2^\kappa)^+\).

Recall that \( \kappa \) is small for \( \lambda \) if for every family \( \mu_i (i < \kappa) \) such that \( \mu_i < \lambda \).

1.1.5. Definition: We write \([\kappa, \lambda] \rightarrow [\mu, \nu] \) whenever \([\kappa, \lambda] - \text{compactness implies } [\mu, \nu] - \text{compactness}\). Similarly for conjunctions of compactness properties implying other such properties:

The following lemma collects some simple but useful facts:

1.1.6. Lemma:
(i) \([\kappa, \lambda] \rightarrow [\mu, \lambda] \) for \( \mu < \kappa \).
(ii) \([\kappa, \lambda] \rightarrow [\kappa, \nu] \) for \( \nu > \lambda \).
(iii) \([\mu] \land [\kappa, \mu^+] \rightarrow [\kappa, \mu] \).
(iv) \([\kappa^+] \land [\kappa, \mu] \rightarrow [\kappa^+, \mu] \).
(v) If \([\beta] \) and for every \( \alpha < \beta \) \([\kappa, \alpha] \) and \([\kappa, \mu] \) then \([\sum_{\alpha < \beta} \kappa, \mu] \).
(vi) \( [cf(\kappa)] \rightarrow [\kappa] \).

Proof:
Trivial for (i) and (ii).
(iii), (iv) and (v) follow from definition (i).
(vi) follows from definition (ii). QED

1.1.7. Proposition:
(i) A logic \( L \) is \([\kappa, \lambda] - \text{compact iff } L \) is \([\mu] - \text{compact for every } \mu, \lambda \leq \kappa \).
(ii) A logic \( L \) is \([=, \kappa] - \text{compact iff } L \) is \( (=, \kappa) - \text{compact} \).

Proof: For (i) we use lemma 1.1.6. and (ii) follows from definition. QED.

[Manilla 1982] has investigated what results from topology give us refinements of theorem 1.1.7. He showed that results from [Alexandroff-Urysohn 1929] and [Vaughan, 1975] can be translated into our framework and one obtains

1.1.8. Proposition:
(i) A logic \( L \) is \([\kappa, \omega] - \text{compact iff } L \) is \([\mu] - \text{compact for every regular } \mu, \omega \leq \kappa \).
(ii) Assume \( cf(\kappa) \geq \omega \). A logic \( L \) is \([\kappa, \lambda] - \text{compact iff } L \) is \([\mu, \lambda] - \text{compact for every regular } \mu, \lambda \leq \kappa \).

Proposition 1.1.8. was first stated in [Makowsky-Shelah 1983], where it was derived from lemma 1.1.6.

Using the methods developed in 1.3, 1.4. this can be sharpened to:

1.1.9. Theorem: Let \( \lambda \) be a cardinal and \( L \) a logic. The following are equivalent:
(i) \( L \) is \([\mu] - \text{compact for every regular } \mu \leq \lambda \).
(ii) \( L \) is \([\mu] - \text{compact for every } \mu \geq \lambda \).
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(iii) \(L\) is \([\kappa, \lambda]\)-compact.

(iv) \(L\) is \([\kappa, \lambda]\)-compact.

Proof: (ii) implies (iii) by 1.1.6.(i), (iii) is equivalent to (iv) by 1.1.7.(i) and (iii) implies (i) by 1.1.6.(i) and 1.1.6.(ii). So we have to prove that (i) implies (ii). Assume (i) and that \(\lambda\) is singular. So \(L\) is \([\lambda^+]\)-compact. Now we use the Abstract Compactness Theorem 1.3.9(ii) which gives us a uniform ultrafilter \(F\) on \(\mu\). By 1.3.11(i) \(F\) is \([\lambda^+, \lambda]\)-regular, so by 1.3.9.(i) \(L\) is \([\lambda^+, \lambda]\)-compact, and therefore \([\lambda]\)-compact. QED.

We have put this proof here, though it uses material from section 1.3., to illustrate the power of the Abstract Compactness Theorem, which gives rise to various transfer results. We shall see more transfer results in section 1.5.

We shall call logics \(L\) satisfying any of equivalent properties above \textit{ultimately compact}.

1.2. Cofinal extensions:

One useful tool for the study of \([\kappa]\)-compactness is its characterization via the non-characterizability of certain ordered structures. In chapter 2 proposition 5.2.4 we have seen the paradigm of this procedure: A logic \(L\) is \([\kappa, \omega]\)-compact iff its well-ordering number is \(\omega\). Here the well-ordering is replaced by the cofinality of some linear order.

1.2.1. Definition:

(i) Let \(A\) be an expansion (possibly with new sorts) of the structure \(\langle \kappa, \langle \rangle \rangle\) and \(B\) an \(L\)-extension of \(A\). \(B\) extends \(A\) beyond \(\kappa\) if there is an element \(b \in B \cap \text{dom}(\langle \rangle)\) such that for every \(a \in A \cap \text{dom}(\langle \rangle)\), \(B \models a \prec \langle b \rangle\). If there is no such element, we call \(B\) a cofinal extension of \(A\).

(ii) Let \(L\) be a logic and \(\kappa\) a regular cardinal, \(L\) cofinally-characterizes \(\kappa\) or \(\kappa\) is cofinally characterizable in \(L\) if there exists an expansion \(A\) (possibly many - sorted with additional sorts) of the structure \(\langle \kappa, \langle \rangle \rangle\) such that every \(L\)-extension \(B\) of \(A\) is a cofinal extension of \(A\).

1.2.2. Theorem: Let \(\kappa\) be a regular cardinal. A logic \(L\) is \([\kappa]\)-compact iff \(\kappa\) is not cofinally characterizable in \(L\).

Proof: Like in chapter 2, proposition 5.2.4. QED.

Theorem 1.2.2. gives a quick proof of lemma 1.1.6.(iv). It can be used, together with a classical result due to Rabin and Keisler [Keisler 1964] (cf. also [Chang-Keisler 1973, theorem 6.4.5]), to study the existence of \(L\)-maximal structures.

Recall that a complete structure \(A\) is a one-sorted structure where every subset \(X \subseteq A^n\) is the interpretation of some relation symbol \(R_X\). In the case of many-sorted structures we have to allow also relations with mixed arities.

1.2.3. Theorem (Rabin-Keisler)

Let \(A\) be a complete structure of cardinality \(\lambda < \text{first uncountable measurable cardinal}\), \(P^A\) be a countable predicate of \(A\) and \(B\) be a proper \(L_{\mu, \lambda}\)-extension of \(A\). Then \(P^A \subseteq P^B\).
One can now easily prove from 1.2.2, and 1.2.3, a generalization of a result of [Malitz-Reinhart 1972] and independently [Shelah 1987 Master thesis]:

1.2.4. **Proposition:** If a logic \( L \) is not \([\omega]\)-compact then there are arbitrarily large \( L \)-maximal structures of cardinality less than the first uncountable measurable cardinal.

Recall a structure is \( L \)-maximal if it has no proper \( L \)-extensions. \( L \)-extensions are further studied in section 3.

### 1.3. Ultrafilters, ultrapowers and compactness

In first order logic compactness is intimately related to the ultrapower construction. One can turn this observation easily into a characterization theorem for \( L_{\omega,\omega} \).

1.3.1. **Definition:** Let \( L \) be a logic. \( L \) is said to have the **Los property** if for every family of \( \tau \)-structures \( A_t (t \in I) \) and every ultrafilter \( F \) and every formula \( \varphi \in L[\tau] \)

\[ \prod_{t \in I} A_t / F = \varphi \iff \{ t \in I : A_t \models \varphi \} \in F. \]

1.3.2. **Theorem:** Let \( L \) be a regular logic which has the Los property. Then \( L = L_{\omega,\omega} \).

**Proof:** By coding a family of structures in one structure and using the Keisler-Shelah theorem, that elementarily equivalent structures have isomorphic ultrapowers, the proof is straightforward. QED.

1.3.3. **Remark:** Theorem 1.3.2 was folklore already around 1972. A detailed version may be found in [Sgro 1977] and [Monk 1976, exercise 25.53].

To study compactness for abstract logics we need a generalization of the Los property.

1.3.4. **Definitions:**

(i) Let \( L \) be a logic and \( F \) be an ultrafilter over \( I \). We say that \( F \) relates to \( L \) if for every \( \tau \) and for every \( \tau \)-structure \( A \) there exists a \( \tau \)-structure \( B \) extending \( \prod_{t \in I} A_t / F \) such that for every formula \( \varphi \in L[\tau], \varphi = \varphi(\bar{x}_1, x_2, \ldots, x_k), i < \alpha \) with a many free variables and every \( f_i \in A_t, i < \alpha \) we have:

\[ B \models \varphi(f_1 / F, f_2 / F, \ldots, f_k / F, \ldots) \]

iff

\[ \{ t \in I : A_t \models \varphi(f_1 (t), f_2 (t), \ldots, f_k (t), \ldots) \} \in F. \]

(ii) We define \( UF(L) \) to be the class of ultrafilters \( F \) which are related to \( L \).

1.3.5. **Remark:** Note that \( B \) is always an elementary extension of \( \prod_{t \in I} A_t / F \).

1.3.6. **Examples:**

(i) Every ultrafilter is in \( UF(L_{\omega,\omega}) \).
(ii) Let \( L \) be \( \mathcal{L}_\omega(Q) \), i.e. first order logic with the additional quantifier "there exist at least \( k \) many". Then every ultrafilter on \( \omega \) is related to \( L \), provided \( \omega \) is small for \( k \).

1.3.7. **Proposition:** \( L \) is compact iff every ultrafilter is related to \( L \).

**Proof:** Let \( \mathbf{M} \) be an \( \tau \)-structure and \( F \) an ultrafilter on a set \( I \). For every \( f \in \mathbb{M}^I \) let \( \phi \) be a new constant symbol not in \( \tau \). Put

\[
T = \{ \phi(c_{x_1, x_2, \ldots}) : \phi \in L[\tau] \text{ and } t \in I : \mathbf{M} \models \phi(f_1(t), f_2(t), \ldots) \} \in F.
\]

If \( L(\tau) \) is a set, so is \( T \) and obviously every finite subset of \( T \) has a model. We just expand \( \mathbf{M} \) appropriately. So let \( \mathbf{N} \) be a model of \( T \). Clearly

\[
\prod_{I} \mathbb{M}/F \subseteq \mathbf{N}
\]

and by the definition of \( T \), \( \mathbf{N} \) satisfies the requirements for \( F \in \text{UF}(L) \).

In the case \( L(\tau) \) is a proper class, we have to take a subclass \( \bar{T} \) of \( T \) which is a set and still guarantees that

\[
\prod_{I} \mathbb{M}/F \subseteq \mathbf{N}
\]

and that \( \mathbf{N} \) satisfies the requirements for \( F \in \text{UF}(L) \). For this we observe that over the structure \( \mathbb{M}^I \) there are only set many inequivalent formulas with less than \( \text{card}(\mathbb{M})^+ \)-many free variables.

The converse is trivial. QED.

The next theorem connects the compactness spectrum \( \text{Comp}(L) \) with the filters in \( \text{UF}(L) \). To be more explicit, we need some more definitions.

1.3.8. **Definitions:** Let \( F \) be an ultrafilter on \( I \), and \( \lambda, \mu \) be regular cardinals with \( \lambda \geq \mu \).

(i) \( F \) is said to be \((\lambda, \mu)\)-regular if there is a family \( \{ X_\alpha : \alpha < \lambda \} \subseteq F \) such that if \( \{ \alpha : \alpha < \lambda \} \) is any enumeration of \( \mu \) subsets of \( \lambda \), then \( \bigcap X_\alpha = \emptyset \). The family \( \{ X_\alpha : \alpha < \lambda \} \) is called a \((\lambda, \mu)\)-regular family.

(ii) \( A : (\lambda, \omega) \)-regular ultrafilter on \( \lambda \) is called regular.

(iii) \( F \) is \( \lambda \)-descendingly incomplete if there exists a family \( \{ X_\alpha : \alpha < \lambda \} \subseteq F \) with \( X_\alpha \subseteq X_\beta \) for \( \alpha < \beta < \lambda \) such that \( \bigcap X_\alpha = \emptyset \).

(iv) \( F \) is uniform on \( \lambda \) if every \( X \subseteq F \) has cardinality \( \lambda \).

1.3.9. **Theorem (Abstract Compactness Theorem):**

Let \( \lambda, \mu \) be cardinals, \( \lambda \geq \mu \), and let \( L \) be a logic:

(i) \( L \) is \((\lambda, \mu)\)-compact iff there is a \((\lambda, \mu)\)-regular ultrafilter \( F \) on \( I = F(\mu, \lambda) \) in \( \text{UF}(L) \)

(ii) If \( \lambda = \mu \) and \( \mu \) regular, then \( L \) is \([\lambda] \)-compact iff there is a uniform ultrafilter \( F \) on \( \lambda \)

in \( \text{UF}(L) \)

The proof of this theorem is delayed to section 1.4.
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Theorem 1.3.9. allows us to use known results from the theory of ultrafilters to understand $[\lambda, \mu]$-compactness. The following lemma collects some simple results from (but not due to) [Comfort and Negrepontis 1974].

1.3.10. Lemma:
(i) If $F$ is $(\lambda, \mu)$-regular and $\mu \leq \mu_1 \leq \lambda$, then $F$ is $(\lambda, \mu_1)$-regular.
(ii) If $\lambda$ is a regular cardinal and $F$ is $\lambda$-descendingly incomplete, then $F$ is $(\lambda, \lambda)$-regular.
(iii) If $F$ is uniform on $\lambda$ the $F$ is $(\lambda, \lambda)$-regular.
(iv) If $F$ is $(c F(\lambda), c F(\lambda))$-regular then $F$ is $(\lambda, \lambda)$-regular.

The abstract compactness theorem and lemma 1.3.10. give us immediately the corresponding statements in lemma 1.1.6.

The next lemma collects some more sophisticated theorems from the literature on ultrafilters. For 1.3.11, (ii) one may also consult [Comfort-Negrepontis 1974, theorem 8.36].

1.3.11. Lemma:
(i) [Kanamori 1978] If $F$ is uniform on $\lambda^+$ and $\lambda$ is singular, then $F$ is $(\lambda^+, \lambda)$-regular.
(ii) [Kunen-Prikry 1971 and Cudnovskii-Cudnovskii 1971] If $F$ is uniform on $\lambda^+$ and $\lambda$ is regular, then $F$ is $\lambda$-descendingly incomplete, and hence $(\lambda, \lambda)$-regular.

This lemma, together with the Abstract Compactness Theorem, is the key to the study of the compactness spectrum in sections 1.5 and 1.6. It is also used in the proof of theorem 1.1.9.

1.4. Proof of the Abstract Compactness Theorem.

Before we prove the Abstract Compactness Theorem we shall give a model theoretic characterization of $(\lambda, \mu)$-regular ultrafilters which will give us the link between $[\lambda, \mu]$-compactness and the existence of $(\lambda, \mu)$-regular ultrafilters. This is implicitly in [Keisler 1987] (cf. also [Comfort-Negrepontis 1974, theorem 13.6]):

Let $H(\lambda)$ denote the set of sets hereditarily of cardinality $< \lambda$ and let $H(\lambda)$ be the structure $< H(\lambda), e >$ where $e$ is the natural membership relation on $H(\lambda)$.

1.4.1. Lemma: (Keisler) For an ultrafilter $F$ on a set $I$ the following are equivalent:
(i) $F$ is $(\lambda, \mu)$-regular
(ii) In the structure $N = \prod_I H(\lambda^*)/F$ there is an element $b = b/F$ where $b : I \to H(\lambda^*)$ is a function, such that $|N| = \mu^N$ and $|N| = \text{card}(b) < \mu^N$ but for every $\alpha < \lambda N = \alpha^N \in b$.

Recall that for an ordinal $\alpha \leq \lambda$, $\alpha^N$ denotes the image of $\alpha$ under the natural embedding into $N$.

Proof: (i) $\Rightarrow$ (ii): Define $b : I \to H(\lambda^*)$ by $b(t) = \{ \alpha \in \lambda : t \in X_\alpha \}$ for $t \in I$ and $X_\alpha : \alpha \in \lambda$ a $(\lambda, \mu)$-regular family.

Now $X_\alpha = \{ t \in I : \alpha \in b(t) \}$ so $|N| = \alpha^N \in b$, since for each $\alpha \in \lambda$, $X_\alpha \in F$. But clearly, $b(t)$
has cardinality $\mu$ for each $t \in I$, since $\{X_\alpha : \alpha \in \lambda\}$ is a $(\lambda, \mu)$-regular family, so $N \models \text{card} (b) < \mu$. Trivially, we have also $N \models b \in \lambda^N$.

(ii) $\rightarrow$ (i): Let $b = b / F$ be the required element in $N$. Define $b'$ by $b'(t) = b(t)$ if $b(t) \in \lambda$ and $\text{card} (b(t)) < \mu$ and $b'(t) = \phi$ otherwise.

Obviously $b / F \approx b' / F$ since $N \models b \in \lambda^N$. We want to construct a $(\lambda, \mu)$-regular family. Put $X_\alpha = \{t \in I : \alpha \in b'(t)\}$ for each $\alpha \in \lambda$. Now suppose that for some $\{\alpha_i : i \in \mu\}$ the intersection $\bigcap X_{\alpha_i} \neq \phi$. So there is a $t \in I$ such that for each $i \in \mu$ $\alpha_i \in b'(t)$, which contradicts the fact that $\text{card} (b'(t)) < \mu$. QED.

1.4.2. Definition: Let $F_1$ be ultrafilter on $\mathbb{L}_1$ and $F_2$ is a projection of $F_1$ if there is a map $f : I_1 \rightarrow I_2$ which is onto and such that $F_1 = \{f^{-1}(X) : X \in F_2\}$.

Projections are closely related to the Rudin-Keisler order on ultrafilters over a fixed set $I$, cf. [Comfort-Ngôthaponta 1974]. We use now lemma 1.4.1. together with complete expansions (i.e., complete structures over their original universe, cf. section 1.2) to get.

1.4.3. Lemma: If $\lambda$ is regular and $F_1$ is $(\lambda, \lambda)$-regular ultrafilter on $I$ then there is uniform ultrafilter $F_2$ on $\lambda$ which is a projection of $F_1$.

Proof: Let $\mathbb{N}$ be the complete expansion of $N = \bigsqcup \mathcal{H}(\lambda^*)$ and $a : I \rightarrow \mathcal{H}(\lambda^*)$ as in lemma 1.4.1. and w.l.o.g. $b'(t) \in \lambda$ for all $t \in I$. Now put $c(t) = \sup (b'(t))$ so $c(t) \in \lambda$ since $\lambda$ is regular, and $N \models b \in c$. Clearly $c : I \rightarrow \lambda$. We define now $F_2$ by $F_2 = \{S \subseteq \lambda^N \mid a \in S\}$ where $S$ is the name of $S$ in $\mathbb{N}$. It is now easy to verify that $F_2$ is a uniform ultrafilter on $\lambda$ which is a projection of $F_1$. QED.

To prove the Abstract Compactness Theorem, we shall prove a slightly more elaborate statement:

1.4.4. Theorem: [Abstract Compactness Theorem]

Let $L$ be a logic, $\lambda, \mu$ be cardinals and $\lambda > \mu$.

(i) The following are equivalent:

(a) There is $(\lambda, \mu)$-regular ultrafilter $F$ on $I = P_\mu(\lambda)$ which is in $\mathcal{U}F(L)$.

(b) For every (relativized) expansion $A$ of $\mathcal{H}(\lambda^*)$, there is an $L$-extension $B$ and an element $b \in B$ such that $B \models \text{card} (b) < \mu^B$ but for every $\alpha < \lambda$ we have $B \models \alpha^B \in b$.

(c) $L$ is $[\lambda, \mu]$-compact.

(ii) Furthermore, if $\lambda$ is regular then the following are equivalent:

(d) There is a uniform ultrafilter $F$ on $\lambda$ which is in $\mathcal{U}F(L)$.

(e) $L$ is $[\lambda]$-compact.

(iii) In particular, we have

(f) If there is a $(\lambda, \mu)$-regular ultrafilter $F$ on any set $I$ which is in $\mathcal{U}F(L)$, then $L$ is $[\lambda, \mu]$-compact.

Proof: (a) $\rightarrow$ (b): Let $F$ be a $(\lambda, \mu)$-regular ultrafilter in $\mathcal{U}F(L)$ and let $M$ be any expansion of $\langle \mathcal{H}(\lambda^*) \rangle_{< \lambda}$. Put $N_0$ to be the ultrapower $\prod_{I} M / F$ and $N_i$ the extension of $N_0$ as required for $F \in \mathcal{U}F(L)$. First we observe that $N_0 < N_1 (L_{\mu, \mu})$ and, by lemma 1.4.1, there is an element $b$ in $N_0$ with the required properties. But then the same element $b$ has
the same properties as also in \( N_i \), since \( N_0 < N_i (L_\omega, a) \). But by the definition of \( N_i \), \( M < N_i (L) \), so we are done.

(b) \( \rightarrow (c) \): Let \( \Delta \Sigma \) be \( L \)-sentences satisfying the hypothesis of \( [\lambda, \mu] \)-compactness. We define an expansion \( M(\Delta, \Sigma) \) of \( <H(\lambda^+), \varepsilon > \) to apply. (ii). For this purpose let \( \{S_\alpha : \alpha < \lambda^+\} \) be an enumeration of all the subsets of \( \Sigma \) of cardinality less than \( \mu \). \( A_\alpha \) be a model of \( \Delta \cup S_\alpha \) and \( \{S_\alpha : \alpha < P_\mu(\lambda)\} \) an enumeration of all the subsets of \( \lambda \) of cardinality less than \( \lambda \). Finally we put \( \nu = \{(\text{sup}(\text{card} (A_\alpha)))^+ \lambda^+\} \), and define \( \lambda_\alpha = \text{card} (A_\alpha) \).

We now define \( M(\Delta, \Sigma) \) to be \( <H(\nu), d_\alpha, R, P > \), \( \alpha, \mu, \varepsilon \in L \) such that \( d_\alpha \) is the name of \( \alpha < \lambda \), \( R \) is a binary predicate not in \( L \) and the domain of \( R \) is \( \lambda \). We arrange it such that for each \( \alpha < \lambda \) the set \( R_\alpha = \{x \in H(\nu) : (a, x) \in R \} \) has cardinality \( \lambda_\alpha \) and such that \( \langle R_\alpha, P > \alpha, \mu, \nu \in L \rangle \). In other words put all the models \( A_\alpha \) into \( M(\Delta, \Sigma) \) in way, that when we now apply (ii) we get a model for \( \Delta \cup \Sigma \). More precisely, we observe that for each formula \( \varphi \in \Delta \):

(a) \( M(\Delta, \Sigma) \models \text{card}(b) < \varphi^{R} \)

and for each \( \beta \lambda^+ \) and for \( \Sigma = \{\alpha : \varphi, \alpha \lambda^+\} \) an enumeration of \( \Sigma \) we have

(b) \( M(\Delta, \Sigma) \models (d_\alpha < \varphi^{R}) \).

Now let \( N, \alpha \in N \) be as in the conclusion of (ii) \( \sum M(\Delta, \Sigma) / F \).

Claim : \( \langle R_\alpha, P > \alpha, \mu, \nu \in L \rangle = \Delta \cup \Sigma \).

This follows from the definition of (a) and (b).

(c) \( \rightarrow (a) \): Suppose \( L \) is \( [\lambda, \mu] \)-compact but no \( (\lambda, \mu) \)-regular ultrafilter on \( P_{<\mu}(\lambda) \) \( F \) is related to \( L \). So for every structure \( F \) there is an \( L_{F} \)-structure \( A_F \) exemplifying this.

We now proceed to construct an ultrafilter \( F_0 \) on \( \lambda \) which contradicts the choice of the \( A_F \)'s. For this we construct first a rich enough structure \( M \) such that

(1) For each \( A_F \) there is a unary predicate \( P_F \) in \( M \) with \( \langle P, \mu, P > \alpha, \mu, \nu \in L \rangle \) \( A_F \).

(2) \( M \) is a model of enough set theory to carry out the argument and

(3) \( M \) is an extension and expansion of \( <H(\lambda^+), \varepsilon > \) (or equivalently \( <H(\lambda^+), \varepsilon > \) is a relativized reduct of \( M \)).

Let \( M^F \) be the complete expansion of \( M \) and put \( \Delta = \theta_\nu L(\lambda) \). the first order theory of \( M^F \) where \( \theta_\nu L(\lambda) \) is the vocabulary of \( M^F \). Furthermore put \( \Sigma = \{b \subseteq d_\alpha \lambda \text{card}(b) < \varphi^{R} \} \). Clearly \( \Delta \cup \Sigma \) satisfies the hypothesis of \( [\lambda, \mu] \)-compactness using the model \( M^F \). So \( \Delta \cup \Sigma \) has a model \( N \). We want to use \( N \) to construct our ultrafilter \( F_0 \). First we observe that \( M^F \) is a model of \( \Sigma \). Let \( a_\rho \) be the interpretation of \( \rho \) in \( N \). We define \( F_0 \) on \( P_{<\mu}(\lambda) \) by \( F_0 = \{R \in P_{<\mu}(\lambda) : N \models a_\rho \in R \} \). This makes sense, since \( M^F \) is a complete expansion and hence, every subset of \( \lambda \) of cardinality \( \mu \) corresponds to a predicate in \( M^F \) (remember \( <H(\lambda^+), \varepsilon > \) is present in \( M^F \)).

To complete the proof we have to verify several claims:

Claim 1: \( F_0 \) is ultrafilter.

Obvious.

Claim 2: \( F_0 \) is \( (\lambda, \mu) \)-regular.

Let \( X_a = \{t \in P_{<\mu}(\lambda) : a \subseteq t \} \). Now \( X_a \in F_0 \) for say \( X_a \) corresponds to \( R_\alpha \) then \( N \models a_\rho \in R_\alpha \) iff \( N \models a_\rho \in F_0 \). which is true for all \( \alpha < \lambda \) by definition of \( a_\rho \). Now \( \{X_a : \alpha < \mu \} \) be a subfamily of the \( X_a \)'s. Clearly, \( \bigcap X_a \neq \emptyset \), since each \( t \) in some \( X_a \) has
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cardinality <\mu>.
Now consider the ultraproduct \[ \prod \mathcal{M} / F_0 = \mathbb{N}_0. \] If \( g \) is an element of \( \mathbb{N}_0 \) then \( g \) is an \( F_0 \)-equivalence class of functions \( g: \mathcal{M}_0(\lambda) \rightarrow \mathbb{M} \), so \( g \) corresponds to a function \( g^0 \) in \( \mathcal{M} \) with name \( g \), (since \( \mathbb{M} \) is the complete expansion) and \( a_0 \in \text{Dom}(g^0) \). So we define an embedding \( f: \mathbb{N}_0 \rightarrow \mathbb{N} \) by \( f(g / F_0) = g_0(a_0) \).

Claim 3: \( f \) is well defined and 1-1.

Let \( g / F_0 = g' / F_0 \). We want to show that this is equivalent to \( \mathbb{N}_0 \models g(a_0) = g'(a_0) \) iff \( Y = \{ t \in P_{\mathcal{M}_0} \mid g(t) = g'(t) \} \in F_0 \). But the latter is true iff \( a_0 \in Y^\mathbb{M} \) which is equivalent to \( g(a_0) = g'(a_0) \).

So we have shown that \( f \) is an embedding of \( \mathbb{N}_0 \) into \( \mathbb{N} \).

Now let \( g = [g_i / F_0] ; <\alpha \} \in \mathbb{N}_0 \).

Claim 4: For every \( L \)-formula \( \varphi \), we have \( \mathbb{N}_0 \models \varphi(g) \) iff \( Y = \{ t \in P_{\mathcal{M}_0} \mid \varphi(g_1(t), g_2(t), \ldots) \} \in F_0 \).

Now \( Y \in F_0 \) iff \( Y^\mathbb{M} \) contains \( a_0 \) iff \( \mathbb{N}_0 \models \varphi(g_1(a_0), g_2(a_0), \ldots) \).

Now look at \( \mathcal{A}_{\mathbb{N}_0} \). By assumption there is no \( \mathbb{N} \) extending \( \prod \mathcal{A}_{\mathbb{N}_0} / F_0 \) satisfying claim 4. But \( \langle P^\mathcal{M}_0, P > \in L_{\mathbb{N}_0} \) is such an \( \mathbb{N} \) by construction. This completes the proof of (i).

(d) \( \Rightarrow \) (e): This follows from the previous, since uniform ultrafilters on \( \lambda \) are \( (\lambda) \)-regular and \( \lambda \) is a regular cardinal by our hypothesis.

(e) \( \Rightarrow \) (d): Here we use lemma 1.4.3. and (a) \( \Rightarrow \) (c). This completes the proof of (ii):

To prove (f) we just observe that in the proof of (a) \( \Rightarrow \) (c) we did not use that \( I = P_{\mathcal{M}_0}(\lambda) \). This completes the proof theorem 1.3.9. QED.

1.5. The compactness spectrum.

In this section we study the structure of the compactness spectrum \( \text{Comp}(L) \) and the regular compactness spectrum \( \text{RComp}(L) \) defined below.

1.5.1. Definition: For a logic \( L \) we define \( \text{Comp}(L) \), \( \text{RComp}(L) \) to be the class of all (regular) cardinals such that \( L \) is \( (\lambda) \)-compact.

1.5.2. Theorem: The first cardinal \( \lambda_0 \) in \( \text{Comp}(L) \) is measurable (or \( \omega \)).

Proof: By Theorem 1.2.2(i) each regular \( \lambda < \lambda_0 \) is cofinally characterizable in \( L \) via a structure \( \mathcal{B}(\lambda) \) with cardinality of \( \mathcal{B}(\lambda) \), let \( \mu \) be defined by

\[ \mu = \sup \{ \kappa \cdot \lambda < \lambda_0 \} + \lambda^\kappa \]

and let \( \mathcal{B} \) be the complete expansion of the structure \( <\mu, \varepsilon> \). Therefore \((*)\) in every \( L \)-extension of \( \mathcal{B} \) all the ordinals smaller than \( \lambda_0 \) are standard.

By \( [\lambda_0] \)-compactness \( \mathcal{B} \) has an \( L \)-elementary extension \( \mathcal{C} \) with some \( c \in \mathcal{C} \equiv \mathcal{B} \) such that \( \mathcal{C} \models c \in \lambda_0^\mathbb{M} \). Since \( \lambda_0 \) is minimal we have no \( \lambda < \lambda_0 \) that \( C \models c \in \lambda_0^\mathbb{M} \). We now define an ultrafilter \( F \) on \( \lambda_0 \) by

\[ F = \{ X \subseteq [\lambda_0]^\mathcal{C} \mid c \in X \} \]

where \( X \) is the name of the set \( X \) in \( \mathcal{B} \). Clearly \( F \) is an ultrafilter. We propose to show that \( F \) is \( \lambda_0 \)-complete.
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Let \( \{X_\alpha : \alpha < \mu < \lambda_0 \} \) be any family in \( F \). The function \( f \) with \( f(\alpha) = X_\alpha \) is a function in \( B \) with name, say, \( f \). Put now \( X = \bigcap \alpha < \lambda_0 X_\alpha \). So \( B \models X = \bigcap X_\alpha \) and therefore
\[
B \models \forall x(\forall \iota (\iota < \alpha \rightarrow x \in f(\iota)) \rightarrow x \in \bigcap f(\iota)).
\]

But by (*) the ordinals \( \alpha < \lambda_0 \) in \( B \) are the same as in \( C \). So \( C \models \exists \alpha \in X \), since \( f \) is a function of \( C \) with \( f^C \models B = f^B \). So \( X \in F \) and therefore \( \lambda_0 \) is measurable. QED.

1.5.3. Example: If \( \kappa \) is a strongly compact cardinal, the logic \( L_{<\kappa} \) is \( (\omega,\kappa) \)-compact and therefore \( [\kappa] \)-compact. But the logic \( L_{\kappa} \) is not \( [\lambda] \)-compact for any \( \lambda < \kappa \).

Note that, as a corollary, we get that strongly compact cardinals are measurable. By [Magidor 1976] it is consistent that the first measurable and the first strongly compact cardinal coincide.

Our next aim is to study the structure of \( \text{Comp}(L) \). The main theorem here is

1.5.4. Theorem: For every cardinal \( \lambda \) and every logic \( L \) \( \lambda^+ \in \text{Comp}(L) \) implies \( \lambda \in \text{Comp}(L) \).

Proof: Use the Abstract Compactness Theorem and lemma 1.3.11. QED.

For \( \lambda \) regular this was first proved in [Makowsky-Shelah 1979], giving a direct proof by relating \( [\lambda] \)-compactness to 'descendingly incomplete' ultrafilters. The general result was proved in [Makowsky-Shelah 1983]. There the connection with ultrafilters was first recognized, on which the presentation here is based.

The next result concerns the structure of \( \text{Comp}(L) \). The following was proven in [Makowsky-Shelah 1979, lemma 8.4(ii)] by an extension of the argument for 1.3.2.

1.5.5. Lemma: Let \( \lambda > \mu \) be two regular cardinals and \( L \) be a logic such that \( \lambda \in \text{Comp}(L) \) but \( \mu \notin \text{Comp}(L) \). Then there is a uniform \( \mu \)-descendingly complete ultrafilter on \( \lambda \).

Consider the following assumption \( A(\lambda) \), where \( \lambda \) is an infinite cardinal.

\( A(\lambda) : \text{"If } F \text{ is a uniform ultrafilter on } \lambda \text{, then } F \text{ is } \mu \text{-descendingly incomplete for every } \mu < \lambda". \)

We denote by \( A(\omega) \) the statement "for every infinite cardinal \( \lambda \), \( A(\lambda) \) holds".

[Donder-Jensen-Koppelberg, 1981] and [Magidor 1987] have studied this assumption. The following theorem summarizes their results (with part (v) being theorem 8.36 in [Comfort-Negrepontis 1974], see also 1.3.11. below):

1.5.6. Theorem:
(i) (Jensen-Koppelberg) Assume \( \neg G^\#: \text{Then for every regular cardinal } \lambda \text{ we have } A(\lambda) \).
(ii) (Donder) Assume there is no inner model of \( \text{ZFC} \) with an uncountable measurable cardinal. Then \( A(\omega) \) holds.
(iii) If \( A(\omega) \) holds then there are no uncountable measurable cardinals.
(iv)" (Woodin) Assume there are uncountable measurable cardinals. Then it is consistent with \( \text{ZFC} \) that \( A(\omega) \) fails.

However, in \( \text{ZFC} \) we already have:
(v) (Kunen-Prikry and Cudnovskii-Cudnovskii) For every \( n \in \omega \) \( A(\omega_n) \) holds.
Magidor has informed us of the yet unpublished result 1.5.8(iv) of Woodin. He had previously proved a similar result, where one has to replace the existence of an uncountable measurable cardinal in the hypothesis by the existence of a supercompact cardinal.

The assumption $A(\omega)$ is intimately connected with compactness properties: It implies that $\text{Comp}(L)$ has no gaps. On the other hand, the existence of strongly compact cardinals allows us to construct logics where $\text{Comp}(L)$ does have gaps. More precisely:

1.5.7. Theorem:

(i) Assume $A(\omega)$ holds. Then $\text{Comp}(L)$ is an initial segment of the cardinals, i.e $\lambda \in \text{Comp}(L)$ and $\mu < \lambda$ implies that $\mu \in \text{Comp}(L)$.

(ii) (Shelah) Let $\mu_1 < \mu_2$ be two uncountable strongly compact cardinals. Then there is a logic $L$ which is $[\kappa]$-compact iff $\kappa < \mu_1$ or $\kappa \geq \mu_2$.

Proof: (i) Assume $\text{Comp}(L) \neq \emptyset$. Since $A(\omega)$ implies that there are no uncountable measurable cardinals, by theorem 1.5.6(iii), the first cardinal in $\text{Comp}(L)$ is $\omega$, by theorem 1. Now, if $\omega < \lambda \in \text{Comp}(L)$ and $\omega < \mu < \lambda$, $\mu \notin \text{Comp}(L)$, $\mu$ regular, we apply lemma 1.5.5. and get a contradiction to $A(\omega)$. If $\mu$ is singular, we apply 1.5.6. to $\mu$ and then use 1.1.6(v).

(ii) will follow from proposition 1.6.7. QED:

The question which remains is whether $\text{Comp}(L)$ is empty or not. Now clearly the logic $L_{\omega, \omega}$ is not compact in any sense, so $\text{Comp}(L_{\omega, \omega})$ is empty. But if we assume that the logic $L$ is bounded in some sense and have some very strong assumption on the existence of large cardinals we can get more specific results. For terminology and results on large cardinals we refer to [Jech 1978].

1.5.8. Definition: A logic is set presentable if (i) there is a cardinal $\kappa$ such that whenever a vocabulary $\sigma \in H(\kappa)$ and $\Sigma \subset L[\sigma]$ has cardinality $< \kappa$ then $\Sigma \in H(\kappa)$ and (ii) for every $\varphi \in L[\tau]$ $\text{Mod}(\varphi)$ is a set theoretically definable class of $\tau$-structures. (Recall that $H(\kappa)$ is the family of sets hereditarily of cardinality $< \kappa$.)

1.5.9. Example: Let $L = L^\omega_\kappa$ be like $n^{th}$ order logic except that we allow conjunctions and disjunctions of less than $\kappa$-many formulas. Clearly $L$ is set presentable and so is every sublogic of it.

1.5.10. Definition: Let $S\text{Comp}(L)$ be the class of cardinals $\kappa$ such that $L$ is $(\omega, \kappa)$-compact and $W\text{Comp}(L)$ be the class of cardinals $\kappa$ such that $L$ is $(\kappa, \kappa)$-compact. Clearly we have $S\text{Comp}(L) \subseteq \text{Comp}(L) \subseteq W\text{Comp}(L)$.

1.5.11. Proposition: [Magidor, 1971]

If $\kappa$ is an extendible cardinal then $\kappa \in S\text{Comp}(L^\omega_\kappa)$.

1.5.12. Definition: The following statement is called Vopenka's principle: Let $\mathcal{C}$ be a proper class of $\tau$-structures for some finite vocabulary $\tau$. Then there are two structures $A, B \in \mathcal{C}$ such that $A$ is (first order) elementary embeddable into $B$.

Now [Magidor, 1971] also shows:

1.5.13. Proposition: If Vopenka's principle holds then the class of all extendible cardinals is closed unbounded.
So propositions 1.5.11. and 1.5.13. give us immediately:

1.5.14. **Theorem** [Magidor-Stavi]: Assume Vopenka's principle holds and that $L$ is a set presentable logic. Then $\mathcal{S}\text{Comp}(L)$ is a non-empty final segment of the cardinals (in other words, $L$ is ultimately compact).

For $\mathcal{W}\text{Comp}(L)$ we do not need Vopenka's principle to prove an analogue of theorem 1.5.14.

1.5.15. **Theorem** [Stavi 1976]: Let $\mu$ be an uncountable measurable cardinal and $F$ be a normal ultrafilter on $\mu$ and $\bar{L}$ be a sublogic of $L^\mu$. Then $\mathcal{W}\text{Comp}(L) \cap \mu \in F$.

Theorem 1.5.15. holds under much weaker assumptions (cf. [Stavi 1976, section 5]) and is also discussed and proved in chapter 17 section 4.2.

The structure of $\text{Comp}(L)$ definitely deserves further investigation. We combine the content of lemma 1.1.6 (v), and theorems 1.5.2, 1.5.4, and 1.5.14 into statement:

**Theorem**: For a logic $L$ we have:

(i) $\kappa \succ (\kappa) \in \text{Comp}(L) \rightarrow \kappa \in \text{Comp}(L)$

(ii) $\kappa^+ \in \text{Comp}(L) \rightarrow \kappa \in \text{Comp}(L)$

(iii) The first cardinal in $\text{Comp}(L)$ is measurable (or $\omega$).

(iv) If $L$ is set presentable and Vopenka's principle holds, then $\text{Comp}(L)$ contains a final segment of the class of all cardinals.

Our last theorem illustrates that Vopenka's principle is the right large cardinal assumption in this context.

1.5.17. **Theorem**; (Makowsky). The following are equivalent:

(i) Vopenka's principle

(ii) For every finitely generated logic $L$, $\mathcal{S}\text{Comp}(L)$ contains a closed-unbounded class of cardinals.

(iii) For every finitely generated logic $L$, $\mathcal{S}\text{Comp}(L) \ni \phi$.

(iv) For every finitely generated logic $L$, $\text{Comp}(L) \ni \phi$.

**Proof**: We only have to prove (iv) $\rightarrow$ (i): Let $C$ be a proper class of $\tau$-structures and let $Q_\tau$ be the Lindstrom quantifier defined by $C$ and $L = L_{\omega,\omega}(Q_\tau)$. Clearly $C$ contains a proper subclass $C^*$ of the form $C^* = \text{Mod}(T)$ where $T$ is a complete $L[\tau]$-theory. Assume that $\kappa \in \text{Comp}(L)$ and let $A \in C_\kappa$ be of cardinality $\geq \kappa$. Using $[\kappa]$-compactness we now find $B \models T$ which is an (first-order) elementary extension of $A$ and clearly $B \in C$. QED.

1.6. Gaps in the compactness spectrum.

In this section we want to study a family of examples of logics with gaps in the compactness spectrum. These examples will also be used in the subsequent sections to illustrate various phenomena concerning dependence numbers and amalgamation properties (see 2.2.5. and section 3.5.).
1.6.1. Example: Let $\kappa$ be a cardinal and $\mathcal{F}$ be an ultrafilter on $\kappa$. We define a logic $L = L_{\mathcal{F}, \kappa}$ by adding to first order logic $L_{\omega, \kappa}$ the following formation rule: If $\{ \varphi_i : i < \kappa \}$ is an indexed family of $L$-sentences, then $\bigcap \{ \varphi_i : i < \kappa \}$ is an $L$-sentence. We additionally assume that $L$-formulas have $< \omega$ many free variables. Satisfaction for $L$ is defined by the additional clause: If $A$ is a $\tau$-structure then $A \models \bigcap \{ \varphi_i : i < \kappa \}$ iff $\{ i < \kappa : A \models \varphi_i \} \in \mathcal{F}$.

1.6.2. Proposition: Let $\mu$ be a measurable cardinal and $\mathcal{F}$ be a $\mu$-complete non-principal ultrafilter on $\mu$.

(i) $L_{\mathcal{F}, \mu} \leq L_{\{ \omega \mu \}}$.

(ii) $L_{\mathcal{F}, \mu}$ is not $[\mu]$-compact.

(iii) $L_{\mathcal{F}, \mu}$ is $[\lambda]$-compact for every $\lambda < \mu$.

Proof: (i) and (ii) are left to the reader. To prove (iii) we make use of the Abstract Compactness Theorem (1.3.9) and we show that every ultrafilter $D$ on $\lambda$ is in $UF(L)$. Let us spell this out precisely.

1.6.3. Lemma: Let $\lambda = L_{\mathcal{F}, \mu}$ and $D$ be any ultrafilter on $\lambda < \mu$. Furthermore let $\{ A_i : i < \lambda \}$ be a family of $\tau$-structures. $\varphi \in L[\tau]$ and $\{ f_j : j \leq \omega < \mu \}$ be a family of functions in $\prod A_i$. Then the following are equivalent:

(i) $\prod_i A_i / D \models \varphi(f_1(t), \ldots, f_{\omega}(t))$.

(ii) $X_\varphi = \{ i \in \lambda : A_i \models \varphi(f_1(t), \ldots, f_{\omega}(t)) \} \in D$.

Proof: Like Los' theorem for first order logic. QED.

Example 1.6.1 can be still further extended.

1.6.4. Example*: Let $\mu_1 > \mu_2$ with $\mu_1$ measurable and $\mu_2$ strongly compact. Let $\mathcal{F}$ be a $\mu_1$-complete non-principal ultrafilter on $\mu_1$. We define the logic $L_{\mathcal{F}, \mu_2}$ as above, but we allow existential quantification over sequences of variables $\{ x_j : j < \kappa < \mu_2 \}$.

1.6.5. Proposition*: (Shelah)

(i) $L_{\mathcal{F}, \mu_2} \leq L_{\mu_1, \mu_2}$.

(ii) The logic $L_{\mathcal{F}, \mu_2}$ is $[\kappa]$-compact for every $\kappa < \mu_1$ and $\kappa < \mu_2$.

Proof: (i). Clearly, the operation $\bigcap$ can be expressed by conjunctions and disjunctions in $L_{\mu_1, \mu_2}$, since $\mu_2$ is a strong limit cardinal and $\mu_1 < \mu_2$.

(ii) For $\kappa < \mu_1$ this is similar to lemma 1.6.3. and for $\kappa < \mu_2$ this follows from (i) and the fact that $\mu_2$ is strongly compact. QED.

Clearly, in proposition 1.6.2, $[\mu_1]$-compactness fails. But it is not clear whether for any $\kappa$ with $\mu_1 < \kappa < \mu_2$, we have $[\kappa]$-compactness. However, we can construct a more refined example.
1.6.6. Example*: (Shelah) Let $D(\mu_1, \mu_2)$ be the set of $\mu_1$-complete ultrafilter $F$ on some set $I \subseteq \mu_2$ such that $\mu_1 \leq \text{card}(I) < \mu_2$. Instead of allowing $\bigcap_F$ for one ultrafilter we can no form a logic $L_{D(\mu_1, \mu_2)}$ as follows: We close first order logic $L_{\omega_1, \omega}$ under all the operations $\bigcap_F$ for $F \in D(\mu_1, \mu_2)$ as in the previous example. Additionally we close under existential quantification over strictly less than $\mu_2$ many individual variables.

The next proposition is proved exactly as 1.6.2.

1.6.7. Proposition*: (Shelah) Let $\mu_1$ be measurable and $\mu_2$ be a strongly compact cardinal bigger than $\mu_1$. Then

(i) $L_{D(\mu_1, \mu_2)} \subset L_{\mu_1, \mu_2}$

(ii) $L_{D(\mu_1, \mu_2)}$ is $[\kappa]$-compact for every $\kappa < \mu_1$ and $\kappa \geq \mu_2$ and

(iii) $L_{D(\mu_1, \mu_2)}$ is not $[\kappa]$-compact for any $\kappa$ with $\mu_1 \leq \kappa < \mu_2$.

This also establishes 1.5.7(ii). Using the same type of examples we can actually find logics with a compactness spectrum containing various gaps. How far we can go with this, is described in the following theorem:

1.6.8. Theorem*:

(i) Assume there are arbitrarily large measurable cardinals. Then there is a $[\omega]$-compact logic $L$ such that both $\text{Comp}(L)$ and its complement are cofinal in the class of all cardinals.

(ii) Assume there are arbitrarily large strongly compact cardinals. Then there is a $[\omega]$-compact logic $L$ such that both $\text{Comp}(L)$ and its complement are cofinal in the class of all cardinals and consist of intervals whose length is a strongly compact cardinal.

Proof: Combine the examples 1.6.1 and 1.6.4 respectively. QED.

Note however, that for set-presentable logics $L$, Vopenka's principle (Theorem 1.5.14) implies that $\text{Comp}(L)$ is a final segment of all cardinals.
2. THE DEPENDENCE NUMBER

2.1 Introduction

In this section we develop further an idea briefly mentioned in chapter 2, section 5.1, namely the meaning of the assertion that a formula \( \varphi \in L[\tau] \) depends only on a subset \( \sigma \subset \tau \). Let us recall a definition.

2.1.1. Definitions: Let \( L \) be a logic and \( \varphi \in L[\tau] \).

(i) \( \varphi \) depends only on (the symbols in) \( \sigma \subset \tau \) if for all \( \tau \)-structures \( AB \) such that \( A \models \sigma \equiv B \models \sigma \) we have \( A \models \varphi \) iff \( B \models \varphi \).

(ii) We say that \( \varphi \in L[\tau] \) depends on \( \sigma \subset \tau \), if there is some \( \sigma_0 \subset \sigma \) such that \( \varphi \) depends only on \( \sigma_0 \).

(iii) A logic \( L \) is weakly regular, if \( L \) satisfies the basic closure properties 1.2.1. and the relativization property 1.2.2. of chapter 2.

The difference between weakly regular and regular is the absence of the substitution property 1.2.3. of chapter 2.

If \( \varphi \in L[\tau] \) does only depend on \( \sigma \subset \tau \), one would generally expect, that there is a \( \psi \in L[\sigma] \) which is equivalent to \( \varphi \). If this is the case, we say that the logic \( L \) is occurrence normal. However, our definition of a weakly regular logic does not imply this. Nevertheless we have:

2.1.2. Proposition*: (Makowsky) For every weakly regular logic \( L \) there is a logic \( L_1 \) such that

(i) \( L = L_1 \) and

(ii) if \( \varphi \in L[\tau] \), \( \sigma \subset \tau \) and \( \varphi \) depends only on \( \sigma \), then there is a \( \psi \in L[\sigma] \) such that for every \( \sigma \)-structure \( A \) \( A \models \psi \) iff every expansion of \( A \) to a \( \tau \)-structure \( A_1 \) \( A_1 \models \varphi \).

Proof: We just add new atomic formulas and consider them as being of the required vocabulary. QED.

Regular logics are closed under substitutions of formulas for atomic predicate letters. For one sorted logics there is no problem in stating this directly, for many sorted logics we have to be a bit careful about the sorts. Gaifman pointed out that the definition of a regular logic ensures that \( L_1 \) actually is \( L \).

2.1.3. Proposition*: (Gaifman) Every regular logic is occurrence normal.

Proof: One-sorted case: Assume \( \varphi, \sigma \) and \( \tau \) as in the definition of occurrence normal above. To construct \( \psi \) we first make use of the eliminability of function symbols (which follows from the substitution property, chapter 2.1.2.3.) and assume that \( \tau - \sigma \) contains only relation symbols. Next we construct for every predicate symbol \( R \in \tau - \sigma \) a formula of first order logic \( \theta_R \) with equality only and with free variables according to the specifications of the many sorted arity of \( R \). We now obtain \( \psi \) by substituting \( \theta_R \) for every occurrence of \( R \) in \( \varphi \), using the substitution property again. Note that we do not need the relativization property here.

In the case of many sorted logics, the definition of the substitution property 1.2.3. from chapter 2 has to be modified. There is no difficulty in doing this so that it implies...
occurrence normality. We leave this as an exercise to the reader. QED.

In the light of propositions 2.1.2 and 2.1.3 we can restrict ourselves for the rest of this chapter to occurrence normal or regular logics. For such logics we can define the concept of an dependence number in a semantical way. In chapter 2 (1.2.3) a syntactic concept of occurrence property was introduced.

2.1.4. Definition:

(i) Given a regular logic $L$, we define a cardinal $\alpha(L) = \kappa$ to be the smallest cardinal such that every formula $\varphi \in L[\tau]$ depends only on some subset $\tau_0 \subset \tau$ with $\text{card} (\tau_0) < \kappa$. If no such $\kappa$ exists we write $\alpha(L) = \infty$. If $\alpha(L) = \omega$ we also say that $L$ has Finite Dependence or has the Finite Dependence Property.

(ii) Given a regular logic $L$, we define a cardinal $\text{OC}(L) = \kappa$ to be the smallest cardinal such that for every formula $\varphi \in L[\tau]$ there is $\sigma \subset \tau$ with $\text{card} (\sigma) < \text{OC}(L)$ and $\varphi \in L(\sigma)$.

In chapter 2 (1.2.3) the Occurrence Property was introduced, which is the syntactic counterpart of our Finite Dependence Property. In our terminology the Occurrence Property is equivalent to $\text{OC}(L) = \omega$. Using proposition 2.1.3 one easily sees that every logic $L$ which has the Finite Dependence Property contains a sublogic $L_0$ equivalent to it which has the Occurrence Property in the syntactic sense. In fact, more generally we have:

2.1.5. Proposition*: Let $L$ be a regular logic with dependence number $\alpha(L)$. Then there is a regular logic $L_1$ with $\text{OC}(L) = \alpha(L)$ which is equivalent to $L$.

Proof. Similar to 2.1.2. QED.

The above proposition shows that up to equivalence of logics, the occurrence number and the dependence number coincide. In [Makowsky-Shelah 1983] the dependence number is, indeed, called occurrence number. The change in terminology was motivated by the requirements of chapter 2 and by the notion of the dependence structure, introduced in section 2.4.

2.1.6. Examples:

(i) In chapter 2 proposition 5.1.3 shows that for a $(\kappa, \lambda)$-compact logic with $\alpha(L) \leq \kappa$ we actually have $\alpha(L) = \lambda$.

This fact was first pointed out in [H. Friedman 1970]...

(ii) Let us look at the logic $L^{\kappa, \lambda}$ defined in 1.8.1. Obviously $\varphi(L) \leq \kappa^*$. But if $\varphi \in L[\tau]$, $\text{card} (\tau) = \kappa$ then there is no smallest $\tau_0 \subset \tau$ such that $\varphi$ depends exactly on the symbols in $\tau_0$. This illustrates that our concept of dependence number is really semantical and not syntactical. For this $L$ there is no reasonable syntactical notion of dependence number.

2.1.7. Substitutes for the dependence number.

The dependence number is a concept which keeps the size of a logic limited. Other assumptions in this direction are:

(i) For every vocabulary $\tau$ with $\tau$ a set $L[\tau]$ is also a set. We call such logics small.

In section 4.3 this concept will be used.

(ii) For every vocabulary $\tau$, if $\tau$ is a set, $\text{card} (L[\tau]) = \text{card} (\tau) + \kappa$ for some fixed cardinal $\kappa$. 
This gives us a special case of a size function, as defined in section 4.3. There we also look at tiny logics, i.e. logics $L$ such that whenever $\text{card}(\tau)$ is smaller than the first uncountable measurable cardinal $\mu_0$, then $\text{card}(L[\tau])$ is also smaller than $\mu_0$.

(iii) The presence of a Lowenheim number $l_\mu(L)$, as introduced in chapter 2.6.2.

For various theorems in abstract model theory such limiting assumptions are needed, as we shall see in the further course of this and the next chapter. Note that from the above properties (ii) $\rightarrow$ (i) and in the presence of an dependence number (iii) $\rightarrow$ (ii), up to equivalence of logics. In fact we have the following

2.1.8. Proposition: Let $L$ be a logic with $\alpha(L) = \mu$ and $l_1(L) = \kappa$ and $\tau$ be a vocabulary with $\text{card}(\tau) = \lambda$ and $\mu \leq \lambda \leq \kappa$. Then there are, up to logical equivalence, only $2^{2^\mu}$ many $\tau$-sentences.

The proof consists of a crude counting argument. Note that we do not get the stronger conclusion $\text{card}(L[\tau]) \leq 2^{2^\mu}$, since there may be many equivalent formulas.

One would actually expect that if $l_1(L) = \kappa$ then $\alpha(L) \leq \kappa^+$ and one might add this to the definition of the Lowenheim number, but it is an open field to determine which model theoretic properties have what impact on the size of the dependence numbers. The only exception is compactness and the rest of section 2 is devoted to this.

2.2. Compactness and dependence numbers.

This section is devoted to the statement of the main theorem and the discussion of several examples. The proof of the main theorem is discussed in the following section but for a technically complete exposition of the proof we refer the reader to [Makowsky-Shelah 1983].

To simplify the statements of the following theorem and its corollaries, we denote by $\bar{\mu}$ the first uncountable measurable cardinal, if there is one, and $=\infty$ otherwise. We stipulate further that if $\bar{\mu} = \infty$, then $\bar{\mu}^+ = \infty$.

2.2.1. Theorem (Finite dependence theorem):
(i) (Global version) Let $L$ be a regular, $[\omega]$-compact logic with dependence number $\alpha(L) < \bar{\mu}$. Then $L$ has the finite dependence property, i.e. $\alpha(L) = \omega$.
(ii) (Local version) Let $L$ be a regular, $[\kappa]$-compact logic, $\tau$ a vocabulary and $\varphi \in L[\tau]$ a formula which depends only on some $\tau_0 \in \tau$ with $\text{card}(\tau_0)$ less than the first uncountable measurable cardinal. Then there is a finite $\tau_1 \subset \tau_0$ such that $\varphi$ depends only on $\tau_1$.

Clearly, (ii) implies (i). The proof of (ii) is presented in section 2.3.

2.2.2. Corollary: Let $L$ be a regular, $[\kappa]$-compact logic, $\kappa < \bar{\mu}$ and $\alpha(L) \leq \bar{\mu}^+$. Then $L$ has the finite dependence property.

Proof of corollary: By theorem 1.5.2, $L$ is $[\omega]$-compact, so we can apply the finite dependence theorem. QED.

As a second corollary we get a representation theorem of some compact logics via Lindström quantifiers (cf. chapter 2.4). Let us recall the definition:
2.2.3. Definition: A logic $L$ is a Lindstrom logic if $L = L_{u,\kappa}(Q,i \in I)$ for some indexed set of Lindstrom quantifiers $Q_i$ ($i \in I$). $L$ is finitely generated if $L$ is a Lindstrom logic and $\text{card}(I) < \omega$.

Note that by theorem 4.1.3. of chapter 2 every regular logic $L$ which has the (syntactic) finite occurrence property is a Lindstrom logic.

2.2.4. Proposition:
(i) Let $L$ be a regular logic with $\alpha(L) = \omega$. Then $L$ is equivalent to a Lindstrom logic.
(ii) If a regular logic $L$ is small, $[\kappa]$-compact and $\alpha(L) = \kappa < \omega^+$, then $L$ is equivalent to a Lindstrom logic.

Proof: Using corollary 2.2.2, we can reduce (ii) to (i). So assume that $L$ has finite dependence. Let $\tau$ be a finite vocabulary. We want to replace every $\phi \in L[\tau]$, which is not equivalent to a first order formula, by a formula consisting of a new quantifier $Q_{\phi}$ applied to a sequence of atomic formulas. The problem is to keep the number of quantifiers so introduced small. But the type of the quantifier does not really depend on the vocabulary $\tau$, but only on the similarity type, i.e. on the number and arities of the symbols in $\tau$. Now there is a countable universal vocabulary $\tau_\omega$ such that for every finite $\tau$ there is $\tau' \subseteq \tau_\omega$ which has the same similarity type as $\tau$. Therefore every $\phi \in L[\tau]$ can be obtained from some $\psi \in L[\tau_\omega]$ by an application of substitution. By our assumption, $L[\tau_\omega]$ is a set. So writing every formula in $L[\tau_\omega]$ as a Lindstrom quantifier, we complete the proof. QED.

Both the theorem and the corollaries have assumptions involving measurable cardinals. In the sequel we shall discuss examples which show that these assumptions are necessary.

2.2.5. Examples:
(i) Let $\mu$ be a strongly compact cardinal. So $L = L_{\mu,\mu}$ is $[\mu]$-compact and $\alpha(L) = \mu$. As noted before, it is consistent that the first strongly compact and the first measurable cardinal coincide, by [Magidor 1976]). This shows that the assumption on measurable cardinals cannot be dropped in the corollaries.

(ii) Let $\mu$ be a measurable cardinal and $F$ be a $\mu$-complete non-principal ultrafilter on $\mu$. We look again at the logic $L = L_{\mu,\mu}$ from example 1.6.1. By 1.6.2, this logic is $[\omega]$-compact, but clearly its dependence number is $\mu^+$. This shows that the assumption on the measurable cardinal cannot be dropped in the Finite Dependence Theorem.

2.3 Proof of the Finite Dependence Theorem.

The proof of the Finite Dependence Theorem uses three lemmas (lemma A, B, C). We do not prove these lemmas here and refer the reader to [Makowsky-Shelah 1983]. Instead, we present the three lemmas without proofs and show how the Finite Dependence Theorem is proved from them. The reader will gain a rather transparent picture of the structure of the proof.

Let us fix a $[\lambda]$ compact logic $L$, a vocabulary $\tau$ and a sentence $\phi \in L[\tau]$. We want to study subsets of $\tau$ on which $\phi$ does not depend. Each lemma introduces a new aspect of the notions involved: Lemma A uses compactness to construct a dummy subset of $\tau$. 
Lemma B builds a function on the power set of $\tau$ which is used to apply lemma C: which makes us conclude that $\text{card}(\tau)$ was measurable.

Lemma A' is an improvement of theorem 5.1.2. in chapter 2, and its proof is very similar.

2.3.1. Lemma A':
(i) For every $T_1 \subset \tau$ with $\text{card}(T_1) \leq \lambda$ there is a $T_0 \subset T_1$ with $\text{card}(T_0) \leq \lambda$ such that $\varphi$ does not depend on $T_1$.
(ii) There is $\mu \leq \lambda$ such that for every $T_1 \subset \tau$ with $\text{card}(T_1) \leq \lambda$ there is a $T_0 \subset T_1$ with $\text{card}(T_0) \leq \mu$ such that $\varphi$ does not depend on $T_1$.

Now lemma A' can be used to prove lemma A.

2.3.2. Lemma A: There is a $T_0 \subset \tau$ with $\text{card}(T_0) \leq \lambda$ such that for every $T_0 \subset \tau$ with $\text{card}(T_0) \leq \lambda$ does not depend on $T_0$.

The second lemma used in the proof of the Finite Dependence Theorem gives us the connection to ultrafilters. Here we use some material from section 1.3., in particular the definition of $\text{UP}(L)$.

2.3.3. Lemma B: Let $\mu$ be a cardinal, $L$ be a logic and $\varphi$ a $L[\tau]$-sentence. If $T_0 \subset \tau$ but for each $T_1 \subset T_0$ with $\text{card}(T_1) \leq \lambda$, $\varphi$ does not depend on $T_1$, then there is a function $f : P(T_0) \to \{0,1\}$ such that:

(i) $f$ is non-constant;
(ii) For every $T_1 \subset T_0$ with $\text{card}(T_1) \leq \lambda$, we have $f(T_1) = f(T_0)$ and
(iii) For every ultrafilter $F \in \text{UP}(L)$ (on $\mu$) $f$ is $F$-continuous.

Recall that if $F$ is an ultrafilter on $\mu$, $\{\sigma_i : \varphi \leq \mu\}$ are subsets of $T_2$ then $\lim F = \sigma$ iff for every $P \in T_2$ the set $\{P \in \sigma_i \leq \sigma_1 \} \in F$ and $f$, is $F$-continuous iff $\sigma = \lim F$ implies that $f(\sigma) = \lim f(\sigma_i)$.

The third lemma, used in the proof of the Finite Dependence Theorem, gives us the connection to measurable cardinals:

2.3.4. Lemma C: If $F$ is a uniform ultrafilter on $\omega$ and $f : P(\omega) \to \{0,1\}$ satisfies (i) - (iii) of the previous lemma, then there is a measurable cardinal $\mu_0$ such that $\omega \leq \mu_0 \leq \omega$.

We are now in a position to prove the Finite Dependence Theorem.

Proof of the Finite Dependence Theorem: Assume $L$ is $[\omega]$-compact and $\omega(L) > \omega$. Then there is an $L[\tau]$-sentence $\varphi$ which does not depend only on a finite subset of $\tau$. So $\text{card}(\tau) = \omega$, and if $\text{card}(\tau) = \omega$ we are done by Theorem 5.1.2 of chapter 2. So $\text{card}(\tau) = \omega$.

By lemma A (for $\lambda = \omega$) we can assume that $\varphi$ does not depend on any countable subset of $\tau$. Now we apply lemma B to construct the function $f$ and by the Abstract Compactness Theorem (1.3.9.) and lemma A we know that $f$ is $F$-continuous for some uniform ultrafilter on $\omega$. So by lemma C we know that $\text{card}(\tau) \leq \mu_0$, the first uncountable measurable cardinal. But this shows that $\omega(L) > \mu_0$, a contradiction.

QED.
2.4. Dependence filters.

So far we studied the concept of a formula depending on some subset of a vocabulary \( \tau \), and our main result was the Finite Dependence Theorem. However, as the examples in section 1.6. and their discussion in 2.1.6. show, this need not be the appropriate notion. We are facing here a similar problem as in the analysis of compactness properties. There it turned out that the more appropriate tool to study compactness is the class of ultrafilters \( UF(L) \). Similarly here, we have to look at dependence filters.

2.4.1. Definition: Let \( \tau \) be an infinite vocabulary and assume, for notational simplicity, that \( \tau = \{ R_i : \alpha \} \), where \( R_i \) are relation symbols. Let \( \varphi \in L[\tau] \) be a formula of some logic \( L \). If \( X \subseteq \lambda \) we write \( \tau_X \) for \( \{ R_i \in \lambda \} \).

(i) Let \( F \) be an ultrafilter on \( \lambda \). We say that \( \varphi \) depends on \( F \) only, if, given two \( \tau \)-structures \( A = (A, R^A_i : \alpha) \) and \( B = (B, R^B_i : \alpha) \), and a set \( X \in F \) such that \( A \upharpoonright \tau_X \models B \upharpoonright \tau_X \) then \( A \models \varphi \iff B \models \varphi \). We call \( F \) an dependence filter for \( \varphi \).

(ii) Let \( Y_0 \cup Y_1 \cup \cdots \cup Y_n \) be a finite partition of \( \lambda \) and \( F_k \), \( (k = 0, 1, \ldots, n) \) be ultrafilters on \( Y_k \) respectively. We say that \( \varphi \) depends on \( F_0, F_1, \ldots, F_n \) only, if, given two \( \tau \)-structures \( A = (A, R^A_i : \alpha) \) and \( B = (B, R^B_i : \alpha) \), and sets \( X_k \in F_k \) such that \( A \upharpoonright \tau_X \models B \upharpoonright \tau_X \), where \( X = \bigcup_{0}^{n} X_k \), then \( A \models \varphi \iff B \models \varphi \). We call \( F_0, F_1, \ldots, F_n \) a finite dependence structure for \( \varphi \).

(iii) We can modify (ii) to allow infinite partitions. In this case we speak of dependence structures for \( \varphi \).

2.4.2. Examples:

(i) If a logic \( L \) has Finite Dependence, \( \varphi \in L[\tau] \), then \( \varphi \) has a principal dependence filter generated by the finite set \( \tau_0 \subseteq \tau \) on which \( \varphi \) only depends.

(ii) Let us return to the logic \( L_{\tau_0} \) from example 2.1.6(ii), introduced in 1.6.1. Recall that \( F \) is an ultrafilter on some set \( J \). Let \( R_i \), \( i \in J \) be relation symbols. The formula \( \bigcap_{F} \{ R_i : i \in J \} \) has among its dependence filters also the ultrafilter \( F \). However, if \( \tau = \{ R_i : i \in J \} \cup \{ S_i : i \in J \} \) then the dependences of the formula \( \bigcap_{F} \{ R_i : i \in J \} \land \bigcap_{F} \{ S_i : i \in J \} \) has to be described by a finite partition of \( \tau \) and a filter on each of the components, which in this case is \( F \).

(iii) If we look at example 1.6.5, it is easy to construct examples of sentences whose dependence is described by more complicated partitions and more complicated ultrafilters.

That those examples are more than accidental is shown by the following theorem from the treasure box [Shelah 1983 Manuscript].

2.4.3. Theorem: (Shelah's Finite Dependence Structure Theorem) Let \( L \) be a \( [\omega] \)-compact logic, \( \tau = \{ R_i : \alpha \} \) a vocabulary and \( \varphi \in L[\tau] \). Then there is a finite partition \( Y_0 \cup Y_1 \cup \cdots \cup Y_n \) of \( \lambda \) and countably complete ultrafilters \( F_k \), \( (k = 0, 1, \ldots, n) \) on \( Y_k \) respectively, such that \( \varphi \) only depends on \( F_0, F_1, \ldots, F_n \). In other words, every \( \varphi \in L[\tau] \) has finite dependence structure.

The proof of the Finite Dependence Structure Theorem consists of elaborations of the lemmas A,B and C in section 2.3. The Finite Dependence Structure Theorem opens
new perspectives in the study of dependence phenomena for compact logics for the case that there are uncountable measurable cardinals.
3. $L$-EXTENSIONS and AMALGAMATION.


Given a logic $L$, it is clear how to define the analogue of elementary equivalence of two structures of the same language $\tau$: They have to satisfy the same $\tau$-sentences. It is more problematic to generalize the notion of elementary embeddings, because already in first order case either free variables or new constant symbols are used in the definition and various definitions are equivalent only because of the Finite Occurrence or even because of compactness. In the general case it is convenient to introduce a cardinal parameter.

Let us recall that the $L$-diagram of an $\tau$-structure $A$ is the set of $L$ sentences true in the structure $<A,A>$, i.e. the structure $A$ augmented with names for all its elements. We denote the $L$-diagram of $A$ by $D_L(A)$.

3.1.1. Definitions:

(i) A $\tau$-structure $B$ is an $L$-extension of a $\tau$-structure $A$, if $A$ is a substructure of $B$ and the two structures $<A,A>$ and $<B,A>$ satisfy the same $L$-sentences. In this case we write $A <_L B$.

(ii) A $\tau$-structure $B$ is a $(\kappa,L)$-extension of a $\tau$-structure $A$, if $A$ is a substructure of $B$ and for every subset $A_0 \subset A$ with $\text{card}(A_0) < \kappa$ the two structures $<A,A_0>$ and $<B,A_0>$ are $L$-equivalent. In this case we write $A <_{L,\kappa} B$.

3.1.2. Examples:

(i) For $L = L_{\omega,\omega}$ without occurrence restrictions we have clearly $A <_L B$ iff $A = B$. Using indiscernibles, it is easy to construct $A,B$ such that $A <_{L,\kappa} B$ for a given $\kappa$.

(ii) If $\omega(L) = \kappa$ then clearly every $(\kappa,L)$-extension is an $L$-extension.

(iii) If $L$ is a compact logic, then we have, by the Finite Occurrence Theorem of the previous section, that $L$-extensions and $(\kappa,L)$-extensions coincide for every $\kappa$.

In model theory extensions are studied extensively and the following three situations are characteristic:

(i) Do models have $(\kappa,L)$-extensions?

(ii) Given a chain of extensions, is the union an extension of each member of the chain?

(iii) Given three $\tau$-structures $A_i, i=0,1,2$ such that $A_0$ is an $L$-substructure of both $A_1$ and $A_2$, does there exist an amalgamating extension $A_3$?

In fact in chapter 20 we shall describe an approach to abstract model theory, which is entirely based on those aspects and not on the notion of formulas and logics. Here however, we shall study logics which allow these constructions.

In this chapter we shall deal with logics which allow one of the above constructions (i)-(iii) universally.

3.1.3. Definition:

(i) A logic $L$ satisfies $\text{EXT}(L)$ or has the Extension Property, if every infinite $\tau$-structure $A$ has an $L$-extension $B$.

(ii) A logic $L$ satisfies $\text{REXT}(L)$ or has the relativized Extension Property, if for every
infinite definable set $X$ in some $\tau$-structure $A$ there is a $\tau$-structure $B$ which is a $L$-extension of $A$ which extends $X$ properly.

Clearly, $\text{REXT}(L)$ implies $\text{EXT}(L)$ for every logic $L$.

3.1.4. Example: Every compact logic $L$ satisfies $\text{REXT}(L)$.

In fact, the following proposition is easily proved by the reader:

3.1.5. Proposition: If a logic $L$ is $[\omega]$-compact the $L$ satisfies $\text{REXT}(L)$.

We shall return to the study of $\text{EXT}$ and $\text{REXT}$ in section 3.2.

3.1.6. Definitions:

(i) A family of $\tau$-structures $A_i, i < \kappa$ is an $L$-chain if $A_i$ is an $L$-extension of $A_j$ for every $j < i < \kappa$.

(ii) A logic $L$ satisfies $\text{CHAIN}(\kappa, L)$ or respects chains of length $\kappa$, if given a $L$-chain $A_i, i < \kappa$ then $\bigcup A_i$ is an $L$-extension of each of the $A_i$'s.

(iii) A logic $L$ satisfies $\text{CHAIN}(L)$ or has the Chain Property, if it satisfies $\text{CHAIN}(\kappa, L)$ for every $\kappa$.

3.1.7. Remark: $\text{CHAIN}(\omega, L)$ was called in chapter 3 the Tarski-union-property.

3.1.8. Examples:

(i) $L_{\kappa,B}$ has the Chain Property.

(ii) If $\kappa$ is regular then $L_{\kappa,B}$ respects chains of length $\lambda$, cf $(\lambda)^{\kappa}$.

In chapter 3 (theorem 2.2.2.) the following result of [Lindstrom 1973] was proved:

3.1.9. Theorem: If a logic $L$ is compact and respects chains of length $\omega$ then $L = L_{\omega,\omega}$.

There are no logics known which are $[\omega]$-compact and satisfy $\text{CHAIN}(L)$. It is open whether this due to a theorem or simple ignorance of more examples. It would be interesting to explore more consequences of $\text{CHAIN}$-properties. In [Tharp 1974] and [Makowsky 1975] "continuous" or "secureable" quantifiers are studied, which, if added to first order logic, give us logics which do satisfy $\text{CHAIN}(L)$.

3.1.10. Definitions:

(i) A logic $L$ satisfies $\text{Am}(\kappa, L)$ or has the $\kappa$-Amalgamation Property if, given three $\tau$-structures $A_i, i = 0, 1, 2$ such that $A_0 \subsetneq A_1, A_0 \subsetneq A_2, j = 0, 1, 2$ there is a $\tau$-structure $B$ such that $A_i' \subsetneq B, i = 0, 1, 2$ and the diagram commutes.

(ii) A logic $L$ satisfies $\text{Am}(L)$ or has the Amalgamation Property, if $\text{Am}(<\kappa, L)$ holds for every $\kappa$.

(iii) A logic $L$ satisfies $\text{JEP}(L)$ or has the Joint Embedding Property if any two $L$-equivalent $\tau$-structures $A_i, i = 1, 2$ have a common $L$-extension $B$.

One can also introduce cardinal parameters for $L$-equivalence and the Joint Embedding Property, but we shall not need this in our exposition.

3.1.11. Theorem:

(i) Every compact logic $L$ has the Joint Embedding Property.

(ii) If a logic $L$ satisfies $\text{JEP}(L)$ then it has the Amalgamation Property.

Proof:

(i) Since $L$ is compact, $L$ has finite occurrence, by the Finite Occurrence Theorem.
we can use compactness again to show that \( D_L(A_i) \cup D_L(A_0) \) has a model \( B \) which is a \((\kappa,L)\)-extension of both the \( A_i \), \( i = 1,2 \).

(ii) Let \( A_i \), \( i = 0,1,2 \) be as in the hypothesis of the Amalgamation Property. Clearly the two structures \( < A_1, A_0 > \), \( < A_2, A_0 > \) are \( L \)-equivalent, so let \( B \) be an \( L \)-extension of both of them. Clearly this \( B \) satisfies the requirements of the Amalgamation Property. QED.

3.1.12. Examples:

(i) If \( \kappa \) is a strongly compact cardinal, then \( L_{\kappa, \kappa} \) satisfies the Joint Embedding Property.

(ii) Let \( L = L_{\omega, \omega} \), but with Finite Occurrence. It is easy to see that \( L \) does not satisfy the Amalgamation Property.

3.1.13. Definition: A logic \( L \) has the Robinson Property if whenever \( \Sigma \subseteq L[\tau_1] \) \( i = 0,1,2 \) are such that \( \tau_0 = \tau_1 \cap \tau_2 \) and \( \Sigma_0 \) is complete and \( \Sigma_0 \cup \Sigma_j \), \( j = 1,2 \) has a model, then \( \bigcup_{i=0}^{2} \Sigma_i \) has a model. Recall that a set of sentences \( \Sigma \) is complete if any two models of \( \Sigma \) are \( L \)-equivalent.

D. Mundici has studied various aspects of the Robinson Property, cf. [Mundici 1981 TAM, 1981 ZMLG]. The Robinson Property is extensively discussed in chapter 19. Here we only note the following theorem:

3.1.14. Theorem: Every logic \( L \), which has the Robinson Property also has the Amalgamation Property.

Proof: Let \( \Sigma_i = D_L(A_i) \) where the \( A_i \) are as in the hypothesis of the Amalgamation Property. QED.

The Amalgamation Property is further studied in section 3.3 and 3.4.

Let us summarize here some rather unexpected consequences of the Amalgamation Property, as they follow from theorem 3.2.1 and the Abstract Amalgamation Theorem (3.3.1).

3.1.15. Theorem: Let \( L \) be a regular logic with occurrence number less than the first uncountable measurable cardinal.

(i) If \( L \) has the Amalgamation Property, then \( REXT(L) \) holds.

(ii) If \( \text{CHAIN}(\omega, L) \) holds and \( L \) has the Amalgamation Property then \( L = L_{\omega, \omega} \).

This theorem stresses the connections between the more "algebraic" properties of logics, as they are at the core of chapter 20: in our context the theorem is trivial. But then, the reader may try to prove (i) directly. The same challenge applies to corollary 3.3.4.

3.2. \( L \)-Extensions

In this section we prove a converse of proposition 3.1.5 and explore further variations of extension properties.

3.2.1. Theorem: A regular logic \( L \) satisfies \( REXT(L) \) if \( L \) is \([\omega]\)-compact.
Proof: Assume $\text{REXT}(L)$ and that $L$ is not $[\omega]$-compact. So by theorem 1.2.2. (or chapter 2, proposition 5.2.4.) $\omega$ is cofinally characterizable in $L$ by some expansion $A$ of $\langle \omega, < \rangle$. But clearly $\omega^A$ is a maximal definable subset of $A$, a contradiction. The other direction was proposition 3.1.5. QED.

We next introduce a cardinal parameter into our extension properties:

3.2.2. Definition: A logic $L$ satisfies $\text{EXT}(\kappa, L)$ if, whenever a $\tau$-structure $A$ has no proper $L$-extension then $\text{card}(A') < \kappa$.

3.2.3. Proposition: If a logic $L$ is $[\lambda]$-compact then $L$ satisfies $\text{EXT}(\lambda, L)$.

The proof is left to the reader.

The next theorem is one of the least constructive theorems in logic: Its proof uses very heavily the replacement axiom. To test our assertion the reader should try to prove theorem 3.2.4. below in $\mathcal{ZC}$ rather than in $\mathcal{ZFC}$. (This problem was suggested by A. Dodd.)

3.2.4. Theorem: Let $\lambda_0$ be an infinite cardinal and $L$ satisfies $\text{EXT}(\lambda_0, L)$ then there is a cardinal $\kappa$ such that $L$ is $[\kappa]$-compact.

Proof: We prove the contrapositive: If $L$ is not $[\kappa]$-compact for any cardinal $\kappa$ then for every cardinal $\lambda_0$ there is a maximal structure $B$ with $\text{card}(B) > \lambda_0$. (Recall that a structure is maximal for $L$ if it has no proper $L$-extensions.) By theorem 1.2.2. every regular cardinal $\lambda$ is cofinally characterizable via some expansion $B$, which we assume without loss of generality of minimal cardinality $g(\lambda)$.

Now let $\mu$ be the first cardinal such that:

(i) If $\nu < \mu$ then $g(\nu) < \mu$.

(ii) $\lambda_0 \leq \mu$.

(iii) $\text{cf} (\mu) = \omega$.

Clearly such a cardinal exist, e.g. the $\omega$-limit of the first fixed points of the function $g(\nu)$. (This is where the replacement axiom is used without control over the complexity of the set theoretic formula involved.)

Let $B$ be the complete expansion of the structure $\langle \mu, \in \rangle$. We claim that $B$ is maximal. For otherwise, let $C$ be an $L$-extension of $B$. If $C$ is proper there is a $c \in C \setminus B$. Remember, $\text{cf} (\mu) = \omega$ and let $\{b_n : n \in \omega \}$ be a cofinal sequence in $B$. Since $\omega$ is cofinally characterizable in $L$ via $B$, $g(\omega) < \mu$ and $B$ is a complete structure, $\{b_n : n \in \omega \}$ is also cofinal in $C$. So clearly, $C = c \in b_k$ for some $k \in \omega$. Now let $d' \in B$ be the smallest (with respect to $\in$) element in $B$ such that $C = c \subset d'$. We note that $d$ is an ordinal. Let $\delta = \text{cf} (d')$ and $\{d_k : i < \delta \}$ be a sequence cofinal to $d'$ in $B$. Again, since $g(\delta) < \mu$ and $\delta$ is cofinally characterizable in $L$ via $B$, $\{d_k : i < \delta \}$ is cofinal to $d'$ in $C$. So, there is a $\xi < \delta$ with $C = c \subset d_\xi$, which contradicts the minimality of $d$. This establishes that $B$ is maximal. Clearly, $\text{card}(B) > \lambda_0$ by our construction, which completes the proof. QED.

If there are no uncountable measurable cardinals, we get the following situation:

3.2.5. Theorem: Assume there are no uncountable measurable cardinals and $L$ is a regular logic. Then the following are equivalent:
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(i) $L$ is $[\omega]$-compact.
(ii) $L$ satisfies $\text{EXT}(L)$.
(iii) $L$ satisfies $\text{REXT}(L)$.

Proof: (i) $\Rightarrow$ (iii) was proposition 3.1.5, and (iii) $\Rightarrow$ (ii) follows from the definitions. To prove (ii) $\Rightarrow$ (i) we apply theorem 3.2.4, and then theorem 1.5.2. QED.

Also the existence of uncountable measurable cardinals is closely related to our extension properties. Let us look at the following example:

3.2.6. Example: A logic $L$ for which $\text{EXT}(L)$ and $\text{REXT}(L)$ do not coincide. Let $Q_{\lambda,\varepsilon}$ be a quantifier of type $\langle 1, 1 \rangle$ with satisfaction defined by $A \Vdash Q_{\lambda,\varepsilon} \varphi(x, y) \iff \text{card}(\varphi^A) < \lambda$ and $\text{card}(\varphi^A) > \varepsilon$.

3.2.7. Lemma: Let $L = L_{\omega,\varepsilon}(Q_{\mu_0,\varepsilon})$ where $\mu_0$ is the first uncountable measurable cardinal.

(i) $L$ is $[\mu_0]$-compact.
(ii) $L$ satisfies $\text{EXT}(L)$.
(iii) $L$ does not satisfy $\text{REXT}(L)$ and therefore is not $[\omega]$-compact.

Proof: We prove (iii) first. For this we look at the structure $A = (2^{\aleph_0})^+, \varepsilon >$. It is straightforward to find an expansion $A'$ of $A$ in which $\omega, \varepsilon >$ is cofinally characterized in $L$, so we apply theorem 3.2.1, together with theorem 1.2.2.

To prove (ii) we distinguish two cases: On structures $A$ with $\text{card}(A) \leq \varepsilon^{\mu_0}$, it is equivalent to first order logic, since the quantifier $Q$ acts trivially, being always false, so first order extensions will do. On structures $A$ with $\text{card}(A) > \varepsilon^{\mu_0}$ we apply (i).

To prove (i) we use the Abstract Compactness Theorem (1.3.9.) and show that every $\mu_0$-complete ultrafilter $F$ on $\mu_0$ is in $\text{UF}(L)$. We need $\mu_0$-completeness to see that finiteness is preserved under ultrapowers over $F$ and we need that $\mu_0$ is small for $(2^{\aleph_0})^+$ to see that the other cardinality restriction is preserved under such ultrapowers. QED.

This example together with theorem 3.2.5. gives us immediately the following characterization of the existence of uncountable measurable cardinals.

3.2.8. Theorem: The following are equivalent:

(i) For every logic $L$ $\text{EXT}(L)$ holds $\iff \text{REXT}(L)$ holds.
(ii) There are no uncountable measurable cardinals.

Finally let us have a look at Hanf numbers. We shall draw some corollaries from results in the previous sections, giving links between existence of some new type of Hanf numbers and various forms of compactness. The existence of this new Hanf number for every finitely generated logic is, as it turns out, equivalent to Vopenka's principle. Let us first recall some definitions from chapter 2.8:

3.2.9. Definitions: Let $L$ be a logic.

(i) Let $\Phi \subseteq L[T]$ be a set of sentences and $\lambda$ be a cardinal. $\Phi$ pins down the cardinal $\lambda$, iff $\Phi$ has a model of cardinality $\lambda$, but $\Phi$ has no models of arbitrary large cardinalities.
(ii) We define a function $h_{\Phi}(L)$ to be the supremum of all cardinals that can be pinned down by a set of $L$-sentences of power $\leq \varepsilon$. $h_{1}(L) = h_{\Phi}(L)$ from chapter 2.6.
(iii) We define $h_{1}(L)$ to be the supremum of all $h_{\Phi}(L)$ if it exists, and otherwise we...
write \( h_w(L) = \omega \). We say that \( L \) has a global Hanf number, if \( h_w(L) < \omega \).

Global Hanf numbers do not necessarily exist, even for finitely generated logics. Clearly compact logics do have global Hanf number \( \omega \). The following clarifies the relationship between compactness and global Hanf numbers.

**3.2.10. Proposition**: (Makowsky) Let \( L \) be a logic.

(i) If \( L \) is \((\omega, \lambda)\)-compact then \( h_w(L) = \lambda \).

(ii) If \( L \) is \( [\omega] \)-compact and \( \text{CHAIN}(L) \) holds, then \( h_w(L) = \omega \).

(iii) If \( L \) has a global Hanf number, then \( \text{Comp}(L) \neq \emptyset \).

**Proof**:

(i) This is a standard application of the method of diagrams.

(ii) Using proposition 3.1.5, we construct an \( L \)-chain of proper \( L \)-extensions. Now \( \text{CHAIN}(L) \) allows us to go as far as we want.

(iii) Let \( \chi_0 \) be the global Hanf number of \( L \). Clearly, every structure of cardinality \( \geq \chi_0 \) has a proper \( L \)-extension, i.e. \( \text{Ext}(\chi_0, L) \) holds. So the result follows from theorem 3.2.4. QED.

**3.2.11. Corollary**: (Makowsky) Assume there are no uncountable measurable cardinals. If \( L \) is a logic which has a global Hanf number then \( L \) has finite occurrence.

**Proof**: By proposition 3.2.10, \( \text{Comp}(L) \neq \emptyset \), so by 1.5.3. and the assumption on measurable cardinals, \( L \) is \( [\omega] \)-compact. Now we apply the finite occurrence theorem (2.2.1.). QED.

The following is an improvement of theorem 1.5.17.

**3.2.12. Theorem**: (Makowsky) The following statements are equivalent:

(i) For every finitely generated logic \( L \) \( \text{Comp}(L) \neq \emptyset \).

(ii) Every finitely generated logic \( L \) has a global Hanf number.

(iii) For every finitely generated logic \( L \) \( \text{Comp}(L) \neq \emptyset \).

(iv) Vopěnka's principle.

**Proof**:

(i) \( \implies \) (ii): This follows from proposition 3.2.10(i) above.

(ii) \( \implies \) (iii): This follows from proposition 3.2.10(ii); above.

(iii) \( \implies \) (iv) and (iv) \( \implies \) (i) both follow from theorem 1.5.17. QED.

Theorem 3.2.12 tells us that there are logics which have no global Hanf number provided Vopěnka's principle is false. Let us end this section with some examples:

**3.2.13. Examples**:

(i) Let \( L \) be \( L_{\omega, \omega} \). Let \( A \) be a complete expansion of a structure of cardinality \( \lambda \). If there are no uncountable measurable cardinals, \( A \) has no proper \( L \)-extensions (see theorem 1.2.3.), so the complete \( L \)-theory of \( A \) pins down \( \lambda \). Hence, assuming there are no uncountable measurable cardinals, \( L \) has no global Hanf number.

(ii) Let \( L_0 \) be the logic \( L_{\omega, \omega}(Q) \) and \( L_1 \) be \( L_{\omega, \omega}(Q) \). In [Malitz-Reinhart 1972] it is shown that \( h_w(L_1) \) \( (\ell = 0, 1) \) is bigger than the first uncountable measurable cardinal.

(iii) Let \( L \) be \( L_{\omega, \omega} \), i.e. second-order logic. By [Magidor 1971] \( h_w(L) \) is smaller than the first extendible cardinal.
3.3. The Amalgamation Property

In this section we present our main theorem in the analysis of the amalgamation properties:

3.3.1. Theorem (Abstract Amalgamation Theorem):
Let \( L \) be a logic with occurrence number \( o(L) = \lambda \) and with the Amalgamation Property. Then \( L \) is ultimately compact. In fact it is \([\lambda, \lambda] \)-compact.

The proof of this theorem will be outlined in section 3.5. Here we mainly illustrate various consequences of this theorem and discuss examples and limitations.

For logics with finite occurrences we immediately get:

3.3.2. Theorem:
For a logic \( L \) with finite occurrence the following are equivalent:
(i) \( L \) is compact
(ii) \( L \) has the Amalgamation Property.
(iii) \( L \) has the Joint Embedding Property.

Proof: We have seen in theorem 3.1.11. that (i) implies (ii) and (iii). And that (iii) implies (ii). So let us assume (ii). From theorem 3.3.1. we get immediately that \( L \) is \([\lambda, \lambda] \)-compact for every regular \( \lambda \) and therefore compact by theorem 1.1.8. QED

D. Mundici has studied the Joint Embedding Property extensively, cf. [Mundici 1982; FM,AMLG]. In general the Joint Embedding Property is not known to be equivalent to the Amalgamation Property. In chapter 19 some consequences of the Joint Embedding Property are studied. Using more of the set theoretic machinery we get

3.3.3. Theorem: If \( L \) is a logic with \( o(L) < \mu_0 \), where \( \mu_0 \) is the first uncountable measurable cardinal, then the following are equivalent:
(i) \( L \) is compact
(ii) \( L \) has the Amalgamation Property.
(iii) \( L \) has the Joint Embedding Property.

Proof: We only have to prove (ii) \( \rightarrow \) (i): Using theorem 3.3.1. we get \([\kappa] \)-compactness for some \( \kappa < \mu_0 \), so by theorem 1.5.2. we get \([\omega] \)-compactness and therefore by theorem 2.2.1. Finite Occurrence. So now the results follows by another application of theorem 3.3.1. QED

3.3.4. Corollary: Let \( L \) be a logic with \( o(L) < \mu_0 \), where \( \mu_0 \) is the first uncountable measurable cardinal.
(i) If \( L \) has the Amalgamation Property (Joint Embedding Property) then every sublogic \( L_0 < L \) has the Amalgamation Property (Joint Embedding Property).
If $L$ has the Amalgamation Property (Joint Embedding Property) then $A(L)$ also has the Amalgamation Property (Joint Embedding Property).

**Proof:** (i) This is clearly true for compactness, so by theorem 3.3.3. also for the Amalgamation Property.

(ii) It is easy to see, that the $\mathcal{A}$-closure of logics preserves compactness and Finite Occurrence.

The reader may try to prove this without using theorem 3.3.3.

3.3.5. **Corollary:** Let $L$ be a logic with $\mathcal{o}(L) < \mu_0$, where $\mu_0$ is the first uncountable measurable cardinal. If $L$ has the Robinson Property, then $L$ is compact.

**Proof:** Use theorem 3.1.14 and theorem 3.3.3. QED.

For Finite Occurrence logics we shall see in chapter 19 another proof of corollary 3.3.5 without using theorem 3.3.1.

The rest of this section is devoted to examples and applications of the above theorems. The first example gives a real application of theorem 3.3.2, for the following result was originally derived from it:

3.3.6. **Example:** Let $L_{\omega,\omega}(Q_\omega)$ be the first order logic with the additional quantifier "there exist at least $\kappa$ many ". Theorem 3.3.2 gives us immediately that this logic satisfies the Amalgamation Property for no cardinal $\kappa$. For $\kappa = \omega$ or $\omega_1$ this was shown by [Malitz-Reinhart 1972], the other cases were open till theorem 3.3.2. was proven.

The next examples all show that the assumption on large cardinals cannot be dropped, in any of the above statements.

3.3.7. **Examples:**

(i) The logic $L_{\omega,\omega}$ has no occurrence number. Since this logic can describe any structure up to isomorphism, one easily verifies that the Robinson Property and the Amalgamation Property hold trivially, but $L_{\omega,\omega}$ has no compactness whatsoever.

(ii) In [Makowsky-Shelah 1983] it is shown that if $\kappa$ is an extendible cardinal, then $L_{\kappa,\kappa}^2$, i.e. second order logic with conjunctions, first order and second order quantification over $< \kappa$ many formulas or variables, satisfies the Robinson Property, and hence the Amalgamation Property and is $[\omega,\kappa]$-compact. Clearly, $\mathcal{o}(L_{\kappa,\kappa}^2) = \kappa$.

(iii) Now let us look at $L_{\alpha,\omega}$ with additionally the Finite Occurrence Property. It is easy to see, that for $\lambda > \omega$ the Amalgamation Property fails. But $L_{\alpha,\omega} < L_{\alpha,\kappa}^2$, so corollary 3.3.4 cannot be improved.

(iv) The logic $L_{\omega,\omega}$ satisfies the Amalgamation Property trivially, but does not satisfy the Robinson Property, as pointed out in [Makowsky-Shelah 1979].

(v) In section 3.5 we present a $[\omega]$-compact logic $L$ which has the Amalgamation Property, but for which $\text{Comp}(L)$ has a large gap. This example presupposes the existence of strongly compact cardinals.
3.4. Proof of the Abstract Amalgamation Theorem.

3.4.1. Synopsis. We first observe that by 1.1.9, it suffices to prove the following weaker theorem:

3.4.2. Theorem:
Let \( \lambda \) be a regular cardinal and \( L \) be a logic with occurrence number \( o(L) \leq \lambda \) and with the Amalgamation Property. Then \( L \) is \( [\lambda] \)-compact.

We give first an outline of the proof, to help the reader. We assume for contradiction that \( \lambda \) is regular and \( L \) is not \( [\lambda] \)-compact. Using theorem 1.2.2, we construct a class \( K \) of linear orderings with additional predicates in which points of cofinality \( \lambda \) are absolute. Inside \( K \) we show the existence of some sufficiently homogeneous structure \( N \). In \( N \) we shall find \( \mathcal{M}_i \) \((i=0,1,2)\) being a counterexample to \( AP \) for \( L \). The occurrence number and the isomorphism axiom will be needed to show that \( \mathcal{M}_0 \not< L \mathcal{M}_i \) \((i=1,2)\) and the absoluteness of "cofinality \( \lambda \)" to show that there is no amalgamating structure.

The counterexample to amalgamation is patterned after the following example: Let \( K \) be the class of dense linear orderings with an additional unary predicate \( \text{Red} \) such that both \( \text{Red} \) and its complement are dense, let \( A < K \) hold if \( A \) is an elementary substructure of \( B \) and the universe of \( A \) is a dense subset of the universe of \( B \). We shall show that \( K \) with this notion of substructure \( <_K \) does not allow amalgamation; For this let \( A_0 \) be the rationals properly coloured, and let \( A_1 \) \((i=1,2)\) the rationals augmented by one element (say \( p \)) coloured \( \text{Red} \) in \( A_1 \) and not coloured in \( A_0 \). Clearly \( A_0 <_K A_i \) \((i=1,2)\), but no amalgamating structure exists, since otherwise \( p \) is simultaneously coloured and not coloured.

3.4.3. The structure \( \mathcal{M} \).

Now, let \( \lambda \geq \text{OC}(L) \) be regular and \( L \) not \( [\lambda] \)-compact. By theorem 1.2.2, \( \lambda \) is cofinally characterize in \( L \) in a structure \( \mathcal{M} \). We need some more information on \( \mathcal{M} \).

Let \( \Sigma, \Sigma_1 = \{ \varphi_{\alpha} : \check{\alpha} \check{<} \lambda \} \) be the counter-example to \( [\lambda] \)-compactness. Put \( \Sigma^1 = \{ \varphi_{\alpha} : \check{\alpha} \check{<} \lambda \} \) and \( \mathcal{M}_i = \Sigma \cup \Sigma^1 \). W.l.o.g., the \( \mathcal{M}_i \)'s are structures of some countable vocabulary \( \tau \) (coding more predicates with parameters), and have the same power \( \mu \geq \lambda \). \( \mathcal{M}_i \subset < \mathcal{M}_i, \zeta_n (n \in \omega) \). We want to code all the \( \mathcal{M}_i \)'s into one structure. So we let \( \mathcal{M} \) be such that:

1. \( \mathcal{M} = < \mathcal{M}, \zeta_n, c_j (n \in \omega, j \in \lambda) > \)
2. \( < \mathcal{M}, < \) is a linear order of cofinality \( \lambda \) such that every initial segment has power \( \mu \) (of order type \( \mu^* + \lambda \), e.g.);
3. \( \{ c_{j, j} : \check{\alpha} \check{<} \lambda \} < \mathcal{M} \) is increasing and unbounded.
4. If \( x \leq c_j \) but \( x > c_i \) for every \( i < j \) then \( \check{\langle} y \in \mathcal{M} : y < x, \zeta_n (x, \ldots, \ldots) \check{=}, \mathcal{M} \).

Let \( T = \mathcal{M}_\tau (\mathcal{M}) \) for some fixed \( \check{\tau} \) as described above.

Claim: Then \( T \) cofinally characterizes \( \lambda \).
This is proved like theorem 1.2.2.

3.4.4. The class \( K(\mathcal{M}) \).

For the rest of this section \( \mathcal{M} \) is fixed. We now define a class of structures \( K(\mathcal{M}) \):

The vocabulary of \( K(\mathcal{M}) \) is that of \( \mathcal{M} \) without the constant symbols for \( c_j \) but with two additional unary predicate symbols \( P \) and \( R \) and one additional binary predicate.
symbol $I$. Actually our main focus is on the order together with $P, R$, and $I$ is used to code copies of $\mathcal{M}$, which we need to guarantee the absoluteness of cofinality $\lambda$.

A model in $K(\mathcal{M})$ is of the form $A = \langle A, \langle \mathcal{G}, P, R, I \rangle \rangle$ with the requirements:

(K1) If $x \in P$ then the cofinality of $x$ in $\langle A, \langle \mathcal{G} \rangle \rangle$ is $\lambda$ with a witnessing sequence $\{c_j(x) : j \in \lambda\}$.

(K2) $(a, x) \in I$ implies that $a < x$.

(K3) $(a, x) \in I$ implies that $x \in P$ and $a \notin P$.

(K4) $P(x)$ implies that $f(\langle c_j(x), x \rangle)$ for every $j \in \lambda$.

Put $J^a_1 = \{a \in A : (a, x) \in I\}$ and $J^a_2$ be the substructure of $\langle A, \langle \mathcal{G} \rangle \rangle$ induced by $J^a_1$.

(K5) The structure $\langle J^a_1, c_j(x) \rangle$ is isomorphic to $\mathcal{M}$.

(K6) $R \subset P$

We call a structure in $K(\mathcal{M})$ pure if additionally

(K7) $\mathcal{Q}$ is false where not defined by the previous requirements.

3.4.5. Comments:

Note that if $A \in K(\mathcal{M})$ is pure and $P$ in $A$ is empty, then $A$ is just a linear ordering, i.e. all the other relations are empty, too, by (K7).

If we add to $\mathcal{M}$ one point at the end, say $z$ and let $P = \{z\}, we get a structure in $K(\mathcal{M})$. We denote this structure by $\mathcal{M}^+.$

In general the structures in $K(\mathcal{M})$ are linearly ordered structures where every point in $P$ has a copy of $\mathcal{M}$ attached. In it in the way that different points have almost disjoint copies of $\mathcal{M}$, and $\mathcal{M}$ cofinally reaches its point in $P$. The choice of $R$ can be any subset of $P$. More precisely:

Fact 1: For every $A \in K(\mathcal{M})$ and every $a, a' \in A$, $J^a_1 \cap J^{a'}_1$ is bounded below both $a, a'$.

This is proved using the fact that $\mathcal{M}$ is of order type $\mu^+ \lambda$. Note that this is first order expressible and could have been stated also as an axiom among (K1)-K7.

Fact 2: If $A \in K(\mathcal{M})$ and $a \in P^A$ and we form $A'$ by changing the truth value of $a \in P^A$, but leaving everything else fixed, then $A' \in K(\mathcal{M})$.

Next we define the notion of $K$-substructure; $A \subset XB$ for $A, B \in K(\mathcal{M})$, by:

(K8) $A \subset B$

(K9) If $x \in P^A$ then $J^a_2 \subset A$.

(K10) If $x \in P^B \cap P^A$ then $\{a \in A : a < x\}$ is bounded below $z$ in $B$; i.e., there is $b_x \in B$ such that $b_x < x$ and for each $a \in A$ with $a < x$ we have $a < b_x$.

The idea behind this is that in $B$, new points in $P_B$ are added to $P^A$ in a way that they are not limits of points in $A$, and that points in $A$ which are of cofinality $\lambda$ also of cofinality $\lambda$ in $B$ with the same copy of $\mathcal{M}$ ensuring this as in $A$.

This ends the definition of $K(\mathcal{M})$ and of $K$-substructures.

3.4.6. Some more facts about $K(\mathcal{M})$.

Before we proceed with the proof of the theorem we collect some more facts:

Definition: If $A_1, A_2 \in K(\mathcal{M})$ we define $A_1 + A_2$ to be the disjoint union of $A_1, A_2$ with the linear ordering of $A_1$ and $A_2$ for their elements and $a_1 < a_2$ for every $a_1 \in A_1, a_2 \in A_2$.

For the other relations we just take their unions.

Fact 3: If $A_1, A_2 \in K(\mathcal{M})$ so $A_1 + A_2 \in K(\mathcal{M})$ and $A_i \subset xA_i + A_i$ ($i = 1, 2$).
This is clear from the definitions.

**Definition:** Denote by $L_\alpha^\varepsilon\{a \in A : a < \varepsilon\}$ and by $L_\alpha^\varepsilon$ the structure $A \upharpoonright L_\alpha^\varepsilon$. If $B \subseteq K(M)$ and $A \subseteq B$ we define a substructure $C(A)$ of $B$ by $C(A) \equiv \bigcup \{ L_\alpha^\varepsilon \mid \alpha \in A \}$.

This makes sense by fact 1 and ensures that:

**Fact 4:** For every $B \subseteq K(M)$, $A \subseteq B$, $C(A) \subseteq K(B)$, but in general $C(A)$ is not pure. Furthermore, if $A$ is bounded in $B$ by $b$, i.e. there is $b \in B$ with $A \subseteq L_b$, so $C(A) \subseteq L_b$ and $C(L_b) = L_b$.

**Fact 5:** If $A \subseteq K(M)$ and $d \in P^d$ then $A \upharpoonright L_d^\varepsilon \subseteq K(A)$.

**Fact 6:** If $\{A_i : i < \alpha\}$ is a sequence of structures in $K(M)$ such that $A_i \subseteq K(A_{i+1})$ then $A = \bigcup A_i \subseteq K(M)$ and $A_i \subseteq K(A_{i+1})$ for each $i < \alpha$.

**Definition:** If $A_1, A_2 \subseteq K(M)$, $B_i \subseteq K(A_i) (i=1,2)$ and $f : B_1 \cong B_2$ is an isomorphism, we define $A_1 + f A_2$ in the following way: Form the disjoint union of $A_1$ and $A_2$ modulo $f$ (i.e. identify elements only via $f$). This makes it into a partially ordered structure: where $a_i \in A_i (i=1,2)$ are comparable only if one of them is in the range or domain of $f$, or there is $b$ between $a_1, a_2$ which has been identified. For incomparable $a_1, a_2$ we extend the order on $A_1 + f A_2$ setting $a_1 < a_2$.

**Fact 7:** If $A_1, A_2 \subseteq K(M)$ and $f : B_1 \cong B_2$, $B_i \subseteq K(A_i) (i=1,2)$ then $A_1 + f A_2 \subseteq K(M)$ and $A_i \subseteq K(A_{i+1})$ if $A_i$.

The proofs of the facts are left to the reader.

### 3.4.7 Two lemmas

The next lemma is crucial for our construction:

**Lemma 1:** If $A \subseteq K(M)$ and $B$ is an $L$-extension of $A$ and $\{d_j : j < \lambda\}$ is cofinal in $J_\varepsilon$ for $\varepsilon \in P^d$, then $\{d_j : j < \lambda\}$ is cofinal in $J_\varepsilon$.

**Proof:** Let $\varepsilon \in P^d$, so $J_\varepsilon \subseteq J_\varepsilon$ by (K5) and by our assumption on $L$ and $M$, $L$ cofinally characterizes $\lambda$ in $M$. Using relativization of $L$ the structure $J_\varepsilon^M$ is an $L$-extension of $M$ so $M$ is cofinal in $J_\varepsilon^M$, hence $\{d_j : j < \lambda\}$ is cofinal in $J_\varepsilon^M$ which proves the lemma. QED.

The next lemma is proved in a similar way as one usually proves the existence of homogeneous structures for Jonsson classes (cf. chapter 20). We omit the proof here and show how one can now complete the proof of the theorem. A detailed proof of the lemma may be found in [Makowsky-Shelah 1983].

**Lemma 2:** There is a structure $N$ in $K(M)$ and $d_1 < d_2 < d_3$ in $N$ with $d_1 \in P^N (i=1,2,3)$, $d_1 \in R^N$, $d_2 \not\in R^N$, such that

1. $N \upharpoonright L_1^d \cong N \upharpoonright L_2^d \cong N \upharpoonright L_3^d$ and
2. $A \subseteq K(N) \upharpoonright L_1^d$ is bounded in $N \upharpoonright L_1^d$ then $N \upharpoonright L_1^d \cong N \upharpoonright L_2^d$ over $A (t=1,2)$.

### 3.4.8 Proof of the Abstract Amalgamation Theorem

Put $M = N \upharpoonright L_1^d (t=1,2,3)$. We have to verify some claims:

**Claim 1:** $M_1 \leq M_2 (t=1,2)$.

**Proof:** Let $\varphi$ be an $L[\tau(M_t)]$-sentence. Since the occurrence number $OC(L) \leq \lambda$, $\varphi$ depends on $< \lambda$ many constants, hence there is $a \in M_t$ and all the constants of $\varphi$ are
in $L_{0}$: So by fact 4 $M_{i} \models L_{0}$ is a bounded $K$-substructure of both $M_{i}$ and $M_{0}$. So, by lemma 2(ii) above, $\langle M_{i}:L_{0}\rangle$ is isomorphic to $\langle M_{0}: L_{0}\rangle$; hence by the basic isomorphism axiom, $\langle M_{i}:L_{0}\rangle \models \varphi$ iff $\langle M_{0}: L_{0}\rangle \models \varphi$.

Now let $f:M_{i} \rightarrow M_{0}$ be the isomorphism from lemma 2(i) above, and $g_{i}:M_{i} \rightarrow M_{0}$ (i=1,2), the $L$-embeddings from the claim i.

Since $L$ has $AP$, let $\mathbf{A}$ be the amalgamation for $g_{1}:M_{1} \rightarrow M_{0}$, $g_{2}:M_{2} \rightarrow M_{0}$.

Claim 2: $A|= d_{1}=d_{2}$.

Proof: $d_{1} \in P^{\mathbf{A}}$ (i=1,2) are both of cofinality $\lambda$ and $g_{i}(M_{i})$ is cofinal in $M_{0} \models L_{A}^{d_{1}}$, and $g_{2}(M_{i})$ is cofinal in $M_{0} \models L_{A}^{d_{2}}$, so by lemma 1 above also in $A|L_{A}^{d_{1}}$ and $A|L_{A}^{d_{2}}$, hence $A|= d_{1}=d_{2}$.

But claim 2 contradicts our assumption of lemma 2 above that $d_{1} \in R^{A}$ and $d_{2} \notin R^{A}$.

This completes the proof of the Abstract Amalgamation Theorem. QED.

In fact the same proof gives also the following versions of the Abstract Amalgamation Theorem:

3.4.9. Theorem*: Let $\kappa$ be a regular cardinal and $L$ be a logic such that:

(i) The Lowenheim number $\lambda(L)$ of $L$ is $\kappa$.

(ii) $Am(\kappa,L)$ holds.

Then $L$ is $(\kappa,\kappa)$-compact.

3.4.10. Theorem*: Let $L$ be a logic with occurrence number $\omega(L) \leq \lambda$. If $Am(\kappa,L)$ holds for every $\kappa \leq \lambda$ then $L$ is $[\omega,\lambda]$-compact.

It is open whether the converse of theorem 3.4.10 also holds.

3.5. An intriguing example.

Let us look now at logics which do have the Amalgamation Property, but have a large occurrence number. One naturally wonders if such a logic has to be an extension of $L_{\kappa}$ for some uncountable $\kappa$, possibly bigger than the occurrence number. This is clearly not the case, provided the logic $L$ is $[\omega]$-compact. The purpose of this section is to present an example of a logic $L$ with occurrence number $\omega(L)$ bigger than the first uncountable measurable cardinal $\mu_{0}$, which is still $[\lambda]$-compact for every $\lambda < \mu_{0}$, satisfies the Amalgamation Property, but is not compact. If, however, a logic $L$ satisfies the Amalgamation Property but is not $[\omega]$-compact, then we know that its occurrence number is bigger than $\mu_{0}$, and therefore, by proposition 1.2.4., every $\tau$-structure $\mathbf{A}$ with $\text{card}(\mathbf{A}) < \mu_{0}$ has an $L$-maximal expansion. This can be used to show that for every $\varphi \in L_{\mu_{0}}(\tau)$, there is $\tau' \subseteq \tau$ and a set $\Sigma \subseteq L[\tau']$ such that $\text{Mod}_{L_{\mu_{0}}}(\varphi) = \text{Mod}_{L(\Sigma)} \upharpoonright \tau$. In the presence of the Robinson Property, $\tau'$ can be assumed to be $\tau$. We develop this idea further in chapter 19, theorem 1.12.

3.5.1. Definitions: Let $\mu$ be a cardinal and $E \subseteq P(\mu)$ a family of subsets of $\mu$.

(i) We say that $E$ is $\langle \kappa \rangle$-closed, $\kappa$ a cardinal, if for every $\lambda < \kappa$ and every ultrafilter $\mathbf{F}$ on $\lambda$ the following holds: Given $\{ A_{i} \subset \mu : i < \lambda \}$, then $\{ t \in \mu : A_{t} \in E \} \in \mathbf{F}$ implies that
\[ \lim_{A_i} \{ \alpha \in \mu : \{ i \in \lambda : \alpha \in A_i \} \in \mathcal{F} \} \in E. \]

(ii) If \( \psi : i \in \mu \) is a family of \( L \)-formulas, we define a connective \( \bigwedge_{i \in \mu} \psi_i \) by
\[ V_{A \in E} \left( \bigwedge_{i \in A} \psi_i \bigvee \bigwedge_{i \notin \mu} -\psi_i \right). \]

3.5.2. Remarks:
(i) If \( E \) is a \( \kappa \)-complete ultrafilter on \( \mu \) then both \( E \) and \( P(\mu) - E \) are \( \langle \kappa \rangle \)-closed.
(ii) The connective \( \bigwedge_{i \in \mu} \psi_i \) is a generalization of the connective \( \bigwedge \) where \( F \) is some ultrafilter.

3.5.3. Definitions:
(i) Let \( \kappa_1, \kappa_2 \) be two strongly compact cardinals. We denote by \( E(\kappa_1, \kappa_2) \) the set of \( \langle \kappa_1 \rangle \)-bi-closed families \( E \subseteq P(\mu) \) with \( \mu < \kappa_2 \).
(ii) Let \( L_{E(\kappa_1, \kappa_2)} \) be the closure of first order logic under all the infinitary operations \( E \bigwedge_{i \in \mu} \psi_i \) for \( E \in E(\kappa_1, \kappa_2) \).
(iii) Recall that \( L = L_{D(\kappa_1, \kappa_2)} \) was defined in example 1.6.6 in a similar way as (ii) above, but instead of \( \langle \kappa_1 \rangle \)-bi-closed sets we only used \( \kappa_1 \)-complete ultrafilters.

3.5.4. Proposition*: (Shelah)
Let \( \kappa_1, \kappa_2 \) be two strongly compact cardinals and \( L = L_{E(\kappa_1, \kappa_2)} \).
(i) \( L_{D(\kappa_1, \kappa_2)} \) is \( \kappa_1 \)-closed.
(ii) \( L \) is \( \kappa_2 \)-compact.
(iii) \( L \) is \( \kappa_2 \)-closed.
(iv) Every ultrafilter \( F \) on \( \mu < \kappa_1 \) is in \( UF(L) \), i.e., is related to \( L \).
(v) For every cardinal \( \mu < \kappa_1 \) is the logic \( L \) \( \langle \mu \rangle \)-compact.
(vi) For no cardinal \( \mu, \kappa_1, \kappa_2 \) is \( L \) \( \kappa_2 \)-compact.

Proof: Essentially the same as in section 1.6. QED.

3.5.5. Theorem*: (Shelah)
Let \( \kappa_1, \kappa_2 \) be two strongly compact cardinals and \( L = L_{D(\kappa_1, \kappa_2)} \). Then \( L \) satisfies the Joint Embedding Property, and therefore the Amalgamation Property.

Outline of proof:
Let \( M_i, M_2 \) be two disjoint \( \tau \)-structures such that \( M_i = \subseteq \mathbb{M}_2 \) and let \( D_i(M_i) \) (\( i = 1, 2 \)) be their \( L \)-diagrams. We want to show that \( D_1(M_1) \cup D_2(M_2) \) has a model. Since \( L \subseteq L_{\kappa_2} \) and \( \kappa_2 \) is strongly compact, it suffices to show that for every subset \( \Gamma_1 \subseteq D_1(M_1) \) and \( \Gamma_2 \subseteq D_2(M_2) \) with \( \text{card} \( \Gamma_1 \) < \kappa_2 \), \( \Gamma_1 \cup \Gamma_2 \) has a model.

Let \( \Gamma_1, \Gamma_2 \) be given and assume \( \Gamma_2 = \{ \phi_i : i < \mu < \kappa_1 \} \). Put
\[ E_0 \subseteq \{ A \subseteq \mathbb{M}_2 : \forall \phi_i \in \{ A \} \text{ has a model} \}. \]

If \( \mu \in E_0 \) we are done. So assume, for contradiction that \( E_0 \not\in E_0 \). Clearly, \( \phi \in E_0 \), since \( \mathbb{M}_1 \) can be expanded to a model of \( \Gamma_2 \).

Claim 1: \( E_0 \) is \( \langle \kappa_1 \rangle \)-closed.
chapter 18.3 Extensions

This can be established using proposition 3.5.4(iv). 
Claim 2: If \( E \subseteq P(\mu) \), \( \mu < \kappa_2 \) is \((< \kappa_1)\)-closed and \( \mu \notin E \), then there is \( E_1 \subseteq P(\mu) \) with \( \mu \notin E_1 \) such that \( E_1 \) is \((< \kappa_1)\)-bi-closed.

This is proved using a reduction to infinitary propositional calculus, with conjunctions of length less than \( \kappa_1 \) and the fact that \( \kappa_1 \) is strongly compact.

Clearly, \( M_2 \models \Lambda \in \mu \varphi_i(\bar{x}) \), and therefore, \( M_2 \models \Lambda \in \mu \varphi_i(\bar{x}) \). Since \( L \) is closed under existential quantification of length less than \( \kappa_2 \), \( \exists \in E \Lambda \in \mu \varphi_i(\bar{x}) \) is an \( L \)-sentence and \( M_2 \models \exists \in E \Lambda \in \mu \varphi_i(\bar{x}) \). So also \( M_1 \models \exists \in E \Lambda \in \mu \varphi_i(\bar{x}) \). Therefore there is \( \bar{b} \) from \( M_1 \) and \( \bar{a} \in E_1 \) such that \( M_1 \models \Lambda \in \mu \varphi_i(\bar{b}) \) which shows that \( \Gamma_1 \cup \{ \varphi_i(\bar{a}) \mid \bar{a} \in A \} \) has a model.

From this we conclude that \( \bar{a} \in E_0 \), contradicting \( E_1 \subseteq P(\mu) \)-\( E_0 \). QED.

Using the Finite Occurrence Structure Theorem and the fact that \( L_{\infty}(\kappa_1, \kappa_2) \omega_2 \) is \([\omega_1]\)-compact, we get now

3.5.6. Proposition*: (Shelah) Let \( \kappa_1 < \kappa_2 \) be two strongly compact cardinals. Then the two logics \( L_{\infty}(\kappa_1, \kappa_2) \omega_2 \) and \( L_{\infty}(\kappa_1, \kappa_2) \omega_4 \) are equivalent.

3.5.7. Corollary*: (Shelah) Let \( \kappa_1 < \kappa_2 \) be two strongly compact cardinals. Then the logic \( L_{\infty}(\kappa_1, \kappa_2) \omega_2 \) has the Joint Embedding Property, and therefore the Amalgamation Property.

3.5.8. Remark: In chapter 19 theorem 1.1. states that, if \( L \) is a small logic with \( s(\omega) = \lambda \) ( \( s \) the size function of \( L \)) which satisfies the Joint Embedding Property, then there are at most \( 2^\lambda \) many regular cardinals \( \mu \) such that \( L \) not \([\mu]\)-compact. Theorem 3.5.4. shows that this is best possible.
4. DEFINABILITY


In model theory one frequently builds new models from a set of given models and it is often very useful to know that the theory of the so constructed model only depends on the theories of the models it was built from. Examples are the ultraproduct construction and various other product-like constructions, which mostly go back to the seminal papers [Mostowski 1952], [Los-Suszko 1957], [Feferman-Vaught 1959] and [Frayne-Morel-Scott 1962]. The possibilities of generalizations of the Los' lemma to logics in general are rather limited, as we have shown in section 1. For simpler constructions, such as disjoint unions or ordered sums, the preservation properties are usually proved with the use of Bäck and Forth arguments, as they are generalized in chapter 19: The first to consider such properties in the context of abstract model theory was S. Feferman in his papers [Feferman 1972, 1974PM, 1974a, and 1975]. The theme was then pursued in [Shelah 1975], [Makowsky, 1978] and [Makowsky-Shelah 1979].

In the context of abstract model theory, in contrast to specific examples of logics, only sum-like operations have played an independent role. They are also used heavily in chapters 12 and 13. For this reason we restrict our exposition here to the description of sum-like operations as they are used in the following subsections and as we think they are of interest for future research. Recent trends in theoretical computer science have shown that abstract model theory offers the appropriate framework to state problems and theorems dealing with specification of abstract data types [Goguen-Burstall 1983] and [Mahr-Makowsky 1983, 1983a], correctness of programs [Harel 1979, 1983], [Makowsky 1980] and [Manders-Daley 1983] and data base theory [Makowsky 1982]. Especially sum-like operations on abstract data types have been recently investigated by [Bergstra-Tucker 1983] to show that some of the concepts in program correctness are probably not stable enough to be transferred from one formalization to another.

4.1.1. Definitions:
(i) (Pair of two structures) Let \( \tau_1, \tau_2 \) be two disjoint one-sorted vocabularies and \( A_1, A_2 \) be \( \tau_i \)-structures respectively. We define the pair \([A_1, A_2]\) to be the two-sorted \( \tau_1 \cup \tau_2 \)-structure with universes \( A_1, A_2 \) and their respective relations, functions and constants. If the vocabularies \( \tau_1, \tau_2 \) are not disjoint, we make them disjoint by a name change and write nevertheless \([\tau_1, \tau_2]\).

(ii) (Pair preservation property) If \( L \) is a logic, we say that \( L \) satisfies the Pair Preservation Property and write \( \text{PPP}(L) \), if whenever \( A_{11}, A_{12}, A_{21}, A_{22} \) are structures such that

\[
A_{11} \models L A_{21} \text{ then } [A_{11}, A_{21}] \models L [A_{12}, A_{22}]
\]

To verify that a given logic satisfies \( \text{PPP}(L) \) it is often useful to use Bäck and Forth type arguments, as described in chapter 2 and more generally in chapter 19. It should
be possible to state a general theorem to the effect of when a Back and Forth property implies the Pair Preservation Property, but this does not seem to be a very rewarding line of thought. For the traditional Back and Forth arguments for infinitary logics this analysis has been carried out in [Feferman 1972].

4.1.2. Examples:
(i) Both $L_{0,\omega}$ and $L_{\omega,0}$ satisfy the Pair Preservation Property.
(ii) $L_{\omega,\omega}$ does not satisfy the Pair Preservation Property ([Mäli71]). $L_{\kappa,\lambda}$ does satisfy the Pair Preservation Property if $\kappa$ is strongly inaccessible ([Mäli71]).
(iii) $L_{\omega,\omega}(\alpha)$ satisfies the Pair Preservation Property by [Wojciechowska 1989].
(iv) For logics with second order quantification, such as stationary logic $L_{\omega,\omega}(\alpha\omega)$ we have to distinguish between the possibility that subsets range over the union of the universes, or that we have also two sorts of set variables. In the former case $L_{\omega,\omega}(\alpha\omega)$ does not satisfy the Pair Preservation Property cf. chapter 4(6.1.2), in the latter case it does [Makowsky-Shelah 1981].

4.1.3. Definitions:
(i) (Algebraic operations) Let $n \in \omega$ and $\tau_1, \tau_2, \ldots, \tau_n, \sigma$ be vocabularies. Let $F: Str(\tau_1) \times \cdots \times Str(\tau_n) \to Str(\sigma)$ be a function. We say that $F$ is an $n$-ary algebraic operation of type $\tau=\langle \tau_1, \tau_2, \ldots, \tau_n, \sigma \rangle$. If $A_i, B_i$ are $\tau_i$-structures and $A_i \equiv B_i$ ($i=1, \ldots, n$) then $F(A_1, \ldots, A_n) \equiv F(B_1, \ldots, B_n)$.
(ii) (L-Projective operations) Let $L$ be a logic. An algebraic operation $F$ of type $\tau$ as above is an $L$-projective operation if the the graph of $F$ is an $L$-projective class.
(iii) (Preservation property for projective operations) We say, a logic $L$ has the Preservation Property for Projective Operations and write $PPPO(L)$, if for every $L$-projective operation $F$ of type $\tau$, if $A_i, B_i$ are $\tau_i$-structures and $A_i \equiv_L B_i$ ($i=1, \ldots, n$) then $F(A_1, \ldots, A_n) \equiv_L F(B_1, \ldots, B_n)$.

4.1.4. Examples:
(i) First order logic satisfies the PPPO by [Feferman 1974].
(ii) The PPPO follows from the Uniform Reduction Property $UR_2$ defined in section 4.2.
(iii) The pair construction in 4.1.1. is a first order projective operation. Therefore PPPO follows from PPPO for any regular logic.
(iv) Various other algebraic operations are studied in [Gaifman 1967,1974], [Isbell 1973], [Hodges 1974, 1975; 1980] and [H.Friedman 1979].

The Preservation Property for Projective Operations seems to be very rare. In fact, it is only known to hold for first order logic, or for logics with Uniform Reduction (see 4.2). For many applications, however, we need much less. A construction somewhere between disjoint unions and general projective operations is enough to obtain interesting theorems in abstract model theory. In the spirit of this section dealing with definability properties in logics, we give both an implicit and an explicit definition.
4.1.5. Definitions:

(i) (Tree-like structure) Let $\tau_{\text{tree}}$ be one-sorted and consist of one unary function symbol $f$ and one constant symbol $c$. A $\tau_{\text{tree}}$-structure $T = \langle T, f, c \rangle$ is a tree-like structure, if the following hold:

(a) For every $x \in T$ there is an $n \in \omega$ with $f^n(x) = c$. The root of $f$.
(b) $f$ is onto.
(c) For every $x \in T$ there is an $n \in \omega$ with $f^n(x) = c$. The root of $f$.

(ii) (Augmented tree-like structure) Let $\tau_{\text{aug}}$ be $\tau_{\text{tree}} \cup \{P\}$, where $P$ is a unary predicate symbol. A $\tau_{\text{aug}}$-structure $T = \langle T, f, c, P \rangle$ is an augmented tree-like structure, if $T \models \tau_{\text{tree}}$ is a tree-like structure.

(iii) (Tree-like sum, implicit version)

Let $\tau$ be a vocabulary with a distinguished predicate symbol $P$ and let $\mathcal{A}, \mathcal{B}$ be two $\tau$-structures. We now define two structures over the vocabulary $\tau \cup \{P\}$. For $x \in T$ we denote by $T_x$ the set $f^{-1}(x) - \{x\}$.

(a) $\mathcal{N} = \text{Tree}_0(A, B)$ is an augmented tree-like structure. We write now $N^i$ for $T_x$ above.
(b) For every $x \in N^i$ there is a bijection $s_x : C \to N^i$ where $C$ is either $A$ or $B$. This bijection makes $N^i$ naturally into a $\tau$-structure which we denote by $N^i$.
(c) For each symbol $R \in \tau$ let $R_x$ be its interpretation in $N^i$. We require now that $R = N^i = \bigcup_{x \in N} R_x$.

(d) We require further that $P = P^i$ be defined by: If $x \in P$ then $N^i_x = A$ and $x \notin P$ then $N^i_x = B$.
(e) If $i = 1$ then $c \in P$ and if $i = 0$ then $c \notin P$.

(iv) (Tree-like sum, explicit version)

To make the definition of the tree-like sum $N^0 = \text{Tree}_0(A, B)$ explicit we proceed as follows:

We let the universe of $N^i$ consist of the set of finite sequences $<a_k : k < n>$ such that:

(a) $a_k \in A \cup B$,
(b) if $i = 0$ then $a_0 \in A$, but if $i = 1$ then $a_0 \in B$,
(c) $a_k \in P^A \cup P^B$ iff $a_{k+1} \in A$.

Next we define $f$, the interpretation of $f$:

(d) For the empty sequence $<$ we put $f (< >) = <$.
(e) $f (< a_k : k < n >) = < a_k : k < n >$.

Finally, for every relation symbol $R \in \tau$ we define its interpretation $R$ by

(f) $(< a_k : k < n >, < b_k : k < n >) \in R$ if $a_k = b_k$ for every $k < n$ and $(a_n, b_n) \in R^A \cup R^B$.

(v) (Tree Preservation Property) Let $L$ be a logic. We say that $L$ has the Tree Preservation Property and write $\text{TPP}(L)$, if whenever $A, B$ are as above, $\tau = \tau_0 \cup \{P\}$ and
additionally

\[ A \models \tau_0 =_L B \models \tau_0 \]

\[ \text{Tree}_L(A, B) \models \tau_0 \cup \tau_{\text{Tree}} =_L \text{Tree}_L(A, B) \models \tau_0 \cup \tau_{\text{Tree}}. \]

4.1.6. Remarks:

(i) The tree-like sum is not, in general, a projective operation, since 4.1.5(c) is not first order definable. However, if the logic \( L \) is such that the structure \( \langle \omega, \prec \rangle \) is \( PC_L \)-characterizable, then the tree-like sum is an \( L \)-projective operation.

(ii) For regular logics \( L \), the Tree Preservation Property implies the Pair Preservation Property, since the pair can be constructed as a relativized reduct of the tree sum.

(iii) If the distinguished predicate \( P \) in the tree-like sum is not unary, we can still define a tree-like sum over \( P \). We just replace \( f \) by a function \( s : T \to T^n \) and define \( s_i \) to be \( s_i \) followed by a projection to the first coordinate. Then we express 4.1.5.(i) (a) and (b) with \( s \) and (c) with \( s_i \).

The construction of the tree-like sum over a predicate \( P \) can sometimes be used to define the predicate \( P \) implicitly. The precise situation where this is possible is given in the following lemma from [Makowsky-Sheh 1979]. The idea goes back to S. Shelah.

4.1.7. Lemma: Let \( L \) be a logic, \( \tau_1 = \tau_0 \cup \{ P_i \}_{i=1,2} \) vocabularies, and \( \varphi \in L[\tau_1] \) be sentences having a model, but such that \( \varphi_1 \land \varphi_2 \) has no model. Then there is a sentence \( \psi \in L[\tau_0 \cup \{ P_i \}_{i=1,2}] \) such that:

(i) Every \( \tau_0 \cup \{ P_i \}_{i=1,2} \)-structure \( A \) has at most one expansion \( A' \models \psi \).

(ii) If \( A_i \) (\( i=1,2 \)) are \( \tau_1 \)-structures and \( A_i \models \varphi_i \) then \( \text{Tree}_L(A_1, A_2) \models \psi \), provided we substitute \( P \) for \( P_1, P_2 \) respectively.

Proof: Let \( \psi = \varphi_0 \land \varphi_1 \land \varphi_2 \) with

\( \varphi_0 \) expresses 4.1.5.(i) (a) and (b).

\( \varphi_1 \) is the \( L \)-formalization of "If \( z \in P \) then \( N_z \models \varphi_1 ."

\( \varphi_2 \) is the first order formalization of "If \( z \notin P \) then \( N_z \models \varphi_2 ."

The latter two involve the appropriate substitutions and relativizations. Clearly (ii) holds, by our construction of \( \text{Tree}_L(A_1, A_2) \). And (i) holds because \( \varphi_1 \land \varphi_2 \) has no model. QED.

We shall use lemma 4.1.7 in section 4.4 to prove some abstract theorems.
4.2. Definability, Interpolation and Uniform Reduction.

We first recall some definitions from chapter 2; section 7.

4.2.1. Definitions:
(i) A logic $L$ has the Interpolation Property, and we write $\text{INT}(L)$, if any two disjoint classes of $T$-structures, which are RPC in $L$, can be separated by some EC-class of $L$.
(ii) A logic $L$ has the $\Delta$-Interpolation Property, and we write $\Delta$-$\text{INT}(L)$, if any class $K$ of $T$-structures, such that $K$ and its complement are RPC in $L$, then $K$ is an EC-class of $L$.
(iii) A logic $L$ has the Weak Beth Property, and we write $\text{WBETH}(L)$, if every strong implicit definition can be replaced by some explicit definition in $L$.
(iv) A logic $L$ has the Beth Property, and we write $\text{BETH}(L)$, if every implicit definition can be replaced by some explicit definition in $L$.
(v) A logic $L$ has the Projective Weak Beth Property, and we write $\text{PWBETH}(L)$, if every implicit definition which is RPC in $L$, can be replaced by some explicit definition in $L$.

The following summarizes the relationship between these properties.

4.2.2. Theorem:
(i) A logic $L$ has the Weak Projective Beth Property iff it has the $\Delta$-Interpolation Property.
(ii) For a logic $L$ the Interpolation Property implies, but is strictly stronger than, the $\Delta$-Interpolation Property (and therefore the Projective Weak Beth Property).
(iii) For a logic $L$ the Interpolation Property implies, but is strictly stronger than, the Beth Property.
(iv) For a logic $L$ the $\Delta$-Interpolation Property implies, but is strictly stronger than, the Weak Beth Property. In fact, the $\Delta$-Interpolation Property does not imply the Beth Property.
(v) For a logic $L$ the Beth Property implies, but is strictly stronger than, the Weak Beth Property, in fact the Beth Property does not imply the $\Delta$-Interpolation Property.

Proof: The implications are all straightforward, (i) is 7.3.3. and (ii) is 7.2.7. in chapter 2. (iii) follows from (v). (iv) is theorem 2.5 in [Makowsky-Shelah 1979] and (v) is proven in [Makowsky-Shelah 1976] and will appear in [Makowsky-Shelah 1987].

4.2.3. Remark: For sublogics of $L_{u,w}$ of the form $L_{\Delta}$ with $\Delta$ primitive recursive closed, the $\Delta$-Interpolation Property implies the Interpolation Property and therefore the Beth Property. This is due to H. Friedman and proved in [Makowsky-Shelah-1976]. See also chapter 8, theorem 6.3.1.

Next we investigate the relationship between the Weak Beth Property and recursive compactness. Of special interest here is that we need an additional assumption, namely either that the logic is finitely generated or the Pair Preservation Property.
4.2.4. Definitions:
(i) A logic $L$ is \textit{finitely generated}, if it is a Lindström logic over a finite set of new quantifier symbols.
(ii) A logic $L$ is \textit{recursively generated}, if it is a Lindström logic over a recursive set of new quantifier symbols.
(iii) A logic $L$ is \textit{recursively compact}, if $L$ is recursively generated and if $\Sigma$ is any recursive set of $L$-sentences such that every finite subset of $\Sigma$ has a model, so $\Sigma$ has a model.

4.2.5. Remarks:
(i) By theorem 5.2.5 in chapter 2, every logic for which validity is \textit{recursively enumerable}, is recursively compact.
(ii) A logic $L$ is recursively compact iff no single sentence $\varphi \in L[\tau]$ with $\tau$ containing a binary relation symbol denoted by $<$, characterizes the structure $<\omega, <$ up to isomorphism among (relativized) reducts of models of $\varphi$. Cf. also chapter 2, section 5.2.

4.2.6. Theorem:
(i) (Lindstrom) Assume a logic $L$ is finitely generated and has the Weak Beth Property, then $L$ is recursively compact.
(ii) Assume a logic $L$ is recursively generated and satisfies the Weak Beth Property and the Pair Preservation Property. Then $L$ is recursively compact.

Proof: The proof of (i) is similar to the proof of 5.2.5 in chapter 2, cf. also chapter 3, remark 2.1.5, or chapter 17 section 4.

To prove (ii), we assume for contradiction that there is a $\varphi \in L[\tau]$ as in the remark (ii) above. Since $L$ is recursively generated, we have at most $2^\omega$ many theories over a countable vocabulary. Now consider the $\tau$-structure $A = (\mathcal{A}; P, Q, \in) \in \mathcal{E}$, where $\mathcal{A} = \bigcup P^n(\omega)$; $P^n$ is the $n$th iteration of the power set operation. $P^n$ are unary predicates with $P^n = P^n(\omega)$, $\in$ is the natural membership relation, and $Q \subseteq P^k$ where $k$ is fixed and such that $\text{BETH}(k)$ is bigger than the number $\kappa$ of inequivalent theories in $L[\tau]$. Now consider the structure $[\mathcal{A}; A]$ with universe of the first sort $A_1$ and universe of the second sort $A_2$ and let $\psi$ be the formula in $L$ which expresses:

(i) $P$ is standard $\omega$. (Here we use $\varphi$.)
(ii) $F$ is a partial map from $A_1$ to $A_2$, where $F$ is a new function symbol.
(iii) $F$ and $F^{-1}$ preserve $\in$.
(iv) $F$ is hereditary, i.e., if $F$ is defined for $x$ and $y \in x$ so $F$ is defined for $y$.
(v) The domain of $F$ is maximal with respect to (i) - (iv).

Clearly, $\psi$ defines $F$ strongly implicitly. Since there are at most $\kappa = 2^\omega$ many theories over $\tau$, we can find two structures $A_1 = (\mathcal{A}, P, Q_1, \in) \in \mathcal{E}$, $A_2 = (\mathcal{A}, P, Q_2, \in) \in \mathcal{E}$, such that $A_1 \equiv_{L[A_2]}$ but $Q_1 \neq Q_2$.

Let $B_1 = [A_1; A_2]$ and $B_2 = [A_1; A_1]$. Now we use $\text{PPP}(L)$ to conclude that $B_1 = B_2$ using the
Weak Beth Property, let $\varphi \in L[\tau]$ define $F$ explicitly: So $\varphi$ defines on $B_1$ a partial map $F_1$ with domain $D_1$. Clearly $Q_1 \subseteq D_1$, and since $B_1 = f B_2$, also $Q_1 \subseteq D_2$. But then we can show by induction on $i$ that $Q_i = Q_2$, contrary to our assumption. Note that, in this proof, we have only used a finite subset of the vocabulary $\tau$. QED.

The same proof actually only requires that the number of theories for a countably vocabulary is smaller than $BETH(\omega^2)$. This can be achieved by assuming either that the Lowenheim number is smaller than $BETH(\omega_\alpha)$ or directly, by assuming that there are not too many different formulas for a given countable vocabulary. One can vary the prove further for logics $L$ such that $card(L[\tau]) < BETH_\alpha$ for countable vocabulary $\tau$. We state the corresponding results without proof:

4.2.7. Theorem:
(i) Assume a logic $L$ satisfies the Weak Beth Property and the Pair Preservation Property, and has a Lowenheim number $l(L) < BETH(\omega^2)$. Then no single sentence $\varphi \in L[\tau]$, with $\tau$ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega, < \rangle$ up to isomorphism among reducts of models of $\varphi$. In other words, the well-ordering number $w_1(L)$ for single sentences of $L$ is $\omega$.
(ii) Assume a logic $L$ satisfies the Weak Beth Property and the Pair Preservation Property, and $card(L[\tau]) < BETH_\alpha$ for countable vocabulary $\tau$. Then no single sentence $\varphi \in L[\tau]$, with $\tau$ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega+\alpha, < \rangle$ up to isomorphism among reducts of models of $\varphi$. In other words, the well-ordering number $w_1(L)$ for single sentences of $L$ is $\omega+\alpha$.

4.2.8. Corollary: Let $A$ be a countable admissible set with $\omega \in A$, or $A = \omega_1$, Then $L_A$ does not satisfy the Pair Preservation Property.

Proof: Clearly $\langle \omega, < \rangle$ is characterizable in $L_A$ and the Interpolation Property holds. QED.

We now want to look at a property introduced in [Feferman 1974, FM] and further studied in [Makowsky 1978], which is a generalization of both the Interpolation Property and some of the Preservation Properties.

4.2.9. Definition:
Let $L$ be a logic and $A_i$ be $\tau_i$-structures $(i=1,2)$ with $\tau$ the vocabulary for $[A_1, A_2]$. We say that $L$ allows uniform reduction for pairs, or has the Uniform Reduction Property for Pairs, and write $URP(L)$, if for every $\varphi \in L[\tau]$ there exists a pair of finite sequences of formulas $\psi_1^1, \ldots, \psi_1^i$ and $\psi_2^1, \ldots, \psi_2^i$ with $\psi_k^i \in L[\tau_i]$ and a Boolean function $B \in \mathbb{Z}^{i \times m_\tau}$ such that for every $\tau_i$-structures $A_i$, $(i=1,2)$ $[A_1, A_2] = \varphi$ iff $B(a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^i) = 1$, where $a_k^i$ is the truth value of $A_k | \varphi$.

4.2.10. Examples:
(i) $URP(L)$ holds for
4.2.11. Definitions:

(i) Let $\tau_0, \tau_1, \ldots, \tau_n$ be disjoint vocabularies and let $R \subseteq \text{Str}(\tau_0) \times \text{Str}(\tau_1) \times \cdots \times \text{Str}(\tau_n)$ be an $n$-ary relation on structures. A sentence $\varphi \in L[\tau_n]$ is said to be invariant on the range of $R$, if for all $A_0.A_1, \ldots, A_{n-1}.A_n$ such that $R(A_0, A_1, \ldots, A_{n-1}, A_n)$ and $R(A_0.A_1, \ldots, A_{n-1}.A'_n).A_n \models \varphi$ iff $A'_n \models \varphi$.

(ii) An $n$-tuple of sequences of sentences $\overline{\psi}_0, \overline{\psi}_1, \ldots, \overline{\psi}_{n-1}$ with $\overline{\psi}_k = (\psi^k_1, \ldots, \psi^k_{m_k})$ and $\overline{\psi}_k \in L[\tau_k]$ together with a Boolean function $B \in \mathbb{B}^{m_1 + \cdots + m_{n-1}}$ is called an UR $n$-tuple for $\varphi$ on the domain of $R$. If, for all $A_0.A_1, \ldots, A_{n-1}.A_n$ we have that $R(A_0, A_1, \ldots, A_{n-1}, A_n)$ implies that $A_n \models \varphi$ iff $B(a^1_0, \ldots, a^1_{m_1}, a^2_0, \ldots, a^2_{m_2}, \ldots, a^n_{m_n}) = 1$ where $a^i_j$ is defined as in 4.2.9.

(iii) We say a logic $L$ satisfies the Uniform Reduction Property for $(n+1)$-ary relations, and we write $UR_n(L)$, if for every relation $R \subseteq \text{Str}(\tau_0) \times \text{Str}(\tau_1) \times \cdots \times \text{Str}(\tau_n)$ which is PC in $L$ and for every $\varphi \in L[\tau_n]$ which is invariant in the range of $R$, there is an UR tuple for $\varphi$ on the domain of $R$.

4.2.12. Remarks:

(i) Clearly $UR_n(L)$ implies $UR(L)$, since the construction of the pair $[A_0, A_2]$ is a PC$_L$-operation, i.e. its graph is a PC$_L$ relation.

(ii) Instead of the pair construct we could consider cartesian product of a fixed finite number $n$ of structures $A_i$ and define similarly uniform reduction for $n$-fold cartesian products ($UR\text{Prod}_n(L)$). Again $UR_n(L)$ implies $UR\text{Prod}_n(L)$.

(iii) Note that $UR(L)$ does not imply $UR_1(L)$. Take example 4.2.10(ii) for a logic with URP, but which does not satisfy the Interpolation Property by chapter 2 (7.1.3.). Now the result follows from theorem 4.2.14. below.

The following clarifies the relationship between PPP and various uniform-reduction properties.

4.2.13. Theorem: Let $L$ be a logic. Then

(i) $UR(L)$ implies PPP($L$).

(ii) $UR_n(L)$ implies PPP$_n(L)$.

If additionally $L$ has an dependence number $\omega(L) = \kappa$ and is $(\mu, \omega)$-compact, with $\mu = \sup\{|\text{card}(L[\tau])| : \text{card}(\tau) < \kappa\}$, then

(iii) [(Shelah 1983 Manuscript)] PPP$_n$($L$) implies URP($L$) and

(iv) [(Shelah 1983 Manuscript)] PPP$_n$($L$) implies $UR_n(L)$ for every $n \in \omega$.

Proof: (i) and (ii) are straightforward. To prove (iii), assume $\varphi$ is a counterexample to URP. So for every pair of sequences of formulas $\overline{\psi}_1 = (\psi^1_1, \ldots, \psi^1_{n_1})$ and $\overline{\psi}_2 = (\psi^2_1, \ldots, \psi^2_{n_2})$
with $\psi_i \in L[\tau_i]$ and every Boolean function $B \in 2^{\tau_1 \times \tau_2}$ there are $\tau_i$-structures $A_i$ such that $[A_1, A_2] |= \phi$. However, $[A_1^2, A_2^2] |= \phi$, but

$$B(a_1(\bar{f}), \ldots, a_2(\bar{f}), 1) = 1$$

where $a_\bar{f}(\bar{f})$ is the truth value of $A_i = \psi_i$.

Claim 1: For every such pair of sequences of formulas $\overline{\psi_1}, \overline{\psi_2}$ there is a function $h: \overline{\psi_1} \cup \overline{\psi_2} \to 2$ such that

$$\Sigma_i^1 = \{\psi\} \cup \{\phi \mapsto h(\phi) : \phi \in \overline{\psi_1} \cup \overline{\psi_2}\}$$

and

$$\Sigma_i^0 = \{\neg \psi\} \cup \{\phi \mapsto h(\phi) : \phi \in \overline{\psi_1} \cup \overline{\psi_2}\}$$

have both models.

If not, for every $h$ as above either $\Sigma_i^1$ or $\Sigma_i^0$ is has no model. We then could construct a boolean function $B$ as follows: Put

$$B_h = \Lambda \{\delta : h(\delta) = 1\} \Lambda \{\neg \delta : h(\delta) = 0\}.$$ 

Now we put

$$B = 1 \vee \{B_h : \Sigma_i^1 \text{ has a model}\}$$

Subclaim:

$$[A_1, A_2] |= \phi$$

iff

$$B = B(a_1, a_1^1, a_2^1, \ldots, a_{n_2}^1) = 1$$

where $a_\bar{f}$ is the truth value of $A_i = \psi_i$.

To see this, assume $[A_1, A_2] |= \phi$. Now put $h_\phi(\psi_1) = a_\bar{f}$. Clearly, $B = 1$. Conversely, if $B = 1$, there is $h$ such that $\Sigma_i^1$ has a model. So, by our assumption, $\Sigma_i^0$ has no model. So $[A_1, A_2] |= \phi$.

Using claim 1, we define $H$ to be the set of functions $h: \overline{\psi_1} \cup \overline{\psi_2} \rightarrow 2$ such that $\Sigma_i^1$ and $\Sigma_i^0$ have both models.

We define a filter $\mathcal{F}_0$ on $H$ with filter basis $\mathcal{U}_\delta = \{h : h(\delta) \in \text{dom}(h)\}$ where $\delta \in L[\tau_1] \cup L[\tau_2]$. Let $\mathcal{F}$ be an ultrafilter extending $\mathcal{F}_0$. Now we define a function $g : \delta \in L[\tau_1] \cup L[\tau_2] \rightarrow 2$ by $g(\delta) = 0$ iff $h(\delta) = 1$ $\forall h \in \mathcal{F}$. Clearly, we have:

Claim 2: For every pair of sequences $\overline{\psi_1}, \overline{\psi_2}$ there is a function $h \in H$ such that $g(\bar{f}) = 0$.

Now we define $\Sigma_i^i$ (i = 0, 1) like the $\Sigma_i^i$'s. Using $\langle \mu, \omega \rangle$-compactness and claim 2 we get:

Claim 3: There are $\bar{A_i}$ (i = 0, 1, 2) such that $[A_i, A_i] = \Sigma_i^i$.

But the latter contradicts $PPP(L)$, since, by the definition of $\Sigma_i^i$, $A_i = A_i |^{L \mathcal{F}}$ (i = 0, 1).

The proof of (iv) is essentially the same. QED.
Uniform reduction is closely-related to the interpolation property. [Feferman 1974] derived $UR_1$ from it and in [Makowsky 1978] the converse was observed.

4.2.14. Theorem (Feferman, Makowsky): Let $L$ be a logic with finite dependence. Then $UR_1(L)$ if $L$ has the Interpolation Property.

Proof: (i) Assume $UR_1(L)$ and let $K_i, K_0 \subseteq Str(\tau_0)$ be two disjoint classes of $\tau_0$-structures which are $PC$ in $L$. So there are vocabularies $\tau_1$ and sentences $\psi_i \in L[\tau_1]$ such that $K_i = Mod(\psi_i) \upharpoonright \tau_0$. Since $L$ has finite dependence all the vocabularies can be assumed finite. We now define $R \subseteq Str(\tau_0) \times Str(\tau_1 \cup \tau_2)$ by $R(A,B)$ if $A \models \psi_1 \quad \tau_1 \quad B$ or $B \models \psi_2 \quad \tau_2 \quad K_0$. Clearly $R$ is $PC_2$ using an additional predicate for the isomorphism and the fact that $\tau_0$ is finite.

Claim: Both $\psi_1, \psi_2$ are invariant in the range of $R$.

This follows from the fact that $K_1 \cap K_2 = \emptyset$.

Now let $\psi_i$ be $UR$ sentences for $\psi_i$, respectively. It is easy to check that $\psi_1 \land \neg \psi_2$ is the desired interpolating sentence.

(ii) Now assume that $L$ has the Interpolation Property. $R$ is a $PC_1$-relation on $Str(\tau_0) \times Str(\tau_1)$ and $\psi \in L[\tau_1]$ is invariant on the range of $R$. Assume $R$ is defined by $\psi \in L[\tau_1]$. Now put $K_1 = Mod(\psi \land \phi) \upharpoonright \tau_1 \quad \tau_2$ and $K_2 = Mod(\psi \land \neg \phi) \upharpoonright \tau_1 \quad \tau_2$.

Claim: $K_1 \cap K_2 = \emptyset$.

This follows from the fact that $\phi$ is invariant on the range of $R$.

So, let $\phi \in L[\tau_0]$ be an interpolating sentence. Therefore, whenever $R(A,B)$ we have that $A \models \phi$ if $B \models \phi$. In other words, $\phi$ is an $UR$ sentence for $\phi$, QED.

Note that in [Feferman 1974; FM] Uniform Reduction is defined for $PC_3$, and 4.2.14(ii) is stated assuming some compactness properties.

4.2.15. Theorem:

(i) For a logic $L$ the following are equivalent:
(a) $UR_2(L)$
(b) $UR_1(L)$ (or equivalently the Interpolation Property) together with $URP(L)$.
(c) $UR_n(L)$, for $n \geq 2$.

(ii) For a compact logic $L$ the following are equivalent:
(a) $UR_2(L)$
(b) $UR_1(L)$ (or equivalently the Interpolation Property) together with $PPP(L)$.
(c) $UR_n(L)$, for $n \geq 2$.
(d) $PPP(L)$.

(iii) $URP$ does not imply $UR_1$, not even for compact logics.

Proof: (i): (a) implies (b) by theorem 4.2.14, and the remarks 4.2.12. (b) implies (c), since $URP$ allows us to reduce $n$-ary relations to binary relations, and (c) implies (a) is trivial. To prove (ii) we combine (i) with theorem 4.2.13.

To prove (iii) we observe that by example 4.2.10(ii) $L_{\omega_1}(Q_\omega)$ satisfies $URP$, but, as
shown in chapter 2 (7.1.3.), it does not have the Interpolation Property. So the result follows from theorem 4.2.14. For a compact counterexample see remark 4.2.17. below. QED.

The last proposition in this section gives us a connection between the Tree Preservation Property and Uniform Reduction, but it is only interesting for logics which are not recursively generated, because the latter hypothesis together with $UR_2$ implies recursive compactness, by theorem 4.2.8(ii).

4.2.16. Proposition: Assume $L$ is a logic in which $<\omega, \in>$ is not characterizable by a single sentence with additional predicates and sorts (in particular $L$ is not recursively compact): Then $UR_2(L)$ implies $TPP(L)$.

Proof: Clearly, we can use the PC-definition of $<\omega, \in>$ to get a PC-definition of the tree construction involved in the Tree Preservation Property. See also remark 4.1.6(iii). QED.

4.2.17. Remark: In section 4.6, we shall present an example of a logic $L$ which satisfies the Beth Property, the Pair Preservation Property, is compact, but does not satisfy the Interpolation Property.

4.3 The Finite Robinson Property.

In section 3.3 we have seen that the Amalgamation Property implies compactness therefore (Corollary 3.3.5.) that the Robinson Property implies compactness. These results depend on some assumptions on the dependence number of the logic. In chapter 19 the Robinson Property is further investigated and instead of the dependence number we have different smallness assumptions on the logic. Here we want to study two weakened version of the Robinson Property. They were studied first in [Makowsky-Shelah 1979], and the assumptions on the logics also did not involve the dependence number.

4.3.1. Definition: Let $L$ be a logic.

(i) $L$ satisfies the Finite Robinson Property ($FROB$), if given a complete set $\Sigma$ of $L[\tau]$-sentences and two sentences $\varphi_1(\varphi_2) \in L[\tau_1](L[\tau_2])$ with $\tau_1 \cap \tau_2=\tau$ such that $\Sigma \cup \{\varphi_1\}$ has a $\tau_1$-model then $\Sigma \cup \{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

(ii) $L$ satisfies the Weak Finite Robinson Property ($WFROB$), if given a complete set $\Sigma$ of $L[\tau]$-sentences and two sentences $\varphi_1(\varphi_2) \in L[\tau_1](L[\tau_2])$ with $\tau_1 \cap \tau_2=\tau$ such that $\Sigma \cup \{\varphi_1\}$ has a $\tau_1$-model then $\{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

4.3.2. Proposition:

(i) Both $FROB$ and $WFROB$ are consequences of the Robinson Property.

(ii) Clearly $FROB$ implies $WFROB$. 

(iii) The Interpolation Property implies $WPROB$.
(iv) If $L$ is compact then the Robinson Property is equivalent to both $FROB, WPROB$ and the Interpolation Property.

The proof is left to the reader. For (iv) cf. chapter 2 theorem 7.1.5.

Our next aim is to study when the Pair Preservation Property suffices to make $FROB$ equivalent to the Robinson Property. The answer is given in theorem 4.3.8.

4.3.3. Definition:
(i) We call a logic $L$ tiny, if for every vocabulary $\tau$ with $\text{card}(\tau)$ smaller than the first uncountable measurable cardinal $\mu_0$ $\text{card}(L[\tau]) < \mu_0$.
(ii) We call a logic $L$ small, if for every vocabulary $\tau$, which is a set, $L[\tau]$ is a set. (Smallness was already introduced in chapter 2, theorem 3.1.4.)

Clearly, if a logic $L$ is tiny, it is also small, provided measurable cardinals exist, if no uncountable measurable cardinals exist, then tiny and small coincide. There are logics with dependence number $\omega(L) = \omega$ which are not small, and it is not difficult to construct logics which are small but have no dependence number. We leave this as an exercise to the reader. The logic defined in example 2.2.5(ii) is tiny, but has a dependence number which is bigger than the first uncountable measurable cardinal.

4.3.4. Theorem: If $L$ has the Robinson Property and is tiny then
(i) $L$ is $[\omega]$-compact and
(ii) $L$ has the Finite Dependence Property.

This differs from corollary 3.3.5, inasmuch as here we do not require that $L$ has an dependence number, whereas in 3.3.5, we require that $\omega(L)$ exists and is smaller than the first uncountable measurable cardinal.

4.3.5. Theorem: If $L$ has the Pair Preservation Property, the Finite Robinson Property and is tiny then
(i) $L$ is $[\omega]$-compact and
(ii) $L$ has the Finite Dependence Property.

Proof: Clearly in both theorems (ii) follows from (i) by 2.2.1. To prove (i) we proceed in parallel and point out the difference in the appropriate places.

Let $B_1, B_2$ be two infinite sets of different cardinality $\beta_1, \beta_2$ smaller than the first uncountable measurable cardinal $\mu_0$. Now we fix $\kappa > \max\{\beta_1, \beta_2\}$ but $\kappa < \mu_0$ and put $A_\kappa = \langle H(\kappa^+), P_1, P_2 \rangle$ where $H(\kappa^+)$ is the complete expansion of $<\kappa^+, \in>$ and $P_1, P_2$ are unary predicates of cardinality $\beta_1, \beta_2$, respectively. Let $\tau_\kappa$ be the vocabulary of $A_\kappa$ and $\Sigma$ the complete $L[\tau_\kappa]$-theory of $A_\kappa$. Assuming that $L$ is not $[\omega]$-compact, we conclude, using the Rabin-Keisler theorem 1.2.3, that $\Sigma$ is categorical. Let $B_i = [A_\kappa, B_i]$ for $i = 1, 2$ be $\tau_\kappa$-structures with $\tau_1 \cap \tau_2 = \tau_\kappa$.

Assumption: $B_1$ and $B_2$ are $L$-equivalent (after appropriate name changing; so that
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both are \( \tau_i \)-structures).

We first finish the proof from the assumption. Let \( \varphi_i \) be the first order formula which says that "\( f \) is a bijection from \( P_i \) onto the universe of the second sort".

Clearly \( B_i \models \Sigma \cup \{ \varphi_i \} \), but \( \Sigma \cup \{ \varphi_1, \varphi_2 \} \) has no model.

To satisfy the assumption the two proofs differ. In the case of theorem 4.3.5. we use

\text{tiny}ness and an argument as in the proof of the existence of Hanf numbers (chapter 2.6.1.) to find \( \beta_1, \beta_2 \) such that for \( \tau = \{ 0 \} \) \( B_1 \) and \( B_2 \) are \( L \)-equivalent. Since \( \tau \) is finite we may assume that \( \beta_1, \beta_2 < \mu_0 \). Now we can use the Pair Preservation Property to conclude that \( B_1 \) and \( B_2 \) are \( L \)-equivalent (after appropriate name changing).

In the case of 4.3.4. we fix a countable universal vocabulary \( \tau_{\infty} \) which has countably many relation symbols for every arity. Using enough constants \( \tau_C \), we can think of \( \Sigma \) as being written over the vocabulary \( \tau \cup \tau_C \). Let \( \Sigma_\infty \be \Sigma \cup \tau_{\infty} \). Using \text{tiny}ness we find, as in the case of theorem 4.3.4., \( \beta_1, \beta_2 \) such that \( B_1 \) and \( B_2 \) are \( L[\tau_{\infty}] \)-equivalent.

Let \( \tau_1 \) and \( \tau_2 \) be two disjoint copies of \( \tau_C \) and put \( \Sigma = \Sigma \cup \{ \varphi_1 \} \) written over \( \tau_\infty \cup \tau_1 \). Clearly \( \Sigma_\infty \cup \Sigma_1 \) has each a model, i.e. \( \Sigma_\infty \cup \Sigma_1 \cup \Sigma_2 \) has not. QED.

4.3.6. Definitions:

(i) We first note that if a logic \( L \) is small then there is function \( s \) on the cardinals such that for every vocabulary \( \tau \) of cardinality \( \lambda \), \( \lambda \leq \text{card}(L[\tau]) < s(\lambda) \). We call this function the size function of \( L \). If \( L \) is \text{tiny} then \( \lambda < \mu_0 \) implies \( s(\lambda) < \mu_0 \).

(ii) Recall that a logic \( L \) is said to be \text{ultimately compact}; if \( L \) is \( (\omega, \lambda) \)-compact for some cardinal \( \lambda \).

4.3.7. Examples:

(i) If a logic \( L \) has a Lowenheim number \( \lambda_1(L) \) then \( L \) is \text{small}.

(ii) In chapter 19 theorem 1.1.1. it is shown that if \( L \) is \text{small} and satisfies the Joint Embedding Property then \( L \) is \text{ultimately compact}.

(iii) If \( L \) has a dependance number and satisfies the Amalgamation Property then \( L \) is \text{ultimately compact}. This holds in particular, if \( L \) satisfies the Robinson Property (Theorem 3.3.1).

Our next theorem shows that already the Finite Robinson Property implies ultimate compactness.

4.3.8. Theorem: (Shelah) Let \( L \) be a tiny logic which satisfies both the Preservation Property for Pairs and the Finite Robinson Property, then:

(i) \( L \) is \text{ultimately compact}. In fact, if \( s \) is the size function of \( L \) and \( s(\omega) < 2^{\omega \omega_\infty} \) then \( L \) is \( [\omega, \omega_\infty] \)-compact.

(ii) If additionally \( L \) is countably generated or \( s(\omega) < \omega_\infty \) for some \( \omega_\infty \in \omega \), then \( L \) is compact and satisfies the Uniform Reduction Properties. \( VR_\infty(L) \).

For the proof we need a lemma. Parts (ii) and (iii) the author has learned from S. Shelah, though others probably have observed them too.
4.3.9. Lemma: (i) (Ulam) Let $\kappa$ be an infinite cardinal. If $S \subseteq \kappa^+$ is stationary, $S$ may be decomposed into $\kappa^+$ disjoint stationary subsets.

(ii) There is a family $S$ of $2^{\kappa^+}$ many stationary subsets of $\kappa^+$ such that for any $S_1, S_2 \in S$ the symmetric difference $S_1 \Delta S_2$ is stationary as well.

(iii) There are $2^{\kappa^+}$ many stationary subsets of $\kappa^+$ such that any finite Boolean combination of them is stationary as well.

Proof: (i) is standard, e.g., theorem 3.2 in chapter B.3 of the Handbook of Mathematical Logic [Barwise 1977]. To prove (ii) let $\{S_\alpha : \alpha < \kappa^+\}$ be the disjoint family of stationary sets from (i). Let $X \subseteq \kappa^+, X \neq \emptyset$. Define $T_X = \bigcup_{\alpha \in X} S_{\alpha^+} \cup \bigcup_{\alpha \in X^+} S_\alpha$. Clearly each $T_X$ is stationary and $X \neq Y$ implies that $T_X \cap T_Y$ is stationary.

The proof of (iii) is similar, but uses a combinatorial result from [Engelking-Karlowicz 1986]. QED.

Proof of theorem 4.3.8: Let $\kappa$ be as required. We can assume it is regular, by theorem 1.5.18. Assume $L$ is not $[\kappa^+]$-compact, so by 1.5.16 again, $L$ is not $[\kappa^+]$-compact, and, by induction, we can assume that $\kappa$ is such that $2^{\kappa}(\omega) < 2^\kappa$. Let $C_\kappa = \{\beta : \beta \in \kappa^+ \text{ and } cf(\beta) = \kappa\}$. For every $S \subseteq C_\kappa$ we define a structure $M_S = (\kappa^+, \in, S)$. By lemma 4.3.9(ii) there are $2^{\kappa^+}$ many stationary sets in $C_\kappa$ with their symmetric difference stationary, too. So, by our assumption on the size function of $L$, and by proposition 2.1.3, there are $S_1, S_2 \in C_\kappa$ with $M_{S_1} \models \mathcal{L}_{\in, S_1, S_2, \mathcal{C}_1}$ and $S_3 = S_1 \Delta S_2$ and $\in, \mathcal{C}_1$ membership and cofinality on $\kappa$.

Let $\mathcal{B}$ be the complete expansion of $\mathcal{A}$. We note that in $\mathcal{B}$ every ordinal of cofinality $\kappa$ or $\kappa^+$ is cofinally characterized by the complete $L$-theory of $\mathcal{B}$. Using that $L$ has the Prikaz Preservation Property, we conclude that $[\mathcal{B}, M_{S_1}] = L[M_{S_2}]$. We want to build a counterexample to PROB. For this purpose let $F_1 : 1 \rightarrow 2$ be new unary function symbols and $\varphi_1$ be the sentence which says that $F_1 \models [\mathcal{B}, M_{S_1}] = L[M_{S_2}]$. We want to build a counterexample to PROB. For this purpose let $F_1 : 1 \rightarrow 2$ be new unary function symbols and $\varphi_1$ be the sentence which says that "$F_1$ is an isomorphism between $\kappa^+, \in, S_2$" and $M_{S_2}$. Clearly $\Sigma \cup \{\varphi_1\}$ is each satisfiable but it is not difficult to show that $\Sigma \cup \{\varphi_1\}$ has no model. QED.

A complete proof may be found in [Makowsky-Shelah 1979].

A combination of the proofs of theorem 4.3.8 and proposition 4.3.2 gives us the following theorem:

4.3.10. Theorem: Let $L$ be a logic which is small and satisfies either the Robinson Property or the Finite Robinson Property together with the Pair Preservation Property. Then $L$ is almost compact.

Combining theorem 4.3.10, with the hypothesis $A(=)$ from section 1.5 we get:

4.3.11. Corollary: For a logic as in theorem 4.3.10, we have:

(i) If $A(=)$ holds then $L$ is compact.
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(ii) If \( L \) is tiny and there are no uncountable measurable cardinals, then \( L \) is compact.

**Proof:** Assume \( A(\omega) \), so there are no uncountable measurable cardinals, by 1.5.4.(iii). Therefore, if a logic \( L \) is small, then it is tiny and by theorem 4.3.4. or 4.3.5. \([\omega]\)-compact. So theorem 4.3.8. together with 1.5.7. give us that \( L \) is compact. This proves both (i) and (ii). QED.

Let us end this section with an open problem:

**4.3.12. Problem** is there a countable logic, different from first order logic, which satisfies both the Robinson Property and the Uniform Reduction Property (as in theorem 4.3.6.)?

4.4. Constructing Counter Examples to the Beth Property.

This last section is devoted to an abstract theorem (4.4.5.) whose main use is to direct us in the construction of possible counterexamples to the Beth Property. For compact logics, it gives a sufficient condition, the Tree Preservation Property, for the Beth and the Interpolation Property to be equivalent. As the example in theorem 4.6.12. shows, the Pair Preservation Property does not suffice. Experience shows that in many cases where we do not have the Interpolation Property, we actually can find a counterexample to the Weak Finite Robinson Property. The following theorem gives some indication on how to transform such a counterexample into a counterexample of the Beth Property.

**4.4.1. Theorem:**

(i) Let \( L \) be a logic which satisfies the Beth Property and the Tree Preservation Property. Then \( L \) also satisfies the Weak Finite Robinson Property.

(ii) If additionally to the Tree Preservation Property \( L \) is compact, then \( L \) has the Beth Property iff it has the Interpolation Property.

Stated in this form the theorem does not have many applications. But its proof still gives directions on how to construct counterexamples to the Beth Property, provided the Interpolation Property fails. In [Makowsky-Shelah 1984?] this approach lead to a proof that \( \Delta(L_{\omega}) \) does not have the Beth Property. Another way of making theorem 4.4.1 more useful is to define all the properties involved for pairs of logics.

4.4.2. Definitions:

(i) Let \( L_1, L_2 \) be two logics such that \( L_1 \leq L_2 \). We define the various Robinson Properties \( ROB, FROB, WFROB \) for the pair \( L_1, L_2 \) and write \( ROB(L_1, L_2), FROB(L_1, L_2), WFROB(L_1, L_2) \), respectively. For \( ROB \) this looks explicitly as follows: If \( \Sigma \) is a complete set of formulas in \( L_2(\tau_0) \), \( \Sigma_1, \Sigma_2 \) are in \( L_1(\tau_1), L_1(\tau_2) \) respectively, \( \tau_1 \cap \tau_2 = \emptyset \) and \( \Sigma \cup \Sigma_i, i=1,2 \) have models each, then \( \Sigma \cup \Sigma_1 \cup \Sigma_2 \) has a model. We leave it to the reader to state the corresponding properties \( FROB, WFROB \).
(ii) Similarly we define the various Beth and Interpolation Properties $\text{BETH}, \text{WBETH}, \text{INT}, \Delta \text{INT}$ for the pair $L_1, L_2$ and write $\text{BETH}(L_1, L_2)$, $\text{WBETH}(L_1, L_2)$, $\text{INT}(L_1, L_2)$, $\Delta \text{INT}(L_1, L_2)$, respectively if the implicit definition, or the formulas to be interpolated are in $L_1$, and the explicit definition or the interpolant is in $L_2$.

(iii) Similarly we define the various Preservation Properties $\text{PPP}, \text{TPP}$ for the pair $L_1, L_2$ and write $\text{PPP}(L_1, L_2)$, $\text{TPP}(L_1, L_2)$, if the given structures are $L_2$-equivalent and the resulting structures are $L_1$-equivalent.

4.4.3. Examples:

(i) While $L_{u,w}(Q_0)$ does not have the Interpolation Property by chapter 2 (7.1.3), $\text{INT}(L_{u,w}(Q_0), L_{u,w})$ does hold.

(ii) The logics $L_{u,w}(Q_0)$ and $L_{u,w}(aa)$ both do not satisfy the Interpolation Property ([Makowsky-Shelah 1981, proposition 6.6] and chapter 2 (7.1.3), but, as we shall see in 4.6.7., $\text{INT}(L_{u,w}(Q_0), L_{u,w}(aa))$ does hold.

(iii) Whereas $\text{INT}(L_{u,w}(Q_1), L_{u,w}(aa))$ does not hold, by chapter 2 (7.1.3) it is shown in [Shelah 1982??] that it is consistent with ZFC that $\Delta - \text{INT}(L_{u,w}(Q_1), L_{u,w}(aa))$ does hold (cf. also 4.8.?).

4.4.4. Proposition: Let PROPERTY be any of the above defined definability properties, and let $L_{10} < L_{11} < L_{20} < L_{21}$ be logics. Then PROPERTY$(L_{11}, L_{20})$ implies PROPERTY$(L_{10}, L_{21})$.

Proof: Obvious.

With these definition we can state a slightly stronger theorem.

4.4.5. Theorem:

(i) Let $L_1 < L_2 < L_3$ be three logics such that $\text{BETH}(L_1, L_2)$ and $\text{TPP}(L_2, L_3)$ hold. Then $\text{WFROB}(L_1, L_2)$ holds.

(ii) If in addition $L_3$ is compact, then $\text{INT}(L_1, L_3)$ holds.

Proof: Let $\varphi_1, \varphi_2$ be two formulas of $L_1(\tau_1)$ respectively, with $\tau_1 = \tau_0 \cup \{ P_1 \}$, which form a countereexample to $\text{WFROB}(L_1, L_3)$. Let $A_1$ be $\tau_1$-structures such that $A_1 \models \tau_0 = L_2 A_0 \models \tau_0$. Without loss of generality we assume that both $P_1$'s are of the same arity. In case they are unary, we apply lemma 4.1.7. directly; otherwise we combine it with remark 4.1.6. So we obtain a formula $\psi \in L_1(\tau_0 \cup \tau_{\text{true}} \cup \{ P \})$ which defines $P$ implicitly. So let $\theta \in L_2(\tau_0 \cup \tau_{\text{true}})$ be an explicit definition of $P$. So we get $\text{Tree}^P(A_1, A_2) = \exists \theta$ but $\text{Tree}^P(A_1, A_2) \models \neg \theta(c)$ which contradicts $\text{Tree}^P(A_1, A_2) \models \tau_0 \cup \tau_{\text{true}} \equiv L_2 \text{Tree}^P(A_1, A_2) \models \tau_0 \cup \tau_{\text{true}}$.

as were required by $\text{TPP}(L_2, L_3)$. QED.

Stating definability and preservation properties for pairs of logics allows us to sharpen results which were previously proven for absolute logics (and therefore for Karp
logics). The reader should also consult chapter 17.

4.4.6. Proposition: (Barwise) If \( L \) is a logic which satisfies \( \text{WFRQ}(L, L_{\omega, \omega}) \), then it has \( \text{Lowenhein} \) number \( \omega \).

**Proof:** Since \( L \subseteq L_{\omega, \omega} \) is a Karp logic, Therefore, if \( L \) properly extends first order logic, there is a sentence \( \varphi \in L[\tau_1] \) such that the relativized reducts of its models are all countably infinite, by lemma 2.1.2 of chapter 3. Assume, for contradiction that there is a sentence \( \psi \in L[\tau_2] \) with \( \tau_1 \cap \tau_2 \models \varphi \), which has only uncountable models. Let \( \Sigma \) be the \( L_{\omega, \omega} \) theory of infinite sets. So \( \Sigma, \varphi, \psi \) form a counterexample to \( \text{WFRQ}(L, L_{\omega, \omega}) \). QED.

4.4.7. Corollary: Let \( L \) be a logic which satisfies \( \text{BETH}(L, L_{\omega, \omega}) \). Then it satisfies

(i) \( \text{WFRQ}(L, L_{\omega, \omega}) \).

(ii) The Lowenhein number \( I_1(L) \) of \( L \) is \( \omega \).

**Proof:** (i) follows from Theorem 4.4.5. and Proposition 4.2.16 and (ii) follows from (i) together with Proposition 4.4.6. QED.

Before we prove theorem 4.4.5. let us give some more concrete examples:

4.4.8. Examples:

(i) The logic \( L_{\omega, \omega}(Q_1) \) from chapter 2 or 7 satisfies the Tree Preservation Property, as one proves easily with a Back and Forth argument. By 7.1.3. of chapter 2 it does not satisfy the Interpolation Property and therefore, since it is countably compact, not the Weak Finite Robinson-Property. So theorem 1 gives us that it does not satisfy the Beth Property.

(ii) The logic \( L_{\omega, \omega}(QF^\omega) \) is compact and does not satisfy the Interpolation Property by 7.1.3. of chapter 2. It is not too difficult to check that \( \text{TPP} \) holds for this logic. So again by theorem 4.4.5. the Beth Property fails.

(iii) The logic \( L_{\omega, \omega}(aa) \) from chapter 4 does not satisfy the Beth Property by [Makowsky-Shelah 1981]. This is shown using the ideas in the proof of theorem 4.4.5., though by example 4.1.2.(iv) \( L_{\omega, \omega}(aa) \) does not satisfy even the Pair Preservation Property. To carry through the proof one has only to verify that it holds for specific structures:

(iv) We cannot replace \( \text{TPP} \) by \( \text{PPP} \) in theorem 4.4.5., as the example in section 4.6. shows.

4.5. Definability and existence of models with automorphisms.

The aim of this section is to explore further the consequences of the assumption that a logic \( L \) satisfies both \( \text{PPP}(L) \) and \( \text{ROB}(L) \). As stated in section 4.3.12. it is an open problem whether such logics exist which properly extend first order logic. The results below may give us directions in solving that problem. Our main theorem is
4.5.1. Theorem: (Shelah) Let \( L \) be a small logic which has the Pair Preservation Property and the Robinson Property. Then every infinite \( \tau \)-structure \( A \) has \( L \)-extensions with arbitrarily-large \( \tau \)-automorphism groups.

For first order logic this is a corollary to the celebrated theorem by Ehrenfeucht and Mostowski concerning indiscernibles. The reader may consult [Chang-Keisler 1973, chapter 3.3] for a detailed exposition. In the proof of theorem 4.5.1 we discern various possibilities of defining abstract model theoretic properties centering around the existence of various automorphisms. Let us explore these first:

4.5.2. Definition: Let \( L_1, L_2 \) be logics.

(i) We say that the pair of logics \( L_1, L_2 \) has the Homogeneity Property, (Homogeneity Property for finite vocabularies), and write \( \text{AUT}(L_1, L_2) \), if for every \( \tau \)-structure \( M \) and \( c_1, c_2 \in M \) such that \( \langle M, c_1 \rangle \equiv L_1 \langle M, c_2 \rangle \) there is model \( \langle N, c_1', c_2' \rangle \) of \( \text{Th}_{L_1}(\langle M, c_1, c_2 \rangle) \) and a \( \tau \)-automorphism \( g \) of \( N \) such that \( g(c_1') = c_2' \). If \( L_1 = L_2 \) we just say that \( L_1 \) has the Homogeneity Property, (Homogeneity Property for finite vocabularies).

(ii) We say that the pair of logics \( L_1, L_2 \) has the Local Homogeneity Property, if for every \( \tau \)-structure \( M \) and \( c_1, c_2 \in M \) such that \( \langle M, c_1 \rangle \equiv L_1 \langle M, c_2 \rangle \) and every \( \varphi \in \text{Th}_{L_1}(\langle M, c_1, c_2 \rangle) \) there is model \( \langle N, c_1', c_2' \rangle \equiv \varphi \) and a \( \tau \)-automorphism \( g \) of \( N \) such that \( g(c_1') = c_2' \). If \( L_1 = L_2 \) we just say that \( L_1 \) has the Local Homogeneity Property.

(iii) We say that \( L \) has the (Local) Automorphism Property, if for every \( \tau \)-structure \( M \) and infinite subset \( P \subseteq M \), the theory \( T \), every sentence \( \varphi \) of the theory \( \text{Th}_{L}(\langle M, P \rangle) \) has a model \( \langle N, P \rangle \) which has an automorphism \( g \) of \( N \) such that \( g \upharpoonright P \neq \text{id} \).

4.5.3. Remark:

(i) If \( L \) is compact, then the Local Homogeneity Property and the Homogeneity Property coincide. The same holds for the Automorphism Property. We shall be mainly interested in the compact case. The local case may be of independent interest for further developments.

(ii) If a logic does not satisfy the Beth Property, one may construct its Beth closure in the natural way. Unlike the A-closure, studied in chapter 2 and chapter 17, the Beth closure cannot easily be proven to preserve compactness. In [Shelah 1983 Manuscript] the properties of the Beth closure were studied extensively. It turns out that stronger forms of the Homogeneity Property yield a sufficient condition for the Beth closure to preserve compactness. In section 4.6 an example of compact logic satisfying \( \text{PPP} \) and the Beth Property is presented, whose proof relies on this idea.

4.5.4. Proposition:

(i) Let \( L \) be a logic which has the Automorphism Property. Then \( L \) satisfies \( \text{REXT}(L) \) and therefore is \( \text{[\lambda]} \)-compact.

(ii) Let \( L \) be a logic which has the Local Automorphism Property. Then \( L \) has well-
ordering number $\omega_1(L)=\omega$. In particular, if $L$ is recursively generated then $L$ is also recursively compact.

**Proof:** (i) We show that $REXT(L)$, which is equivalent to $[\omega]$-compactness by theorem 3.2.1. Let $<M_1,P^M>$ be a $\tau$-structure with $P \in \tau$ and $P^M$ infinite. Let $\tau_1$ be a vocabulary, extending $\tau$, giving every element in $P^M$ a different name and let $M_1$ be the corresponding expansion. Clearly, $<M_1,P^M>$ still satisfies the hypothesis of the Automorphism Property. So let $<N,P^N>$ be a $L[\tau_1]$-extension of $<M_1,P^M>$ with the required automorphism. Clearly, $P^M \subseteq P^N$.

(ii) Here we just use that the standard model of arithmetic is rigid. For the latter remark we apply remarks 4.2.5. QED.

In general the Homogeneity Property does not imply compactness.

**4.5.5. Example:** Let $\kappa$ be a compact cardinal. The pair of logics $L_{\mathfrak{ca}}$, $L_{\mathfrak{ca}}$ has the Homogeneity Property. To see this one uses an ultralimit construction as in [Hodges-Shelah 1987]. Clearly, for $\lambda<\kappa$, these logics are not $[\lambda]$-compact.

However, for compact logics we have:

**4.5.6. Proposition:** If $L$ is a small and compact logic, which has the Homogeneity Property, then $L$ has the the Automorphism Property.

**Proof:** Let $<M,P>$ be a structure with $P$ infinite. Using compactness there are $L$-extensions $<N,P>$ with $P$ of arbitrary large cardinality. Using smallness we can find such an extension with $c_1,c_2 \in P$, $c_1 \neq c_2$ satisfying the same $L$-type. Now we apply the Homogeneity Property. QED.

Now we are in a position to prove the existence of models with many automorphisms.

**4.5.7. Proposition:** Let $L$ be a compact logic with the Automorphism Property. Then every $L$-theory with infinite models has models with arbitrarily large automorphism groups.

**Proof:** Let $\Sigma$ be an $L$ theory and $A$ be an infinite model of $\Sigma$. We want to define by induction vocabularies $\tau_\alpha$ and theories $\Sigma_\alpha$ which are sets such that, if $A\models \Sigma_\alpha$, then $A\models \tau_\alpha \models \Sigma$ and that $A \models \tau$ has at least $\text{card}(\alpha)$ many different automorphisms.

For $\alpha = 0$ we proceed as follows. Since $L$ is small the complete $L$-theory $\Sigma_0$ of $A$ is a set.

Again using smallness together with compactness we can find a model $B$ and $b,b' \in B$ satisfying the same type. So there is a model $M_0$ with a non-trivial automorphism $\phi_0$.

Now we use compactness in the form of proposition 1.1.1. QED.

**4.5.8. Example:** (Shelah) We define a quantifier binding four variables and acting on two formulas (i.e. of type $<2,2>$) in the following way: Let $A$ be a $\tau$-structure.
\[ A = Q^{\text{bool}}_{\text{univ}}(\phi(u,v,z),\psi(u,x,z))[\bar{a}] \]

If \( <A^\mathbb{A},R^\mathbb{A}> \) and \( <A^\mathbb{B},R^\mathbb{B}> \) are partially ordered structures, where the order satisfies the axioms of a boolean algebra and

\[ <A^\mathbb{A},R^\mathbb{A}> \cong <A^\mathbb{B},R^\mathbb{B}> \]

By \( A^\mathbb{A} \) we denote the set \( \{ b \in A : A = \phi[b,b,\bar{a}] \} \) and by \( R^\mathbb{A} \) the relation \( \{ (b,c) \in A^2 : A = \phi[b,c,\bar{a}] \} \), and similarly for \( \psi \).

4.5.9. Theorem: [Shelah 198?MOPAV] Assume GCH. Then the logic \( L_{u,\mathfrak{a}}(Q^{\text{bool}}) \) is compact.

4.5.10. Proposition: There is a sentence \( \psi_{\text{rigid}} \in L_{u,\mathfrak{a}}(Q^{\text{bool}}) \) such that:

(i) Every model of \( \psi_{\text{rigid}} \) is rigid, i.e. has no non-trivial automorphisms.

(ii) \( \psi_{\text{rigid}} \) has models of every infinite cardinality and

Proof: Let \( P \) be a ternary predicate symbol. Define \( \psi_{\text{rigid}} \) to be the conjunction of the following formulas:

\[ \psi_1 = \forall x Q^{\text{bool}} zy(x, y, z) \]

and

\[ \psi_2 = \forall z (x, y, z) \rightarrow Q^{\text{bool}} (x, y, z) \]

To prove (i) let \( A \) be a model of \( \psi_{\text{rigid}} \), \( a \in A \) and let \( h \) be an automorphism of \( A \). Clearly, \( <A^\mathbb{A},R^\mathbb{A}> \) is a boolean algebra by \( \psi_1 \). Since \( h \) is an automorphism, so is \( <A^\mathbb{B},R^\mathbb{B}> \) and they are isomorphic. So, by \( \psi_2 \), \( h(a) = a \).

To prove (ii), let \( \lambda \) an infinite cardinal and \( \{ B_i = < \mathbb{R}_i, \leq_i > : i < \lambda \} \) a family of \( \lambda \) many pairwise non-isomorphic boolean algebras of cardinality \( \lambda \) each. W.l.o.g. \( R_0 = \lambda \). We define a model \( A = <A^\mathbb{A},P^A> \) of \( \psi_{\text{rigid}} \) as follows:

We put \( A = \lambda \) and \( P^A = \{(i, a, b) \in \lambda^3 : a \leq_i b \} \). Clearly, \( A \models \psi_{\text{rigid}} \). QED.

4.5.11. Corollary: (GCH) The logic \( L_{u,\mathfrak{a}}(Q^{\text{bool}}) \) is compact but does not satisfy the Homogeneity Property.

Proof: By 4.5.9, the logic is compact. Assume, for contradiction, the Homogeneity Property. So by 4.5.8. and 4.5.7, we get models with arbitrarily large automorphism groups, contradicting 4.5.10. QED.

4.5.12. Proposition: There is a compact logic \( L \) which does not have the automorphism property.

Proof: Use example above. Then proposition above. QED.

4.5.13. Theorem: (Shelah) Let \( L \) be a logic.

(i) If \( L \) satisfies \( \text{PPP}(L) \) and \( \text{ROB}(L) \), then \( L \) has the Homogeneity Property.

(ii) If \( L \) satisfies \( \text{PPP}(L) \) and \( \text{FROB}(L) \), then \( L \) has the Homogeneity Property for finite vocabularies.
(iii) If \( L \) satisfies \( \text{PPP}(L) \) and \( \text{INT}(L) \), then \( L \) has the Local Homogeneity Property.

**Proof:** We prove only (i), the others being similar. Let \( M \) and \( c_1, c_2 \in M \) be as in the hypothesis of the Homogeneity Property.

Let \( M_1, c_1', c_2' \) be disjoint copies. Put \( N = \langle M, M' \rangle \). Put

\[
T = \forall x \in \text{Nat}(c_1, c_2, c_3, c_4) \langle N, c_1, c_2, c_3, c_4 \rangle.
\]

The equality holds because of \( \text{PPP}(L) \). Let \( c_1, c_2 \) be constant symbols with interpretations \( o_1, o_2 \) and \( o \) be a constant symbol with interpretation \( c'_1 \) or \( c'_2 \) respectively. Let \( F \) be a new function symbol. Let \( \psi_i, (i=1,2) \) be the sentence which says that \( F \) is a \( \tau \)-isomorphism (modulo name changing) mapping the first sort into the second sort which maps \( o_i \) into \( o \). If \( \tau \) is infinite, we need a set of sentences \( \psi_i \) defined similarly.

Clearly, \( T \cup \{ \psi_i \} \) has a model: So by \( \text{ROB}(L) \) or, if \( \tau \) is finite, by \( \text{WROB}(L) \). \( T \cup \{ \psi_1, \psi_2 \} \) has a model \( \langle M_1, M_2 \rangle \) which gives as the required automorphism in \( M_1 \). QED.

**4.5.14. Remarks:**

(i) In proposition 4.5.13. above the three cases coincide for compact logics.

(ii) If we assume that the logics are tiny, the hypotheses in the cases 4.5.13.(i) and (ii) imply that the logics are \( \omega \)-compact and ultimately compact. Assuming that \( L \) has an dependence number \( \alpha(L) \) which is smaller than the first uncountable measurable cardinal, the hypothesis in 4.5.13.(i) actually implies compactness. In 4.5.13.(ii) we need for this, that the logic \( L \) has size function \( s(\omega) < \omega^\omega \) for some \( n \in \omega \). (cf. 4.3.8, and 3.3.5.)

**4.5.15. Corollary:** Let \( L \) be a logic with dependence number \( \alpha(L) \) smaller than the first uncountable measurable cardinal (or, alternatively, with size function \( s(\omega) < \omega^\omega \) for some \( n \in \omega \)). If \( L \) satisfies \( \text{PPP}(L) \) and \( \text{ROB}(L) \), then \( L \) has the Automorphism Property.

**Proof:** We use remark 4.5.14.(ii) above and proposition 4.5.13. QED.

This corollary, together with theorem 4.5.7, gives us a proof of theorem 4.5.1.

**4.6. Some more examples: Stationary logic and its friends.**

In this last section we want to discuss, mostly without proofs, some more examples and consistency results, which all come from [Shelah 1983 Manuscript] and [Keeler-Shelah 1983?, 1983??]. They are all concerned with preservation and definability properties of compact or \( (\omega, \omega) \)-compact logics. Our first example concerns extensions of \( L_{\omega, \omega}(Q_1) \):

Let us recall some facts:

**4.6.1. Proposition:** The logic \( L_{\omega, \omega}(Q_1) \) has the following properties:

(i) \( L_{\omega, \omega}(Q_1) \) is \( (\omega, \omega) \)-compact, but not \( (\omega_1, \omega) \)-compact.

(ii) \( L_{\omega, \omega}(Q_1) \) does satisfy the Pair Preservation Property.

(iii) \( L_{\omega, \omega}(Q_1) \) does not satisfy the \( \Delta \)-Interpolation Property, and therefore neither the...
Interpolation Property.

It remains open whether \( \mathcal{L}_{\omega \omega}(Q_1) \) satisfies the Weak Beth Property. However, there is the following consistency result proved in [Mekler-Shelah 1983?].

**4.6.2. Theorem:** (Shelah) Every model \( M \) of ZFC has a generic extension \( M[G] \) in which \( \mathcal{L}_{\omega \omega}(Q_1) \) satisfies the Weak Beth Property.

For the stronger definability properties there is a consistency result in the other direction. We want to state, that it is consistent with ZFC, that no "reasonable" extension of \( \mathcal{L}_{\omega \omega}(Q_1) \) satisfies both PPP and the Interpolation Property (or equivalently the Uniform Reduction Property \( UR_2 \)). For this we need a definition:

**4.6.3. Definition:** (Definable logics)

(i) A logic \( L \) is definable, if the relations "\( \varphi \in L[\tau] \)" ("\( \varphi \) is a \( L[\tau] \)-formula") and "\( M \models \varphi \)" ("\( M \) is a model of \( \varphi \)") are definable by a formula of set theory without parameters.

(ii) A logic \( L \) is \( \lambda \)-definable, for \( \lambda \) a cardinal, if the relations "\( \varphi \in L[\tau] \)" ("\( \varphi \) is a \( L[\tau] \)-formula") and "\( M \models \varphi \)" ("\( M \) is a model of \( \varphi \)") are definable by a formula of set theory with a parameter \( A \subset \lambda \).

**4.6.4. Remark:** In chapter 17 absolute logics were introduced. This notion is not quite comparable with the above definition. For a logic to be absolute definability with parameters is allowed, but definability is restricted to \( \Delta_1 \)-definability.

**4.6.5. Examples:**

(i) Logics of the form \( \mathcal{L}_{\omega \omega}(Q^i) \) are definable, provided the quantifiers are set presentable in the sense of 1.5.8.

(ii) The logics \( \mathcal{L}_{\omega \lambda} \) are definable.

(iii) Not all logics are definable without parameters. Especially some of the fragments \( L_d \subset \mathcal{L}_{\omega \omega} \) are not definable, but they are \( \omega_1 \)-definable with parameter \( A \subset \omega_1 \). If \( A \) is a countable admissible fragment which has a code in \( \omega \) then \( L_d \) is even \( \omega \)-definable.

(iv) The logic \( \mathcal{L}_{\omega \omega} \) from section 1.6. is definable, provided the ultrafilter \( \mathcal{F} \) is definable.

The definability of this filter may very well depend on the set theoretic assumptions under consideration.

**4.6.6. Theorem:** (Shelah) For every model \( M \) of ZFC that there is a generic extension \( M[G] \) such that no definable logic \( L \) extending \( \mathcal{L}_{\omega \omega}(Q_1) \) satisfies both PPP(\( L \)) and the Interpolation Property (or; equivalently, the Uniform Reduction Property \( UR_2 \)) in \( M[G] \).

It was widely believed that the \( \Delta \)-closure of \( \mathcal{L}_{\omega \omega}(Q_1) \) is a rather untackable logic. That this need not be the case is shown by the next consistency result from [Mekler-Shelah 1983??]. Let us first recall some facts about the logic \( \mathcal{L}_{\omega \omega}(\omega \omega) \) from chapter 4.4. and chapter 2 theorem 7.1.3.
4.6.7. Proposition:

(i) The logic $L_{u,0}(aa)$ is $(\omega,\omega)$-compact, i.e. for validity, but does not satisfy the Interpolation Property.

(ii) $L_{u,0}(Q_1)$ is a sublogic of $L_{u,0}(aa)$.

(iii) $\text{INT}(L_{u,0}(Q_1), L_{u,0}(aa))$ does not hold.

Inspired by 4.6.2, we can state the following problem:

4.6.8. Problem: (Shelah) Does every model $M$ of ZFC have a generic extension $M[G]$ in which $\Delta-\text{INT}(L_{u,0}(Q_1), L_{u,0}(aa))$ holds?

In contrast to this it is shown in chapter 2 (7.1.3.) that $\text{INT}(L_{u,0}(Q_1), L_{u,0}(aa))$ does not hold.

The next example involves the logic $L_{b,0}(\omega^\omega)$.

4.6.9. Proposition:

(i) The logic $L_{u,0}(\omega^\omega)$ is compact, i.e. for validity, but does not satisfy the Interpolation Property.

(ii) $L_{b,0}(\omega^\omega)$ is a sublogic of $L_{u,0}(aa)$.

Proof: From chapter 2 (2.4.), and [Makowsky-Shelah 1981] we know (i). To see (ii) we axiomatize the class of orderings of cofinality $\omega$ by the $L_{u,0}(aa)$-sentence which says that the ordering has no last element, but that almost every countable set $P$ is unbounded. QED.

The next theorem shows that $L_{u,0}(aa)$ behaves more like second-order logic, than originally suspected, since it provides interpolating formulas for the logic $L_{u,0}(\omega^\omega)$.

4.6.10. Theorem: (Shelah) $\text{INT}(L_{u,0}(\omega^\omega), L_{u,0}(aa))$.

The proof may be found in [Mekler-Shelah 1983??].

4.6.11. A generalization: The pair of logics in 4.6.10 can be generalized to higher cardinals. For $L_{u,0}(\omega^\omega)$ this gives us the logics $L_{u,0}(\omega^\omega)$ which requires the ordering to be of infinite cofinality less or equal to $\lambda$. As shown in [Makowsky-Shelah 1981] this logic is still compact, but does not satisfy the Interpolation Property. For $L_{u,0}(aa)$ we have to define a logic $L_{u,0}(aa)$ for an appropriate filter $D_\lambda$. A detailed exposition may be found in [Mekler-Shelah 1983??]. What is important here, is a theorem of Shelah which states that the pair $L_{u,0}(\omega^\omega)$ and $L_{u,0}(aa)$ satisfies a strong form of the Homogeneity Property, as defined in section 4.5. As mentioned in section 4.5., such Homogeneity Properties can be used to prove that the Beth closure preserves \(\text{PPP}\) and compactness.

Using this line of thought Shelah proved the following theorem:

4.6.12. Theorem: (Shelah) The Beth-closure $L$ of the logic $L_{u,0}(\omega^\omega)$ is a compact logic which satisfies

(i) $\text{PPP}(L)$, (and therefore, by compactness, $\text{URP}$).
(ii) has the Beth Property but
(iii) does not satisfy the Interpolation Property (and therefore, by compactness, none of the Robinson Properties).

This shows, that in theorem 4.4.5, the Tree Preservation Property cannot be weakened to the Pair Preservation Property. For otherwise, since the logic is compact, the Beth Property would imply the Interpolation Property. It also shows that the Uniform Reduction Property for Pairs, does not imply even the Uniform Reduction Property $U_R$, which, by theorem 4.2.12, is equivalent to the Interpolation Property.

This example is also the first example so far, which exhibits a compact logic satisfying the Beth Property. Note that it is easy to construct compact logics, which satisfy the Weak Beth Property or the $\Delta$-Interpolation Property by the construction of the $\Delta$-closure or Weak Beth closure, as described in chapter 2 (7.2.5.) and, in more detail, [Makowsky-Shelah-Stavi 1979].
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