A GOLDEN RATIO CONTROL POLICY
FOR A MULTIPLE-ACCESS CHANNEL

by

Alon Itai and Zvi Rosberg

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Alon Itai
Zvi Rosberg

Computer Science Department
Technion - IIT, Haifa, Israel

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Abstract

Consider \( n \) stations sharing a single communication channel. Each station has a buffer of length one. If the arrival rate of station \( i \) is \( \lambda_i \), then \( 1 - \prod (1 - \lambda_i) \) is shown to be an upper bound (over all policies) on the throughput of the channel. Moreover, an optimal policy always exists and is stationary and periodic.

The throughput of two policies, the Random-Control policy and the Golden-Ratio policy, are analyzed for a finite and infinite number of stations. The latter is shown to approach a limit which is within at least 98.4% of the upper bound.

Keywords: Communication protocols, golden ratio, Fibonacci numbers, multiple-access channel, optimal control.
1. Introduction

Consider $n$ transmission stations sharing a single communication channel. Each station contains a buffer capable of storing a single packet; a message is lost if it arrives at a station whose buffer is full. The channel is assumed to be slotted, i.e. the channel time is divided into equal segments called slots.

In order to avoid collisions, each slot is allocated to a single station. When given permission, the station transmits a packet within the slot, if its buffer is not empty; otherwise, no message is sent. A policy $\pi$ allocates to each slot a station; $\pi$ depends on the slot number, $t$, and the state of the buffers.

Suppose that there is probability $r_i$ that a packet arrives at station $i$ during a time slot. Without loss of generality $r_1 \geq r_2 \geq \ldots \geq r_n > 0$.

Let $r=(r_1, r_2, \ldots, r_n)$ and $\bar{V}_t(r, \pi)$ be the expected number of packets successfully transmitted during the first $T$ slots using policy $\pi$. Define the throughput of the channel (under policy $\pi$)

$$\bar{V}(r, \pi) = \liminf_{T \to \infty} \frac{\bar{V}_T(r, \pi)}{T}$$

(1.1)

Finally, let

$$\bar{V}(r) = \sup_{\pi} \bar{V}(r, \pi)$$

$\bar{V}$ is the value function. A control policy $\pi^*$ is optimal for $r$ if $\bar{V}(r, \pi^*) = \bar{V}(r)$.

This study is a continuation of Rosberg [R], where $n=2$. There it was shown that the optimal policy does not contain conflicts (permission of more than one station to transmit in a slot). Moreover, the optimal policy permits one station to transmit every $\ell$ slots while the other station gets permission the rest of the time.
Finding an optimal policy for \( n \geq 2 \) turned out to be a difficult combinatorial problem. Here we shall discuss only conflict-free policies and conjecture that for every \( r \) there exists an optimal policy which does not contain conflicts. This conjecture is partially supported by the results of Rosberg [R] and the following reasoning: One would benefit most from conflicts when the \( r \)'s are very small. In which case, we should use \( \pi_{AT} \), the policy which gives permission to all the stations to transmit all the time, since the probability that more than one station will actually transmit is negligible. However, for equal \( r \) the following calculation shows that \( \pi_{AT} \) is inferior to \( \pi_{RR} \), the round robin policy (station \( i \) is permitted to transmit at slot \( t \) if \( t \equiv i \mod n \)):

\[
\bar{V}(r, \pi_{RR}) = 1 - (1-r)^n > n r (1-r)^{n-1} \bar{V}(r, \pi_{AT})
\]

Regardless of the conjecture, conflict-free policies are important for some communication networks.

In section 3 we show that attention can be restricted to loop policies, policies for which there exists an \( N \) such that for all \( t \), the station allocated to slot \( t \) is also allocated to slot \( t+N \). For such policies the mean buffer length and the mean packet delay in equilibrium under a given policy were studied by Kosovych [K], using unjustified simplifications.

Our model is similar, but not identical, to that of Schoute [S] and Varaiya and Varand [VW] (for details see Rosberg [R]). Other related works are for example Berman [B], Konheim [K], and Hajek and Van Loon [HV].

The goal of this paper is to estimate \( \bar{V}(r) \), investigate \( \pi^* \) and give policies whose values, \( \bar{V}(r, \pi) \), approach \( \bar{V}(r) \). In Section 2 our problem is formulated as a dynamic programming problem. Section 3, gives an upper bound to the value function. The next two sections discuss particular poli-
cies: Section 4 analyzes a random control policy, whereas Section 5 discusses a deterministic policy - the Golden Ratio Policy, which approaches a limit (as the number of stations tends to infinity) greater than 96.4% of the upper bound.

2. Dynamic Programming Formulation

For every station \(i\) define the following r.v.'s whose values are 0, 1.

\[
X_i(t) = 1 \text{ iff the buffer is full at slot } t.
\]

\[
V_i(t) = 1 \text{ iff a new packet arrived at slot } t.
\]

\[
u_i(t) = 1 \text{ iff the station had permission to transmit at slot } t.
\]

Since at most one station has permission to transmit at time \(t\), \(\sum_{i=1}^{n} u_i(t) \leq 1\).

From the definitions:

\[
X_i(t+1) = V_i(t) + X_j(t)[1-V_{i}(t)](1-u_{i}(t)), \quad 1 \leq i \leq n,
\]  

(2.1)

where \(\{X_i(0), V_i(t) : 1 \leq i \leq n, t = 0, 1, \ldots\}\) are Bernoulli independent r.v.'s and \(P(V_i(t) = 1) = P(X_i(0) = 1) = \rho_i\).

Let the immediate reward at slot \(t\), \(w(t)\), be the number of packets successfully transmitted during the slot.

\[
w(t) = w(X(t), u(t)) = \sum_{i=1}^{n} X_i(t) u_i(t)
\]  

(2.2)

where \(X(t) = (X_1(t), \ldots, X_n(t))\) and \(u(t) = (u_1(t), \ldots, u_n(t))\). The total expected number of packets successfully transmitted during the first \(T\) slots using policy \(\pi\) is

\[
V_T(r, \pi) = E_n \sum_{t=1}^{T} w(t) = \sum_{i=1}^{n} E_i w(t).
\]  

(2.3)

where \(E_i w(t)\) is the expected immediate reward at slot \(t\) using policy \(\pi\).

As in Berman [B] and Rosberg [R], we use the following sufficient statistic (for sufficient statistics in optimum control of stochastic systems, see e.g. Striebel [S]): Let \(h_i(t) (i = 1, 2, \ldots, n)\) be the elapsed time since
station $i$'s last permission to transmit. Define $k^{(i)}(0)=1$ ($i=1, \ldots, n$). We have

$$k^{(i)}(t+1)=1+k^{(i)}(t)(1-u_i(t)), \quad i=1, \ldots, n.$$  \hfill (2.4)

Let $u^{t-1}=(u(1), u(2), \ldots, u(t-1))$. Since the $V_i$'s are independent, collisions are avoided and $X_i(t)$ depends only on $u^{t-1}$ and $V_i$, we have the following:

Lemma 2.1

(i) Given $u^{t-1}$, $\{X_1(t), X_2(t), \ldots, X_n(t)\}$ are mutually independent r.v.'s.

(ii) $P(X_i(t)=1 \mid u^{t-1}) = 1-(1-\tau_i)^{k^{(i)}(t)}$, $i=1, \ldots, n$.

Let $p_i(k) = 1-(1-\tau_i)^k$.  \hfill (2.5)

The assumption of no conflicts, (2.2), (2.3) and Lemma 2.1 imply that

$$E(w(t) \mid u^{t-1}) = \sum_{k=0}^{n_i} p_i(k^{(i)}(t))u_i(t).$$  \hfill (2.6)

where $u_i(t)$, $i=1,2, \ldots, n$ are the control actions taken by policy $\pi$.

A dynamic programming problem is defined by the state space $S$, the action space $A= X_{\pi}$, the law of motion $q$ and the reward function $w$.

For every $t$, $X(t)$ is a random variable whose probability distribution depends only on $u^{t-1}$. If $X(t)$ were known it could serve as a state. However, we are interested in a decentralized control policy, therefore $X(t)$ can be considered by all the stations only as a random variable whose distribution depends on $u^{t-1}$, which is common information (since all the stations know the policy).

From Lemma 2.1 the distribution of $X(t)$ is completely defined by the parameters $k(t)=(k^{(1)}(t), \ldots, k^{(n)}(t))$. Therefore, we consider the state space

$$S = \{k=(k^{(1)}, \ldots, k^{(n)}) \mid k^{(i)}=1,2, \ldots, i=1,2, \ldots, n\}.$$
and the action space at state $s \in S$.

$$A_j = \{a_{ij} = (u_1, u_2, \ldots, u_n) \mid u_i = 0, 1 \text{ and } u_i u_j = 0 \text{ for } 0 \leq i < j < n \}.$$  

From (2.4), the law of motion (transition probabilities) becomes:

$$q((k^{(1)}+1, \ldots, k^{(n)}+1) \mid (k^{(1)}, \ldots, k^{(n)}), u = (0, \ldots, 0)) = 1$$

$$q((k^{(1)}+1, \ldots, 1, \ldots, k^{(n)}+1) \mid (k^{(1)}, \ldots, k^{(n)}), u = (0, \ldots, 1, \ldots, 0)) = 1$$

$$q((k^{(1)}+1, \ldots, k^{(n)}, u)) = 0 \text{ otherwise.}$$

Note that the law of motion is deterministic. Finally, from (2.6) the expected immediate reward is

$$w(k, u) = \sum_{i=1}^n p_i(k^{(i)}) u_i.$$  \hspace{1cm} (2.8)

Thus, from (2.3)

$$V^*(r, n) = \sum_{i=1}^n \sum_{t=1}^T p_i(k^{(i)}(t)) u_i(t).$$  \hspace{1cm} (2.9)

**Lemma 2.2** Let $\pi$ be a policy in which station $i$ is permitted to transmit only a finite number, of times. Then there exists a policy $\pi'$ such that

$$V^*(r, n) > V^*(r, n')$$

**Proof:** Let $0 \leq s_1 < s_2 < s_3 \leq 2n$. Because of the convexity of $p_j(k)$

$$f_j(s^1, s^2, s^3) \overset{\text{def}}{=} p_j(s^3-s^1) + p_j(s^3-s^2) - p_j(s^3-s^1) < 1.$$  

Let $f = \max_{1 \leq j \leq m} \max_{0 \leq s_1 \leq s_2 \leq s_3 \leq 2n} f_j(s^1, s^2, s^3)$ and $r \geq 2n$ be such that $p_j(r) > \bar{f}.$

Let $t_0$ be the last slot policy $\pi'$ permitted station $i$ to transmit. Define $t_m = t_0 + m \tau$. In each interval $[t_{2m+1}, t_{2m+1}]$ there is a station $j_m \neq i$ which is permitted to transmit three times: $s^1_m, s^2_m, s^3_m$.

Consider the policy $\pi'$ which is identical to $\pi$ except that at slots $s^2_m$ station $i$ is permitted to transmit instead of station $j_m$.

From (2.9) the net gain in the immediate reward for the interval $[t_{2m}, t_{2m+1}]$ is...
\[ p_i(s_{n+1}^{-} - s_n^{-}) - (p_{in} (s_n^{-} - s_{n-1}^{-}) + p_{im} (s_{n}^{-} - s_{n-m}^{-}) - p_{jm} (s_m^{-} - s_{n-m}^{-})) > p_i(\tau) - f > 0. \]

Since this gain repeats every interval of length \( 2\tau \)

\[ \bar{V}(r, \pi') \geq \bar{V}(r, \pi) + \frac{p_i(\tau) - f}{2r} > \bar{V}(r, \pi) \]

Let \( d^{(i)}(\pi) = \sup_{r} k^{(i)}(r, \pi) \), where \( k^{(i)}(r, \pi) \) is \( k^{(i)}(t) \) under policy \( \pi \). The following lemma is proved using techniques similar to those of Lemma 2.2:

**Lemma 2.3** For every \( r \), there exists a \( \tau \) such that for every policy \( \pi \), there exists a policy \( \pi' \) for which

(i) \( d^{(i)}(\pi') \leq \tau \quad i = 1, \ldots, n \);

(ii) \( \bar{V}(r, \pi) \leq \bar{V}(r, \pi') \).

**Remark 2.1** The weak inequality of Lemma 2.3 (ii) cannot be strengthened since the long run average is not sensitive to fluctuations of zero weight.

Since the state space can be restricted to the finite space,

\[ S_0 = \{(k^{(1)}, k^{(2)}, \ldots, k^{(n)}) | k^{(i)} \leq \tau, i = 1, 2, \ldots, n \} \]

for some \( \tau \), the action space is finite and we have (see e.g. Derman [D]):

**Theorem 2.1** There exists a non-randomized stationary control policy \( \pi^* \) such that

\[ \bar{V}(r) = \sup_{\pi} \bar{V}(r, \pi) = \bar{V}(r, \pi^*). \]

**Remark 2.2** It should be noted that the above non-randomized stationary policies depend only on the \( k^{(i)} \)'s and \( r \). If a state appears at slot \( t \) and \( t' \), then for every \( m \) the states of slot \( t + m \) and \( t' + m \) are equal. Also, since the state space is finite, some state must repeat, thus the policies are ultimately periodic, i.e. they are loop policies.
3. An upper bound on the value function.

An empty action is a slot in which no permission is given. Since for every policy which has empty actions there is a policy with no empty actions and at least the same throughput, (replace every empty action by a permission to some station), we shall investigate policies with no empty actions.

Let \( \pi \) be a loop policy of period \( N \) and station \( i \) be given \( N^{(i)} \) permissions. Let \( d_f^{(i)}(j=1, \ldots , N^{(i)}) \) be the distances between two consecutive permissions to station \( i \) in the loop. Clearly, \( \sum_j d_f^{(i)} = N \).

From (1.1), (2.9), the definition of \( d_f^{(i)} \) and the mean ergodic theorem:

\[
\bar{v}(r, \pi) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} p_i(d_f^{(i)}) = \frac{1}{N} \sum_{i=1}^{N} \frac{N^{(i)}}{j=1} N^{(i)} \frac{1}{N^{(i)}} p_i(d_f^{(i)}) \quad (3.1)
\]

Because of the convexity of \( p_i \),

\[
\bar{v}(r, \pi) \leq \frac{1}{N} \sum_{i=1}^{N} N^{(i)} \frac{1}{N} \sum_{j=1}^{N^{(i)}} d_f^{(i)} = \frac{1}{N} \sum_{i=1}^{N} N^{(i)} \frac{1}{N} \sum_{j=1}^{N^{(i)}} d_f^{(i)}. \quad (3.2)
\]

Theorem 3.1 For every policy \( \pi \),

\[
\bar{v}(r, \pi) \leq \prod_{i=1}^{n} (1 - r_i) \quad \text{def} = U(r)
\]

Proof: From (3.2) we have \( \bar{v}(r, \pi) \leq U(r) \), where \( U(r) \) is the solution to the following optimization problem:

\[
U(r) = \max \sum_{i=1}^{n} x_i^{(i)}(1 - r_i)^{1/z_i^{(i)}} = 1 - \min \sum_{i=1}^{n} x_i^{(i)}(1 - r_i)^{1/z_i^{(i)}}
\]

subject to \( \sum_{i=1}^{n} x_i^{(i)} = 1 \).

We shall see that the optimum solution to the following problem satisfies

\[
\min_{f \{x^{(1)} , \ldots , x^{(n)} \}} = \min \sum_{i=1}^{n} x_i^{(i)}(1 - r_i)^{1/z_i^{(i)}} \quad (3.4)
\]
subject to \( \sum_{i=1}^{n} x^{(i)} = 1 \).

To solve (3.4), we use Lagrange multipliers:

Let \( F(x^{(1)}, x^{(2)}, \ldots, x^{(n)}; \lambda, \gamma) = \lambda \left( 1 - \sum_{i=1}^{n} x^{(i)} \right) + \sum_{i=1}^{n} x^{(i)} \left( 1 - \tau_i \right)^{1/z^{(i)}} \).

A necessary and sufficient condition to the solution is

\[
\frac{\partial F}{\partial x^{(i)}} = -\lambda + (1 - \tau_i)^{1/z^{(i)}} (1 - \ln(1 - \tau_i)^{1/z^{(i)}}) = 0,
\]

\[
\frac{\partial F}{\partial \lambda} = 1 - \sum_{i=1}^{n} x^{(i)} = 0.
\]

Therefore, the solution must satisfy

\[
(1 - \tau_i)^{1/z^{(i)}} (1 - \ln(1 - \tau_i)^{1/z^{(i)}}) = \lambda, \quad i = 1, 2, \ldots, n
\]

and

\[
\sum_{i=1}^{n} x^{(i)} = 1
\]

(3.5)

Since \( (1 - \tau_i)^{1/z^{(i)}} < 1 \) and the function \( \gamma(z) = z(1 - \ln(z)) \) is monotonically increasing for \( z < 1 \), it follows from (3.5) that for every \( i \),

\[
(1 - \tau_i)^{1/z^{(i)}} = C \quad \text{for some } C
\]

(3.6)

and

\[
\sum_{i=1}^{n} \frac{\ln(1 - \tau_i)}{\ln C} = 1.
\]

Therefore, the minimum of (3.4) is obtained when

\[
x^{(i)} = \frac{\ln(1 - \tau_i)}{\sum_{j=1}^{n} \ln(1 - \tau_j)}
\]

and

\[
C = \prod_{i=1}^{n} (1 - \tau_i)
\]

(3.7)

As promised, these \( x^{(i)} \)'s are nonnegative. From (3.3), (3.4) and (3.7) it fol
The interpretation is that station $i$ is permitted to transmit every $1/x^{(i)}=n$ slots. However, in general, the $x^{(i)}$'s are irrational, thus the $i$-th station cannot transmit every $1/x^{(i)}$ slot. Consequently, the upper bound cannot be obtained. We shall try to approximate the optimal solution by policies which permit station $i$ to transmit $N^{(i)}x^{(i)}$ equally spaced times in a loop of size $N$ (according to Remark 2.2). Thus, we shall first calculate the $x^{(i)}$'s and then consider policies, which gives station $i, N^{(i)}$ permissions. These policies will be close to the optimal only if the permissions to station $i$ are nearly equally spaced.

Thus, we are confronted with the following placement problem:

Given $N^{(1)}, N^{(2)}, \ldots, N^{(n)}$, $\sum_{i=1}^{n} N^{(i)} = N$.

Place the permissions of each station such that

$[N/N^{(i)}] \leq d^{(i)} \leq N/N^{(i)}$

where $d^{(i)}$ and $d^{(i)}$ are the minimum and maximum distance between consecutive permissions to station $i$. 

Corollary 3.1 If $\tau_i=\tau$, $i=1, \ldots, n$, then the "round robin" policy, $\pi_{RR}$, is optimal.

Proof: A straightforward calculation shows that

$U(x) = 1 - \prod_{i=1}^{n} (1 - \tau_i)$

If all the $\tau_i$'s are equal then all the $x^{(i)}$'s are also equal and $x^{(i)}=1/n$.
This problem does not always have a solution. (For example: $n=3$, $N=8$, $N^{(1)}=3$, $N^{(2)}=2$ and $N^{(3)}=1$; ideally, station 1 should be given permission every other time slot, station 2 at time slots $t$ and $t+3 \mod 8$; this is not obtainable.)

Let us consider also the asymptotic case where the number of stations $n \to \infty$ and $r_i = r_i(n) \to 0$. Suppose $\pi$ is a loop policy which permits each station to transmit at least once during a loop of size $N = N(n)$, and let $d_j^{(i)}$ be as above. From (3.1):

$$V(r, \pi) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N^{(i)}} \left[ 1 - (1 - r_i) d_j^{(i)} \right] = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N^{(i)}} \left( d_j^{(i)} r_i + O(r_i^2) \right)$$

(3.8)

$$\approx r + O(r^2),$$

where $r = \sum_{i=1}^{N} r_i$. However,

$$U(r) = 1 - \prod_{i=1}^{N} (1 - r_i) = r + O(r^2)$$

Thus, if also $r \to 0$, $V(r, \pi) \approx U(r)$ for every $\pi$.

Thus, it is important to define the asymptotic behavior non-trivially. That is, when the number of stations $n \to \infty$, the arrival rate to station $i$ (given $n$ stations) $r_i(n) \to 0$ and the total arrival rate $r = \sum_{i=1}^{n} r_i(n)$ remains fixed.

Let $z^{(i)}(n)$ be the proportion of permissions given to station $i$ by policy $\pi$ in the system $(n, r(n))$, where

$$r(n) = (r_1(n), \ldots, r_n(n)).$$

Also let

$$C^{(i)}(n) = \max \left\{ \frac{r}{r_i(n)} - d_j^{(i)}(n) \right\}$$

(3.9)

where $d_j^{(i)}(n)$ are the distances $d_j^{(i)}$ defined above for policy $\pi$ and system $(n, r(n))$. For a given policy $\pi$ we simplify the notation to $z^{(i)}(n)$ and $C^{(i)}(n)$.
Since attention can be restricted to loop policies, a policy \( \pi \) depends only on \( (n, r(n)) \). Denote by \( P \) the set of all loop policies.

**Definition 3.1**: A policy \( \pi: (n, r(n)) \to P \) is asymptotically optimal if

\[
| \bar{V}(r(n), n(n, r(n))) - \prod_{i=1}^{n}(1 - r_i(n)) | \to 0
\]

Note that under the above asymptotic conditions

\[
\lim_{n \to \infty} U(r(n)) = \lim_{n \to \infty} \prod_{i=1}^{n}(1 - r_i(n)) = 1 - e^{-r}.
\]

**Theorem 3.2**: Let \( \pi \) be a loop policy, if

\[
| x^{(i)}(n) - \frac{r_i(n)}{r} | \to 0
\]

and

\[
C^{(i)}(n)r_i(n) \to 0;
\]

then \( \pi \) is asymptotically optimal.

**Proof**: For any given \( (n, r(n)) \), we have from (3.9)

\[
\bar{V}(r(n), n(n, r(n))) = \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{N} p_i(d_j^{(i)}(n))
\]

where \( N, N^{(i)} \) and \( d_j^{(i)} \) are as defined earlier.

Therefore, from (3.9)

\[
\bar{V}(r(n), n(n, r(n))) = \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{N^{(i)}} \left[ 1 - \left( 1 - \frac{r_i(n)}{r} - c^{(i)}(n) \right) \right]
\]

\[
= \sum_{i=1}^{n} \left[ 1 - \left( 1 - \frac{r_i(n)}{r} - c^{(i)}(n) \right) \right].
\]

Since \( \sum_{i=1}^{n} x^{(i)}(n) = 1 \), \( (1 - r_i(n)) \to e^{-r} \) and \( (1 - r_i(n)) - c^{(i)}(n) \to 1 \) we obtain

\[
\lim_{n \to \infty} \bar{V}(r(n), n(n, r(n))) = 1 - e^{-r},
\]

and in conjunction with (3.10) the theorem is proved.
Remark 3.1 Since the average distance between two consecutive permissions to station \( i \), is \( 1/z^{(i)}(n) \), the condition \( C(n)\tau(n) \to 0 \) implies that
\[
|z^{(i)}(n) - \tau(n)| \to 0.
\]

Remark 3.2 From Theorem 3.2 it follows that for a large number of stations, a policy \( \pi \) might still be good, even if its \( d^{(i)} \)'s are far away from the desired distances according to (3.7). Thus it might not be needed to solve the placement problem.

We have not been able to prove that any policy is asymptotically optimal, even though we have a candidate (see Section 6). In the next two sections we analyze two policies.

4. A Random Control Policy

For a given system \((n,r(n))\), let
\[
z^{(i)}(n) = \frac{\ln(1-\tau(n))}{\sum_j \ln(1-\tau_j(n))}.
\]

Definition 4.1 Let \( \pi_R \) be the control policy which at every slot \( t, t=1,2,\ldots \), permits station \( i \) to transmit with probability \( z^{(i)}(n) \), \( i=1,2,\ldots,n \).

This policy can be implemented distributively (without conflicts) if all the stations use the same random number generator.

Let \( D^{(i)} \) be the number of slots between two consecutive permissions to station \( i \). \( D^{(i)} \) (\( i=1,\ldots,n \)) are independent geometrically distributed random variables with probability of success \( z^{(i)}(n) \). Therefore,
\[
V(r(n),\pi_R) = E \sum_{i=1}^{n} (1-(1-\tau_i(n)))D^{(i)}.
\]
The expectation is taken over the common distribution \((I,D)\), where \( I \) is the station which has been given permission to transmit at a random slot and \( D \)
is the distance from the last permission. Clearly,

\[ V(r(n), \pi_R) = E_1 E_{\pi_R} (1 - (1 - r_i)^{p(i)}) = \sum_{i=1}^{n} x^{(i)}(n) E (1 - (1 - \tau_i(n))^z) \]

\[ = 1 - \sum_{i=1}^{n} x^{(i)}(n) \sum_{k=1}^{x^{(i)}(n)} (1 - \tau_i(n))^k (1 - x^{(i)}(n))^{k-1} x^{(i)}(n) \]

\[ = 1 - \sum_{i=1}^{n} x^{(i)}(n) \frac{x^{(i)}(n)(1 - \tau_i(n))}{1 - (1 - \tau_i(n))(1 - x^{(i)}(n))} \]  \hspace{1cm} (4.2)

Under the asymptotic conditions, \( \ln(1 - \tau_i(n)) = \approx \tau_i(n) \), and from L'Hopital's rule,

\[ \lim_{n \to \infty} \sum_{i=1}^{n} x^{(i)}(n) \frac{x^{(i)}(n)(1 - \tau_i(n))}{1 - (1 - \tau_i(n))(1 - x^{(i)}(n))} = \frac{1}{1 + r} \]  \hspace{1cm} (4.3)

Now since \( \sum_{i=1}^{n} x^{(i)}(n) = 1 \) for every \( n \) we have from (4.2) and (4.3)

\[ \lim_{n \to \infty} V(r(n), \pi_R(n, r(n))) = 1 - \frac{1}{1 + r} \]

Note that for small and large \( r \), the random policy \( \pi_R \) is asymptotically close to the upper bound, \( \mathcal{U}(r) = 1 - e^{-r} \).

5. The Golden Ratio Control Policy

Let \( (n, r(n)) \) be a given system and \( x^{(i)}(n) > 0, i = 1, 2, \ldots, n, \sum_{i=1}^{n} x^{(i)}(n) = 1 \) be the desirable proportions of permissions to each of the stations. (When no confusion arises the argument \( n \) will be omitted.) Also, let \( F_k \) be the \( k \)-th Fibonacci number and \( N_k^{(i)}, i = 1, 2, \ldots, n \) be integers such that

\[ [x^{(i)}F_k] \leq N_k^{(i)} \leq [x^{(i)}F_k] \]

and

\[ \sum_{i=1}^{n} N_k^{(i)} = F_k. \]  \hspace{1cm} (5.1)

Thus
For each \( k \), the Golden Ratio policy assigns \( N_k^{(1)} \) slots to station \( i \) and attempts to distribute the permissions uniformly over a loop of size \( F_k \). (The analysis of Section 3 implies that it is desirable to distribute the permissions uniformly.)

Open address hashing confronts a similar problem: To distribute keys uniformly over a hash table. The uniformity of the distribution depends on the hash function. It has been shown that multiplicative hashing with the Golden Ratio multiplicative, \( \phi^{-1} = (\sqrt{5} - 1)/2 \approx 0.6180339887 \), distributes the keys most uniformly (Knuth [Kn vol. 1]). The Golden Ratio policy applies some of these results.

Let \( \text{frac}(y) = y - [y] \), \( a_j = \text{frac}(j\phi^{-1}) \) and \( A_N = \{a_j \mid j = 0, \ldots, N-1\} \). The \( t \)-th smallest point of \( A_N \) is identified with the \( t \)-th slot of the loop.

**Definition 5.1** The Golden Ratio policy, \( \pi_{CR(k)} \), is the policy which assigns to station \( i \) the slots corresponding to the points

\[
\{a_j \mid \sum_{m=1}^{j-1} N_k^{(m)} \leq j < \sum_{m=1}^{j} N_k^{(m)}\},
\]

It will be convenient to identify the points \( 0 \) and \( 1 \), and thus the points \( a_j \) are distributed over a circle \( C \).

**Example 5.1:** Suppose \( n=3 \), \( F_8=8 \), \( N_8^{(1)}=4 \), \( N_8^{(2)}=3 \) and \( N_8^{(3)}=1 \). Then \( \pi_{CR(8)} \) assigns to station 1 the slots corresponding to 0, \( \phi^{-1} \), \( \text{frac}(2\phi^{-1}) \) and \( \text{frac}(3\phi^{-1}) \); to station 2 the slots corresponding to \( \text{frac}(4\phi^{-1}) \), \( \text{frac}(5\phi^{-1}) \) and \( \text{frac}(6\phi^{-1}) \); and to station 3 the point corresponding to \( \text{frac}(7\phi^{-1}) \). Thus the loop policy keeps giving permission to the stations in the following cyclic order: "1,2,1,3,2,1,2,1".

Let \( l_m = \text{frac}((-1)^m F_{\phi^{-1}}^{-1}) \). From Knuth [K, vol. 3, pp. 506-549] we can deduce the following:
Theorem 5.1 Let $N = F_m + s \ (0 \leq s < F_{m-1})$.

(i) $A_N$ partitions $C$ into

- $s$ intervals of length $l_{m+1}$;
- $F_{m-2} + s$ intervals of length $l_m$;
- $F_{m-1} - s$ intervals of length $l_{m-1}$.

(ii) An additional point, $a_N \ (N < F_{m+1})$, breaks an interval of length $l_{m-1}$ into one of length $l_m$ and one of length $l_{m+1}$.

(iii) The lengths of the subintervals decreases, moreover $l_{m+1} = \varphi^{-1} l_m$.

Corollary 5.1 For each station $i$ (with $N_i^{(k)} = s_i^{(k)} + F_{i_k}$) there are at most three types of intervals:

- $s_i^{(k)}$ of length $l_{k_i+1}$;
- $F_{k_i-2} + s_i^{(k)}$ of length $l_{k_i}$;
- $F_{k_i-1} - s_i^{(k)}$ of length $l_{k_i-1}$.

Proof: Theorem 5.1 considered starting at $a_0 = 0$. However, the same result holds if we make a circular shift and start at $a_q$ where $q = \sum_{m=1}^{r} N_k^{(m)}$.

In order to compute $V(r, \pi(q(k)))$, we shall find the number of slots between two consecutive permissions to station $i$. Thus we consider the number of points of $A_{F_k}$ in intervals of lengths $l_{k_i-1}$, $l_{k_i}$ and $l_{k_i+1}$.

Consider the partitioning of $C$ by $A_{F_k}$ into subintervals. A subinterval is atomic if it is not partitioned into any smaller subinterval. Let $X^1$ and $X^2$ be two intervals of length $|X^j| = l_m$ and whose endpoints belong to $A_{F_k}$. Let $g_k(X)$ denote the number of atomic subintervals in $X^j$. 

Technion - Computer Science Department - Technical Report CS0256 - 1983
From Theorem 5.1 the atomic subintervals have lengths \( l_k \) and \( l_{k-1} \), thus

\[
l_k g_k(\mathcal{X}_1^j) + l_{k-1} g_{k-1}(\mathcal{X}_1^j) = I_m, \quad j = 1, 2.
\]

From Theorem 5.1 (iii)

\[
\varphi^{-1} g_k(\mathcal{X}_1^1) + g_{k-1}(\mathcal{X}_1^1) = \varphi^{-1} g_k(\mathcal{X}_2^2) + g_{k-1}(\mathcal{X}_2^2),
\]

and

\[
\varphi^{-1} (g_k(\mathcal{X}_1^1) - g_k(\mathcal{X}_2^2)) = g_{k-1}(\mathcal{X}_2^2) - g_{k-1}(\mathcal{X}_1^1).
\]

Since \( g_k(\mathcal{X}_1^j) \) are integer and \( \varphi^{-1} \) is irrational:

\[
g_{k-1}(\mathcal{X}_1^1) = g_{k-1}(\mathcal{X}_2^2)
\]

\[
g_k(\mathcal{X}_1^1) = g_k(\mathcal{X}_2^2).
\]

Thus \( g_m \) is independent of the particular choice of \( \mathcal{X}_1^j \), and \( d_k(l_m) \), the number of atomic subintervals in an interval of length \( l_m \), given the partitioning \( A_{P_k} \), satisfies

\[
d_k(l_m) = g_k(\mathcal{X}_1^1) + g_{k-1}(\mathcal{X}_1^1).
\]

To compute \( d_k(l_m) \), notice that

\[
d_k(l_k) = d_k(l_{k-1}) = 1.
\]

From Theorem 5.1 each interval of size \( l_m \) is partitioned into an interval of size \( l_m+1 \) and one of size \( l_m+2 \). Thus,

\[
d_k(l_m) = d_k(l_m+1) + d_k(l_m+2).
\]

Comparing this to Fibonacci's recursive formula yields

Lemma 5.1

\[
d_k(l_m) = F_{k+1-m}.
\]

Remark 5.1 Note that \( d_k(l_{k-1}) \), \( d_k(l_k) \) and \( d_k(l_{k+1}) \) are the number of slots between two consecutive permissions to station \( i \).

Lemma 5.2 Let \( j_i \) satisfy \( \varphi^{-1} x(1) < \varphi^{-1} x^2 + 1 \), if \( x(1) \neq \varphi^{-1} x^2 + 1 \) then for almost all \( k \).
\[ k^{(i)} = k - j_i. \]

**Proof:** From (5.2) and the definition of \( N_k^{(i)} \) and \( s_k^{(i)} \)

\[
\lim_{k \to +\infty} \frac{F_k + s_k^{(i)}}{F_k} = x^{(i)}
\]

Consequently, for almost all \( k \)

\[
\varphi^{-j_i} \left\{ \frac{F_k + s_k^{(i)}}{F_k} \right\} \varphi^{-j_{i+1}} < F_k \varphi^{-j_i} < F_k \varphi^{-j_{i+1}}.
\]

(5.3)

From Knuth [K]

\[
F_k = \varphi^k - \left( 1 - \varphi \right)^k \sqrt{5}
\]

Since \( |1 - \varphi| < 1 \)

\[
\lim_{m \to \infty} F_m - \frac{\varphi^m}{\sqrt{5}} = 0.
\]

Thus

\[
\lim_{k \to +\infty} F_k \varphi^{-j_i} - F_k^{-j_i} = 0.
\]

Thus from (5.3) for almost all \( k \)

\[
F_k^{-j_i} < F_k^{j_i} + s_k^{(i)} < F_k^{-j_{i+1}}.
\]

The result follows since \( s_k^{(i)} < F_k^{-j_i} \).

**Remark 5.2** The above proof is not valid for \( x^{(i)} = \varphi^{-j_i} \). However, in this case we may choose \( N_k^{(i)} = F_k^{-j_i} \).

**Corollary 5.2**

\[
\lim_{k \to +\infty} \frac{F_k - q}{F_k} = \frac{F_k - q}{F_k} = \varphi^{-j_i+q}
\]

\[
d_k(l_{k-1+q}) = F_k^{j_i+1-q} \quad q = 0, 1, 2,
\]

\[
\lim_{k \to +\infty} \frac{s_k^{(i)}}{F_k} = x^{(i)} - \varphi^{-j_i}.
\]
Theorem 5.2

\[ \overline{V}(r, \pi_{GR}) \overset{\text{def}}{=} \lim_{k \to \infty} \overline{V}(r, \pi_{GR(k)}) = 1 - \sum_{i=1}^{n} \left[ (x^{(i)} - j_i)(1-r_i) \right]_{F_{i+1}}^{
}
+ (x^{(i)} - \varphi^{-i-1})(1-r_i)_{F_{i+1}}^{
} + (\varphi^{-i-1} - x^{(i)})(1-r_i)_{F_{i+1}}^{
} \]

Proof: From Corollary 5.1, Remark 5.1 and (3.1)

\[ \overline{V}(r, \pi_{GR(k)}) = \frac{1}{F_k} \sum_{i=1}^{n} \left[ s_k^{(i)}(1-(1-r_i) d_k(\ell_{i+1})) \right]_{F_{i+1}}^{
} + (F_{k-2} + s_k^{(i)})(1-(1-r_i) d_k(\ell_{i+1}))_{F_{i+1}}^{
} + (F_{k-1} - s_k^{(i)})(1-(1-r_i) d_k(\ell_{i+1}))_{F_{i+1}}^{
} \]

The theorem follows from Corollary 5.2.

We now investigate the asymptotic behavior of the throughput for increasing \( n \) and fixed \( r \):

Theorem 5.3 If \( r_i(n) \to 0 \) and \( \sum_{i=1}^{n} r_i(n) = r \) then

\[ \lim_{n \to \infty} \overline{V}(r(n), \pi_{GR}) = 1 - (1 - \varphi^{-1})e^{-r\sqrt{r}/\sqrt{5}} - \varphi^{-1}e^{-r^{2}/\sqrt{5}} \]

Proof: /* From Lemma 5.2

\[ j_i = -\log_\varphi x^{(i)} + a_i \]

where \( a_i = \text{frac} \left( \log_\varphi x^{(i)} \right) \). Thus

\[ \varphi^{-j_i} = x^{(i)} \varphi^{-a_i} \]

Therefore by Theorem 5.2,

\[ \overline{V}(r(n), \pi_{GR}) = 1 - \sum_{i=1}^{n} x^{(i)}(n) h^{(i)}(n) \]

where \( h^{(i)}(n) = (1 - \varphi^{-a_i})(1-r_i(n))_{F_{i+1}}^{
} + (1 - \varphi^{-a_i})(1-r_i(n))_{F_{i+1}}^{
} + (\varphi^{-a_i+1}-1)(1-r_i(n))_{F_{i+1}}^{
} \)

We show that the right-hand-side of (5.4) attains its minimum for \( 0 \leq a_i \leq 1 \).

/* In the proof we omit the argument \( n \), whenever it is understood.
for $i=1,2,\ldots, n$, when $a_i = 1$.

$r_i(n)/r \to 0$ implies that $x^{(k)}(n) \to 0$, hence $\lim_{n \to \infty} F_{ji} = \infty$. Therefore,

$$F_{ji+q} = \frac{\varphi^{ji+q} - (1 - \varphi)^{ji+q}}{\sqrt{f_0}} = \frac{\varphi^{a_{ji+q}}}{x(n) \sqrt{f_0}} + O(1), \quad q = 0, 1, 2.$$ Let $c_q = (1 - r_q) \varphi^{(0)(n) \sqrt{f_0}}$. Clearly, $0 \leq c_q < 1$.

Thus the $h^{(i)}(n)$'s are approximately /**

$$(1 - \varphi^{-a_i}) c_i^q \varphi^{a_q} + (1 - \varphi^{-a_i-1}) c_i^q \varphi^{a_q+1} + (\varphi^{-a_{i+1}} - 1) c_i^q \varphi^{a_q+2}$$

(5.5)

Using

$$\varphi^{a+2} = \varphi^a + \varphi^{a+1}$$

and

$$c_i^q a_q+1 > 0,$$

it is sufficient to show that

$$g(a) = (1 - \varphi^{-a}) c_i^q \varphi^{a_q} + (1 - \varphi^{-a-1}) c_i^q \varphi^{a_q+1} + (\varphi^{-a_{i+1}} - 1) c_i^q \varphi^{a_q+2}$$

attains its maximum when $a = 1$.

Since

$$(1 - \varphi^{-a}) + (1 - \varphi^{-a-1}) + (\varphi^{-a_{i+1}} - 1) = 0,$$ it follows that

$$g(a) = (1 - \varphi^{-a})(c_i^q \varphi^{a_q} - c_i^q) + \varphi^{-a_{i+1}}(c_i^q \varphi^{a_q+1} - 1).$$

Clearly, the first summand of $g(a)$ increases in $a$. By differentiation and using the fact that $z(1-lnz)$ increases for $0 < z < 1$ it can be shown that the second summand also increases in $a$.

Thus for almost all $n$, (5.4) and (5.5) imply

/** The correct $h^{(i)}(n)$ is obtained by replacing $c_i$ by $c_i(1 + O(1))$, however this approximation is justified because $h^{(i)}(n)$ is continuous, and for large $n$ the error approaches zero.
\[ \bar{V}(r(n), \pi_{GR}) = 1 - \sum_{i=1}^{n} x^{(i)}(n) \left[ (1 - \phi^{-1}) c_i^p + \phi^{-1} c_i^p \right]. \]

To finish the proof note that \( c_i \to 1 / \phi \sqrt{6} \) and \( \sum_{i=1}^{n} x^{(i)}(n) = 1. \)

Finding the minimum over \( r \) of the ratio between \( \lim \inf \bar{V}(r(n), \pi_{GR}) \) and \( \lim U(r(n)) \) yields:

**Corollary 5.3** Under our asymptotic conditions, for almost all \( n \)

\[ \frac{\bar{V}(r(n), \pi_{GR})}{U(r(n))} > 0.984. \]

6. Conclusions

Theorem 2.1 implies that there always exists a stationary periodic optimal policy, hence, there is always an asymptotically optimal policy. Even though Theorem 3.2 gives sufficient conditions for a policy to be asymptotically optimal, we have not been able to demonstrate one. The Golden-Ratio policy of section 5 is shown to be very close of being asymptotically optimal (see corollary 5.3). These results depend on \( k \)-th Fibonacci number being sufficiently large. For practical implementation \( k \) must be finite. We believe that when \( F_k \gg \frac{1}{x^{(i)}} \) the throughput is sufficiently close to the limit (see (3.7)).

The above results do not imply that we should always use the Golden Ratio policy. For a specific \( r \) the placement problem may be easily solvable. For example: when the \( \eta_i \)'s are equal, then \( x^{(i)} = \frac{1}{n} \) implying that the Round-Robin policy is optimal.

We conjecture that the following policy is asymptotically optimal:
Let

\[ \alpha_j(t) \] be the number of slots station \( i \) was permitted to transmit until time \( t \):

\[ \nu_i(t) = 1 \text{ iff } i \text{ is the station for which } \min_j \left( \frac{\alpha_j(t)}{t} \right) \text{ is obtained.} \]

Numerical calculations indicate that this policy is promising.
REFERENCES


