POSITIVE RESULTS IN ABSTRACT MODEL THEORY:
A THEORY OF COMPACT LOGICS

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Technical Report #254
December 1982

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(Revised version December 5, 1982)

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Abstract:
We prove that compactness is equivalent to the amalgamation property, provided the occurrence number of the logic is smaller than the first uncountable measurable cardinal. We also relate compactness to the existence of certain regular ultrafilters related to the logic and develop a general theory of compactness and its consequences.

AMS Classification:
03C95, 03C20, 03E55, 03C55, 03C30.

1 Partially supported by Swiss National Science Foundation Grant No. 82.820.0.80 (1980-82) and Minerva Foundation (1978/79).

2 Partially supported by US-Israel Binational Foundation Grant No. 1110
0. Introduction

Abstract model theory, after its promising successes due to Lindström, Barwise and Feferman\(^1\), is widely considered to be in a crisis. Indeed, the expectations were high: The theory should give us more information on why which model theoretic properties hold in what logics, should characterize most logics or logic-families as "natural" or should explain to us why certain properties of logics are rare. But instead, counterexamples emerged refuting many reasonable conjectures.

Many more logics appeared on the scene like Shelah's various compact quantifiers [Sh1, MS2], stationary logic [MKM, Ma1\(^1\)], Souslin-logic [E1] and infinitary propositional connectives [Fr1, Ha] and topological logics [Zi, Ma2]. Many researchers turned pessimistic: There seemed no hope for positive results. In particular one problem (P1) remained open, whether Craig's Interpolation theorem and compactness characterize first order logic (cf. [Fr2, MS1]).

It turned out that, though this problem is still unsolved, it did contribute quite a bit to the positive development of abstract model theory. In this paper we present a rather satisfying theory of compact logics. Let us trace its development.

In [MS1] we had proved that (under some strange set theoretic hypothesis) some version of Robinson's Consistency lemma (RCL 1) together with a Feferman-Vaught-type theorem for pairs of structures implies compactness. In the light of (Pi) it is still possible that this is an "empty" theorem, i.e. the hypothesis is only satisfied by first order logic.

A preprint of D. Muridici [Mu] drew our attention to a different version of Robinson's Consistency lemma (RCL) and we realized, while proof-reading, [MS1]. In spring 1979, that under the same set theoretic hypothesis as before, RCL was

\(^1\) For a bibliography we refer to [MS1, 2] and especially to [BF].
equivalent to compactness together with Craig's Interpolation theorem.

Unsatisfied by the set theoretic hypothesis we started to search for improvements. New properties were singled out to be crucial and finally a rather pleasant picture evolved.

Friedman [Fr3] already had observed that compactness implied, that every formula depended essentially only on a finite number of non-logical symbols. This lead us to define the occurrence number \( OC(L) \) of a logic \( L \) to be the smallest cardinal \( \kappa \) such that every \( L \)-sentence depends on less than \( \kappa \) many non-logical symbols (if it exists). With the help of this one realizes that the amalgamation property \( AP \) for \( L \)-embeddings is both a consequence of either compactness or \( RCL \). But assuming the existence of extendible cardinals, there are logics which have \( AP \) and are not compact. However, here is a surprise: If there are no uncountable measurable cardinals, no such logic exists. Then compactness is equivalent to \( AP \). In fact, it suffices to assume the logic \( L \) has occurrence-number \( OC(L) \leq \mu_0 \), the first uncountable measurable cardinal. Another surprise: This result solves an open problem of Malitz and Reinhart [MR]. The logics \( L_{\mu_0}(Q_\kappa) \) with the quantifier "there exist at least \( \kappa \) many (\( \kappa \) infinite) do not satisfy \( AP \).

Already from [MSi] we knew that "chopping up compactness in \( (\lambda, \kappa) \) -compactness" was not a good idea. We introduced there the notion of \( \kappa \)-c.c. (\( \kappa \)-relative compact) which we generalize here to \( (\kappa, \lambda) \)-compactness all denote it \( (\kappa, \lambda) \)-compactness following the literature in topology [AU, VA]. It turns out that this is a very natural notion and we develop a general theory connecting it with certain regular ultrafilters (and ultrapowers over them) introduced by Keisler [Ke1].

More consequences of compactness, such as the upward Lowenheim-Skolem theorem and non-existence of maximal models (in both relativized and non-relativized versions) are examined. Here \( (\omega, \omega) \) -compactness and again the
existence of measurable cardinals play an important role. The latter comes via a theorem of Rabin and Keisler [Ke2] on elementary extensions of complete structures.

The paper is organized as follows: In chapter 1 we collect the preliminaries, define the central concepts and discuss their obvious interrelations.

In chapter 2 we discuss consequences of compactness and their interrelations. The results are collected in figure 1, and the end of section 2.4.

In chapter 3 we develop the general theory of $[\kappa, \lambda]$-compactness and $(\kappa, \lambda)$-regular ultrafilters. We define the class of ultrafilters $\mathcal{UF}(L)$ related to a logic $L$ and show that $L$ is $[\kappa, \lambda]$-compact iff some $(\kappa, \lambda)$-regular ultrafilter $D$ is in $\mathcal{UF}(L)$. An ultrafilter $D$ is related to a logic $L$ if, roughly speaking, some generalization of Los' lemma for ultrapowers over $D$ holds for $L$. This correspondence theorem is the key to results like $[\kappa^+, \lambda^+]$-compactness implies $[\kappa, \kappa]$-compactness for every $\kappa$ or $[\text{cf}(\kappa), \text{cf}(\kappa)]$-compactness implies $[\kappa, \kappa]$-compactness.

In chapter 4 we introduce the occurrence number $\text{OC}(L)$ of a logic $L$. Our main result here is: (Assume there are no uncountable measurable cardinals) If $L$ is a logic which is $[\kappa, \kappa]$-compact then $\text{OC}(L) = \omega$.

In chapter 5 we prove finally our main theorem: If $\text{OC}(L) \leq \mu_0$ (the first uncountable measurable cardinal) and $L$ satisfies $AP$ then $L$ is compact. We also consider the Joint Embedding Property ($JEP$) and $RCL$. The main theorem relies heavily on the results from chapters 2, 3 and 4. But behind it is a theorem which can be stated and proved independently of the previous chapters (Theorem 5.3) and for which the main tool in the proof is the construction of "homogeneous" models of some non-elementary class $K$. In spirit, though not in detail, this is similar to the method Shelah deals with abstract elementary classes in [Sh2].
The order of the chapters is dictated by their interdependence. Although proofs are rather detailed every chapter is independently readable (provided one believes the quoted results). In fact the casual reader may want to read chapter 5 first and then backtrack for the needed material.

We hope to convince the reader that abstract model theory is still a challenging branch of model theory and logic and that this paper positively supplements the work presented in [MS1, 2 and 3].

Postscript November 1982: The present paper was completed in September 1979 and circulated as a preprint. The first author has presented the whole paper in the seminar of the "Model Theory Year 1980/81" in Jerusalem and is indebted to all the participants for various suggestions and corrections of inaccuracies, but especially to J.Baldwin and M.Magidor. We Are also indebted to U.Kueker and S.Buechler for their careful reading and their valuable comments.

In the time between completion and revision of the present paper a book has been prepared under the untiring editorship of J.Barwise and S.Feferman [FB] which puts much of what we tried to say in the introduction into a larger perspective. What should be mentioned here is that chapter 18 of [FB] is to some extent based on this paper.
1. Preliminaries

1.1. The framework.

We work again in the framework of abstract model theory as described in [Bal] and [MSS; MS1,2]. In many respects this paper is an expansion and continuation of [MS1, Sec.6]. We follow closely [Bal] for notation; with one exception: We denote by $L$ vocabularies (called languages by Barwise), by $L$ logics (hence omitting the star in $L^*$ of Barwise) and we use $L(L)$ to denote the class of $L$-formulas in the logic $L$. Furthermore, we assume in most of the cases, unless otherwise mentioned, that $L(L)$ is a set whenever $L$ is a set. To avoid confusion we nevertheless write $L_{\omega,\omega}$ to denote predicate calculus and to denote the set of first order $L$-formulas. But the context will always make clear if we are thinking of the $L$-formulas or the logic.

We use:

- $A,B,C,M,N$ for $L$-structures,
- $A,B,C,M,N$ for their underlying sets,
- $L_0,L_1,...$ for vocabularies,
- $\kappa,\lambda,\mu,\nu,...$ for cardinals,
- $\alpha,\beta,\gamma,\delta,...$ for ordinals.

Other letters are used freely, but their usage should be clear from the context.

1.2. Compactness properties.

A logic $L$ is $(\kappa,\lambda)$-compact if, given a set $\Sigma$ of $L$-sentences of cardinality $\kappa$ such that every $\Sigma_0 \subset \Sigma$ of cardinality $<\lambda$ has a model, then $\Sigma$ has a model. A logic $L$ is compact (fully compact) if $L$ is $(\kappa,\omega)$-compact for every $\kappa$. A logic $L$ is $[\kappa,\lambda]$-compact if, given any set $\Sigma$ of $L$-sentences and a set $\Sigma_1$ of $L$-sentences of cardinality $\kappa$ such that for every $\Sigma_0 \subset \Sigma_1$ with $\text{card}(\Sigma_0) < \lambda$, $\Sigma_0 \cup \Sigma$ has a model, then $\Sigma_1 \cup \Sigma$ has a model. Here $\Sigma$ plays the role of the diagram (or a fragment thereof).
of a given structure. Note that \([\kappa, \kappa]\)-compactness was called \(\kappa\text{-r.c.}\) in [MS1].

Clearly, \([\kappa, \lambda]\)-compactness implies \([\kappa, \lambda]\)-compactness, compactness implies \([\kappa, \lambda]\)-compactness for every \(\kappa \geq \lambda\) and some trivial monotonicity properties hold as well. Less trivial consequences may be found in chapters 2 and 3, in particular theorems 2.8, 3.11 and 3.12.

1.3. Elementary-embeddings.

Let \(L\) be a logic and \(A, B\) be two \(L\)-structures. We say that \(A\) and \(B\) are \(L(L)\)-equivalent and write \(A \equiv B(L(L))\) if for every \(L(L)\)-sentence \(\varphi \models L\varphi \iff B \models \varphi\).

We denote by \(Th_{L(L)}(A)\) the set \(\{\varphi \in L(L) : A \models \varphi\}\).

We say that \(A\) is an \(L(L)\)-substructure of \(B\) and write \(A \leq_{L(L)} B\) if \(A\) is a substructure of \(B\) and \(\langle A, a \in A \rangle \equiv \langle B, a \in A \rangle (L(L))\) where \(L_a\) is the vocabulary \(L\) augmented by names for each element of \(A\). Note that for \(L = L_{\omega, \omega}\) \(A \leq_{L} B\) implies that \(A = B\).

We say that a logic \(L\) has the Amalgamation Property (AP), if whenever \(A, B_1, B_2\) are \(L\)-structures such that \(A \leq_{L} B_i\) (\(i = 1, 2\)) then there is an \(L\)-structure \(C\) such that \(B_i \leq_{L} C\) amalgamating \(A\).

We say that \(L\) has the Joint Embedding Property (JEP) if whenever \(A_1, A_2\) are \(L\)-structures such that \(A_i = A_0(L(i))\) then there is an \(L\)-structure \(B\) such that \(A_i \leq_{L(L)} B\) (\(i = 1, 2\)).

Proposition 5.1 gives some easy consequences of these definitions.

1.4. The Robinson consistency lemma.

Let \(\Sigma\) be a set of \(L(L)\)-sentences. We say that \(\Sigma\) is complete if given two models \(A, B\) of \(\Sigma\) we have \(A = B(L(L))\). We also say \(L(L)\)-complete if the context is not clear.

Let \(L\) be a logic. We say that \(L\) satisfies the Robinson Consistency Lemma (RCL) if the following holds:
Let \( L_i \) \((i = 0, 1, 2)\) be vocabularies such that \( L_0 = L_1 \cap L_2 \) and let \( \Sigma_i \) \((i = 0, 1, 2)\) be sets of \( L_i \)-sentences. If \( \Sigma_0 \) is complete and \( \Sigma_0 \cup \Sigma_1 \) and \( \Sigma_0 \cup \Sigma_2 \) have models then \( \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \) has a model.

We say that \( L \) satisfies \( RCL1 \) for the case where \( \Sigma_1 \) and \( \Sigma_2 \) are finite.

Proposition 5.1 shows that \( RCL \) implies \( AP \). Trivially \( RCL1 \) is a consequence of \( RCL \). If \( L \) is compact then \( RCL \) is a consequence of Craig’s Interpolation theorem (cf. [MS1]).

1.5. **Models of arbitrary large cardinality.**

Let \( L \) be a logic. \( L \) is said to satisfy the **Upward Lowenheim Skolem theorem** (ULS) if every set of \( L \)-sentences which has an infinite model, has arbitrary large models.

This is equivalent to the following: Given an infinite \( L \)-structure \( A \), there are \( L \)-structures \( B \) of arbitrary large cardinality, such that \( A \leq_{L(L)} B \).

Note that for many-sorted structures the cardinality is defined as the sum of the cardinalities of its sorts. The logic \( L \) is said to satisfy the **Relativized ULS (RULS)** if, given an \( L \)-structure \( A \) with distinguished unary predicate \( P^A \), \( P^A \) infinite, there are \( L \)-structures \( B \) with \( P^B \) of arbitrary large cardinality such that \( A \leq_{L(L)} B \).

Trivially, **RULS** implies **ULS**.

These notions are studied in Chapter 2.

1.6 **Maximal models**

Let \( L \) be a logic and \( A \) and \( L \)-structure. \( A \) is \( L(L) \)-**maximal** if there is no proper \( L \)-extension \( B \) of \( A \).

If \( P^A \) is a unary predicate on \( A \), we say that \( P^A \) is \( L \)-**maximal** if for every \( L \)-extension \( B \) of \( A \) \( P^B = P^A \).

We say that \( L \) satisfies **MAX** if there are no infinite \( L \)-maximal models. \( L \) satisfies **RMAX** if there are no infinite \( L \)-maximal predicates in any \( L \)-structure.
Trivially $ULS$ implies $MAX$, $RULES$ implies $RMAX$ and $RMAX$ implies $MAX$, and all of them are consequences of compactness. Chapter 2 is devoted to the study of these properties.

Let $A$ be an $L$-structure. The complete expansion $A^+$ of $A$ is the structure with universe $A$ and for every $X \subseteq A^n$ there is a new predicate symbol $P_X$ whose interpretation is $X$. Note that in the many-sorted case we do not add sorts to form the complete expansion.

We say a structure is complete if $A \models A^+$.

The following is a version of the Rabin-Keisler theorem, (cf. [Ke1]) which we present here with an outline of a proof:

**Theorem 1.1:** Let $A$ be a complete structure of cardinality $\lambda < \mu_0$, where $\mu_0$ is the first (uncountable) measurable cardinal. $P^A$ be a countable predicate of $A$ and $B$, a proper $L_{\omega, \omega}$-extension of $A$. Then $P^A \subseteq P^B$.

**Proof:** Let $c \in B - A$ and put $F = \{X \subseteq A : B \models P_X(c)\}$, where $P_X$ is the predicate of $A$ representing $X$. Clearly $F$ is an ultrafilter. Now assume $P^A = P^B$. We want to show that $F$ is countably complete to conclude that $\text{card}(A) \leq \mu_0$. So let $\{X_n : n \in \omega\}$ be a family with $X_n \in F$ for every $n \in \omega$. Let $P^A = \{a_n : n \in \omega\}$ be an enumeration of $P^A$. Since $A$ is complete there is a predicate $P^A$ (with predicate symbol $F$) such that $P^A \subseteq P^A \times A$ and $\{a \in A : A \models P(a, a)\} = X_n$ for each $n \in \omega$.

Now we use $P^A = P^B$ and conclude that $B \models \forall x (P(x) \rightarrow F(x; c))$ and hence $\bigcap X_n \in F$. QED.

**Corollary 1.2:** Let $L$ be a logic and $P^A$ a countable $L$-maximal predicate in some structure $A$ of cardinality $\lambda < \mu_0$. Then there are arbitrarily large $L$-maximal structures $B$ of cardinality $< \mu_0$.

Note that if $\text{card}(A) \geq \mu_0$, $\text{card}(P^A) < \mu_0$ and $F$ is a $\mu_0$-complete ultrafilter on $\mu_0$, then $\prod A / F \models B$ is a proper $L_{\omega, \omega}$-elementary extension of $A$, but $P^A = P^B$. 

The following proposition is easy: (cf. [F1]).

**Proposition 1.3**: Let $L$ be a logic. The following are equivalent:

(i) $L$ is not $[\omega, \omega]$-compact and.

(ii) there is a countable $L$-maximal predicate.

**Corollary 1.4**: If $L$ is not $[\omega, \omega]$-compact, then there are arbitrary large $L$-maximal structures of cardinality less than $\mu_0$.

The reader should compare these results with Lemmas 2.5, 2.6 and Theorem 2.7.
2. Some Consequences of Compactness and their Interrelations.

2.1 Characterizing Cardinals.

Let \( L \) be a logic and \( \lambda \) an infinite cardinal.

**Definition:** A cardinal \( \lambda \) is \textit{cofinally \: characterizable in} \( L \) if there is an expansion \( \mathcal{A} \) of \( \langle \lambda, \in \rangle \) of the form \( \mathcal{A} = \langle \lambda, U^A, E^A, \ldots \rangle \) of the vocabulary \( L \) such that:

(i) \( U^A, E \upharpoonright U \models \langle \lambda, \in \rangle \) and

(ii) whenever \( \mathcal{B} \) is an \( L \)-structure and \( \mathcal{A} \models \langle L(U), \mathcal{B} \rangle \) then \( U^A, E \upharpoonright A \) is cofinal in \( U^B, E \upharpoonright B \), i.e. for every \( b \in U^B \) there is \( a \in U^A \) with \( b \models E(a, b) \).

A cardinal \( \lambda \) is \textit{cardinallike \: characterizable in} \( L \) if there is an expansion \( \mathcal{A} \) of \( \langle \lambda, \in \rangle \) of the form \( \mathcal{A} = \langle A, U^A, E^A, \ldots \rangle \) in the vocabulary \( L \) such that:

(i) \( U^A, E \upharpoonright U \models \langle \lambda, \in \rangle \) and

(ii) whenever \( \mathcal{B} \) is an \( L \)-structure and \( \mathcal{A} \models \langle L(U), \mathcal{B} \rangle \) then \( U^B, E \upharpoonright B \) is a \( \lambda \)-like order.

We shall say that \( \lambda \) is \textit{cofinally \: (cardinallike) \: characterizable in} \( L \) via \( \mathcal{A} \).

In [M61] we have proved:

**Proposition 2.1:**

(i) \( L \) is \([\lambda, \lambda]\)-compact, \( \lambda \) regular, iff \( \lambda \) is not cofinally characterizable in \( L \).

(ii) \( L \) is compact iff no infinite \( \lambda \) is cofinally characterizable in \( L \).

Note that (ii) follows immediately from (i).

**Proposition 2.2:** \( \omega \) is cofinally characterizable in \( L \) iff \( \omega \) is cardinallike characterizable in \( L \).

**Proof:** If \( \omega \) is cofinally characterizable in \( L \), \( L \) is not \([\omega, \omega]\)-compact, hence, by Proposition 1.3, \( \omega \) is cardinallike characterizable. The other direction follows from the fact that no proper elementary extension \( \langle L_\omega, \in \rangle \) of \( \langle \omega, \in \rangle \) is \( \omega \)-like. QED.
Note that \([\lambda, \lambda]\)-compactness is essential here. If we want similar theorems for \((\lambda, \lambda)^\ast\)-compactness, we have to replace the complete theory of \(\langle A, A \rangle\) in \(L\) by a theory of cardinality \(\lambda\).

**Definition:** \(\lambda\) is strongly cofinally (cardinallike) characterizable in \(L\) if \(\lambda\) is cofinally (cardinal-like) characterizable in \(L\) via \(A\) such that \(A = \bigcup^A\).

We shall use strong characterizability in section 2.3.

### 2.2. The existence of maximal models.

We first characterize the properties \(RMAX\) and \(MAX\) for a logic \(L\):

**Theorem 2.3:** \(L\) satisfies \(RMAX\) iff \(L\) is \([\omega, \omega]\)-compact.

**Proof:** Assume for contradiction \(L\) is not \([\omega, \omega]\)-compact. By proposition 2.1 (ii) \(\omega\) is cardinallike characterizable via \(A\) for some \(L\)-structure \(A\). Clearly then \(A\) has an expansion with a unary predicate \(U = \omega\) which is \(L\)-maximal. So \(L\) does not satisfy \(RMAX\).

To prove the other direction let \(B\) be a structure with an infinite \(L\)-maximal predicate \(R\). Expand \(B\) to \(\bar{B}\) with a new unary predicate \(P^B \subset R^B\) and \(P^B\) countable and let \(\{b_i : i \in \omega\}\) be an enumeration of the names of elements of \(P^B\). Let \(\Sigma\) be the \(L\)-diagram of \(\bar{B}\) and \(\Sigma_1 = \{c \neq b_i : i \in \omega\} \cup \{P(c)\}\), where \(c\) is a new constant symbol not in the language of \(\bar{B}\). By \([\omega, \omega]\)-compactness \(\Sigma \cup \Sigma_1\) has a model \(C\) which extends \(\bar{B}\) since \(P^C\) properly extends \(P^B\), hence it extends \(R^\bar{B}\). QED.

To characterize \(MAX\) we introduce a new definition:

**Definition:** Let \(L\) be a logic and \(\lambda\) a cardinal. \(L\) satisfies \(MAX(\lambda)\) if all \(L\)-maximal models have cardinality \(\leq \lambda\). Clearly \(MAX(\omega)\) is the same as \(MAX\).

**Proposition 2.4:** If \(L\) is \([\lambda, \lambda]^\ast\)-compact then \(L\) satisfies \(MAX(\lambda)\).

The proof is similar to the proof of theorem 2.3 and is left as an exercise.
Lemma 2.5: Let \( \lambda_0 \) be an infinite cardinal, and let \( L \) satisfy \( \text{MAX}(\lambda_0) \). Then \( L \) is \( [\chi, \chi] \)-compact for some infinite cardinal \( \chi \).

Proof: We prove the contrapositive. Assume, by proposition 2.1 (i), all regular cardinals \( \lambda \) are cofinally characterizable in \( L \) via some structure \( B(\lambda) \), which we assume without loss of generality of minimal cardinality \( g(\lambda) \). Let \( \mu \) be the first cardinal such that:

(i) If \( \nu < \mu \), \( \nu \) a cardinal, then \( g(\nu) \leq \mu \);  
(ii) \( \lambda_0 \leq \mu \);  
(iii) \( \text{cf}(\mu) = \omega \).

Clearly such a cardinal exists. Let \( B \) be the complete expansion of the structure \( <\mu, \in> \). We claim that \( B \) is maximal. For otherwise, let \( C \) be an \( L \)-extension of \( B \). If \( C \) is a proper extension, there is \( c \in C - B \). Remember \( \text{cf}(\mu) = \omega \) and let \( \{b_n : n \in \omega\} \) be a cofinal sequence in \( B \). Since \( \nu \) is cofinally characterizable in \( L \), \( g(\omega) \leq \mu \) and \( B \) is a complete structure \( \{b_n : n \in \omega\} \) is also cofinal in \( C \). So \( C|c = c \in b_k \) for some \( k \in \omega \). Now let \( d \in B \) be the smallest (with respect to \( \in \) ) element in \( B \) such that \( C|d = c \in d \). So \( d \) is an ordinal. Let \( \delta = \text{cf}(d) \) and \( \{d_i : i < \delta\} \) be a sequence cofinal to \( d \) in \( B \). Again since \( g(\delta) \leq \mu \) and \( \delta \) is cofinally characterizable in \( L \), \( \{d_i : i < \delta\} \) is cofinal to \( d \) in \( C \). So there is \( j < \delta \) with \( C|c = c \in d_j \), which contradicts the minimality of \( d \). By our definition of \( \mu, \lambda_0 \leq \mu \), so we proved that \( L \) does not satisfy \( \text{MAX}(\lambda_0) \). QED.

Lemma 2.6: Let \( L \) be a logic and let \( \lambda_0 \) be the first cardinal such that \( L \) is \( [\lambda_0, \lambda_0] \)-compact. Then \( \lambda_0 \) is measurable (or = \( \omega \)).

Proof: By proposition 2.1(i) each regular \( \lambda < \lambda_0 \) is cofinally characterizable in \( L \) via a structure \( B(\lambda) \) with \( \gamma_\lambda \) the cardinality of \( B(\lambda) \). Let \( \mu \) be define by

\[
\mu = [\gamma_\lambda : \lambda < \lambda_0]^+ + \lambda_0^+
\]

and let \( B \) be the complete expansion of the structure \( <\mu, \in> \). By \( [\lambda_0] \)-compactness \( B \) has an \( L \)-elementary extension \( C \) with some \( c \in C - B \) and such
that
\[ C \models c \in \lambda^B. \] Since \( \lambda_0 \) is minimal we have for no \( \lambda < \lambda_0 \) that \( C \models c \in \lambda^C \). We now define an ultrafilter \( F \) on \( \lambda_0 \) by
\[ F = \{ X \subseteq \lambda_0 : C \models c \in X \} \]
where \( X \) is the name of the set \( X \) in \( B \). Clearly \( F \) is an ultrafilter. We propose to show that \( F \) is \( \lambda_0 \)-complete. For this it suffices to show that for every \( \mu < \lambda_0 \) the filter \( F \) is \( \mu \)-descendingly complete.

Let \( \{ X_\alpha : \alpha < \mu \} \) be any family in \( F \). The function \( f \) with \( f(\alpha) = X_\alpha \) is a function in \( B \) with name, say, \( \bar{f} \). Put now \( X = \bigcap_{\alpha < \mu} X_\alpha \). So \( C \models c \in X \) since \( f \) is a function of \( C \) with \( \bar{f} \upharpoonright B = f^B \). So \( X \in F \) and therefore \( F \) is \( \mu \)-descendingly complete. An easy induction now gives that \( \lambda_0 \) is measurable. QED.

We are now ready to characterize \( \text{MAX} \).

**Theorem 2.7:** Assume there are no uncountable measurable cardinals. Then a logic \( L \) satisfies \( \text{MAX} \) iff \( L \) satisfies \( \text{RMAX} \) iff \( L \) is \([\omega,\omega] \)-compact.

**Proof:** It suffices to show that \( L \) satisfies \( \text{MAX} \) if \( L \) is \([\omega,\omega] \)-compact, by theorem 2.3. So one direction is proposition 2.4 and the other direction is lemma 2.5 and 2.6. QED.

From the lemma 2.6 we get in fact more:

**Theorem 2.8:** Assume \( \omega \) is the only measurable cardinal: If \( L \) is \([\lambda,\lambda] \)-compact for some \( \lambda > \omega \), then \( L \) is \([\omega,\omega] \)-compact.

Let us end this section with a counter-example.

**Example 2.9:** Let \( \mu_0 \) be the first uncountable measurable cardinal and let \( \forall x \forall y (\phi(x),\psi(y)) \) be a binary quantifier (of type \( <1,1> \) ) which is defined by \( A \models \forall x \forall y (\phi(x),\psi(y)) \) if \( \{ a \in A : A \models \phi(a) \} \) is finite and \( \{ a \in A : A \models \psi(a) \} \) is of cardinality bigger than \( \mu_0 \).

Let \( L \) be the logic \( L_{\omega,\omega}(Q) \). Clearly \( L \) does not satisfy \( \text{RMAX} \), since there is an
expansion of \(<(2^{\aleph_0})^+, \varepsilon >\), in which \(<\omega, \varepsilon >\) is \(L\) - characterized. Now if \(A\) is an \(L\) - structure of cardinality \(\leq 2^{\aleph_0}\), \(Q\) is trivially false, so every \(L_{\omega, \varepsilon}\) - extension of \(A\) of cardinality less than \(2^{\aleph_0}\) is an \(L\) - extension. If \(\text{card}(A) > 2^{\aleph_0}\), let \(F\) be a \(\mu_0\) - complete ultra-filter on \(\mu_0\) and form \(\bigcup A/F = B\). Since \(\mu_0\) is small for \((2^{\aleph_0})^+\), \(B\) is a proper \(L\) extension of \(A\) where definable sets of cardinality \(\leq 2^{\aleph_0}\) are preserved. Since \(F\) is \(\mu_0\) - complete, finiteness is preserved, so \(B\) is an \(L\) - extension of \(A\), hence \(L\) satisfies \(\text{MAX}\) and is \([\mu_0, \mu_0]\) - compact.

With this we get:

**Theorem 2.10:** The following are equivalent:

(i) For every logic \(L\) \(\text{MAX}\) holds iff \(\text{RMAX}\) holds.

(ii) There exists an uncountable measurable cardinal.

**Proof:** Theorem 2.7 and the example above. QED.

2.3. The upward Lowenheim-Skolem theorems.

Let us first characterize the properties \(\text{ULS}\) and \(\text{RULS}\) for a logic \(L\).

**Theorem 2.11:** Let \(L\) be a logic. Then

(i) \(L\) satisfies \(\text{RULS}\) iff no infinite cardinal is cardinal-like characterizable in \(L\);

(ii) \(L\) satisfies \(\text{ULS}\) iff no infinite cardinal is strongly cardinal-like characterizable in \(L\).

**Proof:** (i) Clearly, if \(\kappa\) is cardinal-like characterizable in \(L\) via \(A\) then \(U^A\) has cardinality \(\kappa\) and so has \(U^B\) for every \(B\) which is an \(L\) - extension of \(A\). This contradicts \(\text{RULS}\). In the other direction let \(A\) be an \(L\) - structure with \(U^A\) contradicting \(\text{RULS}\), i.e. for every \(B\) with \(A \leq_{L(U)} B\), \(\text{card}(U^B) \leq \kappa\) for some cardinal \(\kappa\). Put \(S(A) = \{\mu \leq \kappa : \text{there is } B, A \leq_{L(U)} B\ \text{ and } \text{card}(U^B) = \mu\}\). If \(S(A)\) has a last element \(\kappa_0\), then \(\kappa_0\) is cardinal-like characterizable in \(L\) (in some structure \(C\) coding the situation). Otherwise let \(\kappa_1 = S(A)\). Then \(\kappa_1\) is cardinal-like characterizable in \(L\).
The details and the proof of (ii) are left to the reader. QED.

2.4 Some Examples.

Example 2.12: Let $Q_\kappa$ be the quantifier "there exist at least $\kappa$ many". Let $L$ be $L_{\omega,\omega}[Q_\kappa]$ with $\kappa > 2^\omega$. Since $\varphi$ is small for such a $\kappa$, Los' lemma holds for $L$ for any ultra-filter over $\omega$ (cf. also chapter 3, example 3.5) so $L$ is $[\omega,\omega]$-compact, hence satisfies, by theorem 2.7, RMAX. Obviously RULS, even ULS do not hold for $L$.

Example 2.13: Let $Q^{\exists x,y\varphi(x,y)}$ be the quantifier which, say, that $\varphi$ is a linear ordering of its domain of cofinality $\geq \kappa$, $\kappa$ a regular cardinal. Let $Q_\kappa: y(\varphi(x),\varphi(y))$ be the quantifier which holds in $A$ iff $A \models Q_\kappa(\varphi(x))$ and $A \models Q^{\exists x,y\varphi(x,y)}$ where $\mu_0$ is the first uncountable measurable cardinal. Put $L=L_{\omega,\omega}(Q)$. Obviously $L$ does not satisfy RULS. We want to show that $L$ satisfies ULS. Again, as in example 2.9, if $A$ is an $L$-structure of cardinality less than $2^{\omega_0}$ the quantifier is trivially false on $A$, hence every $B$ with $A \subseteq L_{\omega,\omega} B$ is an $L$-extension of $A$. So w.l.o.g. card$(A) > 2^{\omega_0}$. Let $\varphi$ be a $L$-formula in two variables $x,y$ and parameters from $A$, $\varphi(\varphi(x),\varphi(y))$. We define $cf(\varphi,A)$ to be the cofinality of the linear ordering defined by $\varphi$ in $A$. If $\varphi$ does not define a linear ordering we put $cf(\varphi,A)=0$. If $cf(\varphi,A)=\delta$, let $\{c(\varphi,A,\gamma) : \gamma < \delta\}$ be a witnessing cofinal sequence in $A$. Now let $F$ be a $\mu_0$-complete ultra-filter on $\mu_0$ and form $B=\prod A \upharpoonright F$.

Claim 1: If $cf(\varphi,A) < \mu_0$ or $cf(\varphi,A) > 2^{\omega_0}$ then $\delta=cf(\varphi,A)=cf(\varphi,B)$ and $\{c(\varphi,A,\gamma) : \gamma < \delta\}$ is cofinal in $\varphi$ in $B$.

To prove the claim we use that $\mu_0$ is small for each $\kappa > 2^{\omega_0}$ and that $F$ is $\mu_0$-complete.

Claim 2: If $cf(\varphi,A) < 2^{\omega_0}$ then $cf(\varphi,B) < 2^{\omega_0}$.

Again we use that $\mu_0$ is small for $\kappa > 2^{\omega_0}$. 
Claim 3: If \( cf(\varphi,A) = cf(\varphi,B) = \delta \) and \( \{c(\varphi,A,\gamma) : \gamma < \delta \} \) is cofinal for \( \varphi \) in \( B \) and \( C = \prod B/\sim \) then \( \{c(\varphi,A,\gamma) : \gamma < \delta \} \) is cofinal for \( \varphi \) in \( C \) and hence \( cf(\varphi,C) = \delta \).

Proof: Let \( f \in B_{\mu_0} \) and put \( X_\gamma \{i \in \mu_0 : B = \varphi(c(\varphi,A,\gamma), f(i)) \} \) and assume for contradiction that \( X_\gamma \in F \) for each \( \gamma < \delta \). Now \( B = \varphi(c(\varphi,A,\alpha), f(i)) \) for some \( \alpha < \delta \) since \( \{c(\varphi,A,\gamma) : \gamma < \delta \} \) was cofinal for \( \varphi \) in \( B \). Put now \( g(i) = c(\varphi,A,\alpha_i) \) where \( \alpha_i \) is the smallest index such that \( f(i) < c(\varphi,A,\alpha_i) \). This defines a function \( g \in A^{\mu_0} \) with \( \{i \in \mu_0 : B = \varphi(c(\varphi,A,\gamma), g(i)) \} = X_\gamma \). Since each \( X_\gamma \in F \) we conclude that \( \{c(\varphi,A,\gamma) : \gamma < \delta \} \) is not cofinal for \( \varphi \) in \( B \).

Claim 4: \( L \) does not satisfy \( ULS \).

Proof: Clearly we get from claim 1, that \( A <_L B \). Now let \( \alpha \) be any ordinal and put \( A_0 = A \), \( A_{i+1} = \bigcap A_i / F \) and \( A_\alpha = \bigcup A_\beta \) for \( i < \alpha \) and \( \delta \) limit, \( \delta < \alpha \). From claim 1-3 we get that \( \{A_\beta : \beta < \delta \} \) is an \( L \)-chain. Choosing \( \alpha \) big enough we prove that \( L \) satisfies \( ULS \).

Example 2.14: Now we put \( L = L_{\omega_1} [Q^{\varphi_1, \lambda}] \) for each \( \lambda \leq (\text{Beth}_{\omega_2})^+ = \kappa \) and \( \omega \) small for \( \lambda \), i.e. we add all these quantifiers simultaneously to \( L_{\omega,\omega} \). Clearly \( L \) is not \( [\omega_1, \omega_1] \)-compact. We put

\[ \Sigma = \{\varphi_1, \varphi_2\} \cup \{\psi_\lambda : \lambda < \text{Beth}_{\omega_2}\} \]

with

\( \varphi_1 \) says that " \( \varphi \) is a linear ordering",

\( \varphi_2 \) is \( -Q^{\varphi_1 \varphi_2} \text{xy} \varphi(x,y) \) and

\( \psi_\lambda \) is \( Q^{\varphi_1 \varphi_2} \text{xy} \varphi(x,y) \).

Using similar arguments as in example 2.13 we prove \( RULS \) for \( L \). The details are left to the reader.
2.5 The Equivalence of RULS and ULS.

Let us discuss here if RULS and ULS could possibly be equivalent, as this is the case for MAX and RMAX if there are no uncountable measurable cardinals (Theorem 2.7). For this we digress a bit to the theory of ultrafilters.

Look at the following assumption, where $\lambda$ is an infinite cardinal.

$A(\lambda)$: if $F$ is a uniform ultrafilter on $\lambda$, then $F$ is $\mu$-descendingly incomplete for every $\mu \leq \lambda$.

We denote by $A(\omega)$ the statement "for every $\lambda A(\lambda)$.

Jensen and Koppelberg [JK], Magidor [Mg] and Donder [Do] have studied this assumption. The following theorem resumes there results:

**Theorem 2.15:**

(i) (Jensen-Koppelberg) Assume $\neg \text{O}^\theta$. Then for every regular cardinal $\lambda$ we have $A(\lambda)$.

(ii) (Donner) Assume there is no inner model of ZFC with an uncountable measurable cardinal. Then $A(\omega)$.

(iii) If $A(\omega)$ holds then there are no uncountable measurable cardinals.

(iv) Assume there are supercompact cardinals. Then it is consistent with ZFC that $A(\omega_1)$ fails.

The assumption $A(\omega)$ is intimately connected with compactness properties, as we shall see in chapter 3.

**Theorem 2.16:** Assume $A(\omega)$ and $L$ is a logic, Then $L$ satisfies ULS iff $L$ satisfies RULS.

To prove theorem 2.16 we first state two lemmas:

**Lemma 2.17:** Let $\mathcal{L}$ be a logic which does not satisfy RULS. Then there is an $\mathcal{L}$-structure $M = \langle M, P^M, \ldots \rangle$ and a unary-predicate symbol $P$ such that for each $\mathcal{L}$-structure $N$ with $\mathcal{M} \models \mathcal{L} \iff \mathcal{N}$ we have $\text{card}(P^N) = \text{card}(P^N)$.
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Proof: Let $\mathcal{M}$ be an $L$-structure which is a counter example to ULS, i.e. $\mathcal{M} = \langle M, P^M, \ldots \rangle$ and $\{ \kappa \in \text{Card} : \text{card}(P_N) = \kappa \} = S_N$. 

is bounded. If $S_M$ has a maximal element, there is nothing to prove. So we can assume that $S_M = \{ \kappa \in \text{Card} : \kappa < \alpha \}$ for some limit $\alpha$. We then construct a structure $\mathcal{N} = \langle N, <, P, \ldots \rangle$ such that $\text{card}(\mathcal{N}) = \alpha$ by coding all $S_0$ in it. QED.

Lemma 2.18: Let $L$ be a logic which satisfies ULS', and $\mathcal{M}$ be an $L$-structure. Then there is an $L$-structure $\mathcal{N}$, $\mathcal{M} <_L \mathcal{N}$, and in $\mathcal{N}$ there is a sequence $\{ a_i : i < \text{card}(\mathcal{M})^+ \}$ of elements such that for each first-order formula $\varphi(x, y)$ in two variables and each $a_i < a_j$, $a_k < a_l$ we have that $\mathcal{N} \models \varphi(a_i, a_j)$ iff $\mathcal{N} \models \varphi(a_k, a_l)$.

Sketch of proof: Take $\mathcal{N}$ to be large enough so one can apply a Ramsey-type argument, as they are now standard in model theory, cf. [CK].

Proof of Theorem 2.16: Let $L$ a logic which satisfies ULS but not RULS. By the hypothesis we can apply both lemmas.

Fix $\mathcal{M}$ as in Lemma 2.17 and let $\mathcal{M}^\mathfrak{b}$ be the complete expansion of $\mathcal{M}$ and $\mu = \text{card}(\mathcal{M})^+$. Now let $\mathcal{N}$ and $\{ a_i : i < \alpha \}$ be as in Lemma 2.18 applied to $\mathcal{M}^\mathfrak{b}$. We define an ultrafilter $\mathcal{F}_0$ on $\mathcal{M}$. If $X \subseteq M$ let $R_X$ be the unary predicate symbol of $\mathcal{M}^\mathfrak{b}$ whose interpretation is $X$. Then we put $\mathcal{F}_0 = \{ X \subseteq M : \mathcal{N} \models R_X(a_0) \}$. Clearly $\mathcal{F}_0$ is an ultrafilter.

Let $A \in \mathcal{F}_0$ be a set of smallest cardinality and define an ultrafilter $F$ on $A$ by $F = F_0 / A$. $F$ is a uniform ultrafilter on $A$.

Claim 1: $\text{card}(A) > \text{card}(\text{card}(\mathcal{M}))$.

Assume $\text{card}(A) = \text{card}(\text{card}(\mathcal{M}))$. Then there is a 1-1 function $f : A \rightarrow \mathcal{M}$. Let $\mathcal{F}$ be the function symbol of $\mathcal{M}^\mathfrak{b}$ representing $f$ and $R_A$ the predicate symbol representing $A$. Since $A \in F$ the domain of $\mathcal{F}$ in $\mathcal{N}$ contains $a_0$ and hence all the $a_i$ for $i < \mu$. But since $\mathcal{M}^\mathfrak{b} <_L \mathcal{N}$, $F$ is 1-1 in $\mathcal{N}$, so we conclude that $\text{card}(\text{card}(\mathcal{N})) = \mu$. But by our assumptions (Lemma 2.16), $\text{card}(\text{card}(\mathcal{N})) = \text{card}(\text{card}(\mathcal{M}))$, a contradiction.

Now let $\kappa$ be the smallest cardinal such that $\text{card}(\mathcal{N}) < \kappa$. Put...
\( \beta = \text{card}(P^N) \).

**Claim 2:** \( \kappa < \text{card}(A) \).

Trivially \( \kappa \leq \text{card}(P^N) = \text{card}(P^M) < \text{card}(A) \) using claim 1.

Now we apply our hypothesis: \( A(\lambda) \) for \( \lambda = \text{card}(A) \) and conclude with claim 2 that

**Claim 3:** \( F \) is not \( \kappa \)-descendingly complete.

Let \( \{ A_i : i < \kappa \} \) be a decreasing sequence of elements of \( \mathcal P \) with \( \bigcap A_i = \emptyset \). Now, since \( \text{card}(P^M) < \text{card}(P^M) \), we can find \( c_i = \{ c_{ij} \in \kappa \} \) for each \( i < \beta \) with \( c_{ij} \in P^M \) and \( c_{ij} \neq c_{il} \) for each \( j \neq l \). We now define functions \( f_i : A \to P^M \) for \( i < \beta \) by \( f_i(a) = c_{ij} \) where \( j \) is the least index such that \( a \in A_j \). This is well defined, since \( \bigcap A_i = \emptyset \). Let \( F_i \) be the function symbol representing \( f_i \) in \( M^i \). By the choice of the \( f_i \) \( M^i \models \forall a (F_i(a) \neq F_j(a)) \) for \( i < j \). Since the domain of \( F_i \) is \( A \) we have \( N \models P(F_i(a_0)) \) and \( N \models F_i(a_0) \neq F_j(a_1) \) for \( i < j \). So \( \beta \leq \text{card}(P^N) \), a contradiction. QED.

Figure 1 gives a synopsis of the situation.
Theorem 2.16, Example 2.12

Theorem 2.7 assuming there are no uncountable measurable cardinals

Theorem 2.3, Example 2.9

Theorem 2.8 assuming there are no uncountable measurable cardinals.

Drawn lines show implications which hold, broken lines such which do not hold.
3. Compactness and ultrafilters.

3.1 Introduction

In this section we present a general theory of compactness of logics $L$, centering around the notion of $[\lambda,\mu]$-compactness and relating it to $(\lambda,\mu)$-regular ultrafilters. This section shows clearly that $[\lambda,\mu]$-compactness is a natural notion, in fact much more natural than $(\lambda,\mu)$-compactness. We reprove, with different methods, most of the results of [MS1, section 6] and improve some of them. For non-defined notions on ultrafilters we refer the reader to the monograph [CN].

In section 3.2 we collect some facts about ultrafilters. In section 3.3 we prove our main theorem of this chapter. And in section 3.4 we apply this to study $[\lambda,\mu]$-compactness.

3.2 $(\lambda,\mu)$-regular ultrafilters.

The following notion was introduced by [Ke1], but his notation is reversed. (The reader should think of the brackets as unordered pairs.)

**Definition:** Let $F$ be an ultrafilter on $\mathcal{L}$, and $\lambda,\mu$ be regular cardinals with $\lambda \geq \mu$.

(i) $F$ is said to be $(\lambda,\mu)$-regular if there is a family $\{X_\alpha : \alpha < \lambda \}$ such that if $\{\alpha_i : i < \mu \}$ is any enumeration of subsets of $\lambda$ of cardinality $\mu$, then $\bigcap_{\alpha_i} X_\alpha = \emptyset$.

(ii) A $(\lambda,\omega)$-regular ultrafilter on $\lambda$ is called regular.

(iii) $F$ is $\lambda$-descendingly incomplete if there exists a family $\{X_\alpha : \alpha < \lambda \}$ such that $X_\alpha \subset X_\beta$ for $\alpha \prec \beta \prec \lambda$ such that $\bigcap X_\alpha = \emptyset$.

(iv) $F$ is uniform on $\lambda$ if every $X \in F$ has cardinality $\lambda$. 
Lemma 3.1:
(i) If $F$ is $(\lambda, \mu)$-regular and $\mu \leq \mu_1 \leq \lambda$, then $F$ is $(\lambda_1, \mu)$-regular.
(ii) If $\lambda$ is a regular cardinal and $F$ is $\lambda$-descendingly incomplete, then $F$ is $(\lambda, \lambda)$-regular.
(iii) If $F$ is uniform on $\lambda$, then $F$ is $(\lambda, \lambda)$-regular.
(iv) If $F$ is $(\lambda, \lambda)$-regular, then $F$ is $(\lambda, \lambda)$-regular.

Proof: (i) is obvious.
(ii): Let $\{X_\alpha: \alpha < \lambda\}$ be a decreasing family with empty intersection. Since $\lambda$ is regular, $\{X_\alpha: \alpha < \lambda\}$ is a $(\lambda, \lambda)$-regular family.
(iii) follows from the fact that $F$ is uniform on $\lambda$ if for every $X \subseteq \lambda$ such that $\text{card}(\lambda - X) < \lambda$ we have that $X \in F$, cf. [CK, Exercise 4.3.2].
(iv): Let $\{X_\alpha: \alpha < \text{cf}(\lambda)\}$ be a $(\text{cf}(\lambda), \text{cf}(\lambda))$-regular family and $\{\beta_i: i < \text{cf}(\lambda)\}$ cofinal in $\lambda$. Put now $Y_\xi = X_{\beta_\xi}$ for every $\xi$ with $\beta_\xi \leq \xi < \beta_{\xi+1}$. Then $\{Y_\xi: \xi < \lambda\}$ is a $(\lambda, \lambda)$-regular family. \(\text{QED}\).

This is essentially lemma 7.10 in [CN] and due to [Ko1]. The next lemma is from [Ko, Corollary 2.4], and [CN, Corollary 8.36].

Lemma 3.2:
(i) [Ko] If $F$ is uniform on $\lambda^+$ and $\lambda$ is singular, then $F$ is $(\lambda^+, \lambda)$-regular.
(ii) [Pr,CC] If $F$ is uniform on $\lambda^+$ and $\lambda$ is regular, then $F$ is $\lambda$-descendingly incomplete, and hence $(\lambda, \lambda)$-regular.

We now proceed to give a model theoretic characterization of $(\lambda, \mu)$-regular ultrafilters. Let $\mathcal{H}(\lambda)$ denote the set of sets hereditarily of cardinality $\leq \lambda$, and $\in$ the natural membership relation on $\mathcal{H}(\lambda)$.

Lemma 3.3: For an ultrafilter $F$ on a set $I$ the following are equivalent:
(i) $F$ is $(\lambda, \mu)$-regular
(ii) In the structure
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\[ N = \prod_I H(\lambda^+), \varepsilon \vDash /F \]

there is an element \( a = a/I \) where \( a:I \to H(\lambda^+) \) such that

\[ N \models a \in \lambda^N \& \text{card}(a) < \mu^N \]

but for every \( \alpha < \lambda \)

\[ N \models \alpha^N \vDash a^N \]

**Proof:** (i) \( \to \) (ii): Define \( a:I \to H(\lambda^+) \) by \( a(t) = \{ \alpha \in \lambda : t = X_{a(t)} \} \) for \( t \in I \) and \( \{ X_{a(t)} : \alpha < \lambda \} \) a \( (\lambda, \mu) \)-regular family. Now \( X_{a(t)} = \{ t : \alpha \in a(t) \} \), so \( N \models \alpha \vDash a \), since for each \( \alpha < \lambda \), \( X_{a(t)} \in F \). But \( a(t) \) has cardinality less than \( \mu \) for each \( t \in I \), since \( \{ X_{a(t)} : \alpha < \lambda \} \) is \( (\lambda, \mu) \)-regular, so \( N \models \text{card}(a) < \mu \). Trivially, \( N \vDash a \in \lambda \).

(ii) \( \to \) (i): Let \( a = a/I \) be the required element in \( N \). Define \( a' \) by:

\[ a'(t) = a(t) \text{ if } a(t) \in P_{=\mu}(\lambda) \text{ and } \phi \text{ otherwise.} \]

Obviously \( a/F = a'/F \) since \( N \vDash a \in \lambda \) and \( N \vDash \text{card}(a) < \mu \). We want to construct a \( (\lambda, \mu) \)-regular family. We put \( X_{a(t)} = \{ t : \alpha \in a'(t) \} \) for each \( \alpha < \lambda \). Now suppose that for some \( \{ \alpha_i : i < \mu \} \), \( \cap X_{a_i} \neq \emptyset \). So there is \( t \in I \) such that for each \( i < \mu \), \( \alpha_i \in a'(t) \), which contradicts the fact that \( a'(t) \notin P_{<\mu}(\lambda) \). QED.

**Definition:** Let \( F_i \) be ultrafilters on \( I_i \) (\( i = 1, 2 \)). \( F_2 \) is a projection of \( F_1 \) if there is a map \( f : I_1 \to I_2 \) which is onto and such that \( F_2 = \{ X \subseteq I_2 : f^{-1}(X) \in F_1 \} \). This is also known as the Rudin-Keisler order on ultrafilters.

**Lemma 3.4:** If \( \lambda \) is regular and \( F \) is an \( (\lambda, \lambda) \)-regular ultrafilter on \( I \) then there is a uniform ultrafilter \( \mathcal{F} \) on \( \lambda \) which is a projection of \( F \).

**Proof:** We use lemma 3.3 for \( \lambda \neq \mu \). Let \( \mathcal{N} \) be the complete expansion of \( \mathcal{N} = \prod_I H(\lambda^+) \vDash /F \) and \( a : I \to H(\lambda^+) \) as in lemma 3.3. Put now \( \mathcal{N} \) to be the supremum of the set \( a(t) \), so \( b(t) \in \lambda \) since \( \lambda \) is regular and \( \text{card}(a(t)) < \lambda \); moreover \( N \vDash a \in \mathcal{N} \). Thus \( b : I \to \lambda \) is well defined. We now define \( \mathcal{F} = \{ S \subseteq \lambda : N \vDash \bar{b} \in S \} \) where \( \bar{S} \) is the name of \( S \) in \( \mathcal{N} \). Clearly \( \mathcal{F} \) is an ultrafilter on \( \lambda \).
Claim 1: $F$ is uniform.

Assume for contradiction $S \in F$, but $\operatorname{card}(S) \leq \lambda$. So $S$ is bounded by some $\alpha_S \in \lambda$ and for every $\beta$ in $N$ such that $N^\beta_\alpha \models \beta > \alpha_S$ we have $N^\beta_\alpha \models \neg (\beta \in S)$. But $N^\beta_\alpha \models \alpha \leq \delta$, hence $N \models \neg (\beta \in S)$ a contradiction.

Claim 2: $F$ is projection of $F_1$ by $b : I \to \lambda$.

Clearly $S \in F$ iff $N \models \delta \in S$ iff $\{ t \in I : b(t) \in S \} \in F$ iff $b^{-1}(S) \in F$. QED.

Remark: In lemma 3.3 we can find $F$ on $\lambda$ also if $\mu \leq \sigma f(\lambda)$, like in the proof of 3.4, but $F$ need not be uniform. The function $\alpha$ in lemma 3.3 is from [Ke1].

### 3.3 Ultrafilters and compactness.

The key definition of this section is motivated by lemma 3.3.

**Definition:**

(i) Let $L$ be a logic and $F$ be an ultrafilter over $I$. We say that $F$ relates to $L$ if for every $\tau$ and for every $\tau$-structure $A$, there exists a $\tau$-structure $B$ extending $\prod A/F$ such that for every formula $\varphi \in L[\tau]$, $\varphi = \varphi(x_1, x_2, \ldots, x_i, \ldots), i < \alpha$ and every $f_i \in A_i, i < \alpha$ we have:

$$B \models \varphi(f_1/F, f_2/F, \ldots, f_i/F, \ldots)$$

iff

$$\{ j \in I : A \models \varphi(f_1(j), f_2(j), \ldots, f_i(j), \ldots) \} \in F.$$ 

(ii) We define $\text{UF}(L)$ to be the class of ultrafilters $F$ which are related to $L$.

**Remark:** Note that $B$ is always an elementary extension of $\prod A/F$.

Before we continue let us look at some examples:

**Examples 3.5:**

(i) Every ultrafilter is in $\text{UF}(L_\omega)$.  

(ii) Let $L$ be $L_{\omega, \omega}(Q)$, i.e. first order logic with the additional quantifier "there exist at least $\psi$ many ". Then every ultrafilter on $\omega$ is related to $L$, provided $\omega$ is
small for $\kappa$. This follows from the ultrapower theorem for $L$ as stated in [BS, chapter 13, theorem 2.2].

In the above examples $N = \prod_{\mathcal{F}} M^f / \mathcal{F}$. In general, as we see in the proof of the theorem below, this is not the case.

**Proposition 3.6:** $L$ is compact iff every ultrafilter is related to $L$.

**Proof:** Let $M$ be an $L$-structure and $F$ an ultrafilter on a set $I$. For every $f \in M^I$ let $c_f$ be a new constant symbol not in $L$. Put

$$T = \{ \phi(c_{f_1}, c_{f_2}, \ldots) : \phi \in L(L) \land \{ t \in I : M[t] = \phi(f_1(t), f_2(t), \ldots) \} \in F \}$$

Obviously every finite subset of $T$ has a model. We just expand $M$ appropriately. So let $N$ be a model of $T$. Clearly

$$\prod_{\mathcal{F}} M^f / \mathcal{F} \subseteq N$$

and by the definition of $T$, $N$ satisfies the requirements for $F \in \text{UP}(L)$. The converse is trivial if we code ultraproducts in ultrapowers of modified structures. QED.

**Example 3.7:** There are many compact logics, the simplest being the logics with the cofinality quantifiers, cf. [MS2] and [Sh1].

**Remark:** If all ultrafilters are in $\text{UP}(L)$ and $N = \prod A / \mathcal{F}$ for every $F \in \text{UP}(L)$ then $L = L_{\omega, \omega}$, since, by the Keisler-Selah ultrapower theorem (cf. [CK]) elementarily equivalent structures (in $L_{\omega, \omega}$) have isomorphic ultrapowers, and hence by the isomorphism axiom ($L$ does not distinguish between isomorphic structures) and proposition 3.6 the claim follows easily.

**Lemma 3.8:** $\text{UP}(L)$ is closed under projections.

**Proof:** Let $f : I \rightarrow F$ be the projection of $F$ onto $F$. Choose $g$ be such that $fg$ is the identity on $f$. $g$ induces a map $g^*$ from $A^f$ into $A_I$ such that $g^*(h') = h$ for $hg = h'$ and $g^*$ is compatible with filters, since $f$ is a projection. QED.
We are now in position to state and prove our main theorem, the Abstract Compactness Theorem.

**Theorem 3.9:** (Abstract Compactness Theorem)

Let $L$ be a logic, $\lambda, \mu$ be cardinals and $\lambda \geq \mu$. The following are equivalent:

(i) There is $(\lambda, \mu)$-regular ultrafilter $F$ on $\mathcal{I} = \lambda^\mu$ which is in $\text{UF}(L)$.

(ii) For every expansion $A$ of $\text{H}(\lambda^+)$: there is an $L$-extension $B$ and an element $b \in B$ such that $B|\text{card}(b) < \mu^\beta$ but for every $\alpha < \lambda$ we have $B|\alpha^\beta \notin b$.

(iii) $L$ is $[\lambda, \mu]$-compact.

If $\lambda$ is regular then the following are equivalent:

(iv) There is a uniform ultrafilter $\mathcal{F}$ on $\lambda$ which is in $\text{UF}(L)$.

Furthermore we have: (vi) If there is a $(\lambda, \mu)$-regular ultrafilter $F$ on any set $I$ which is in $\text{UF}(L)$ then $L$ is $[\lambda, \mu]$-compact.

(v) $L$ is $[\lambda, \lambda]$-compact.

**Proof:**

(i) $\Rightarrow$ (ii): Let $F$ be a $(\lambda, \mu)$-regular ultrafilter in $\text{UF}(L)$ and let $M$ be any expansion of $\text{H}(\lambda^+)$, $\in >$. Put $N_0$ to be the ultrapower $\prod_{\mathcal{F}} M/F$ and $N_1$ the extension of $N_0$ as required for $F \in \text{UF}(L)$. First we observe that $\mathcal{N}_0 \prec N_1 (L_{\mu, \omega})$ and, by lemma 3.3, there is an element $\alpha$ in $N_0$ with the required properties. But then the same element $\alpha$ has the same properties also in $N_1$ since $\mathcal{N}_0 \prec N_1 (L_{\mu, \omega})$. But by the definition of $\mathcal{N}_1$, $\mathcal{M} \prec \mathcal{N}_1 (L)$, so we are done.

(ii) $\Rightarrow$ (iii): Let $\Sigma, \Sigma_1$ be $L$-sentences satisfying the hypothesis of $[\lambda, \mu]$-compactness. We define an expansion $\mathcal{M}(\Sigma, \Sigma_1)$ of $\text{H}(\lambda^+)$, $\in > \mathcal{L}_0$ apply (ii). For this purpose let $\{S_\alpha, \alpha < \lambda^\mu\}$ be an enumeration of all the subsets of $\Sigma_1$ of cardinality less than $\mu$, $A_\alpha$ be a model of $\Sigma \cup S_\alpha$ and $\{c_\alpha, \alpha < \lambda^\mu\}$ an enumeration of all the subsets of $\lambda$ of cardinality less than $\mu$. Finally we put $\nu = \text{sup}(\text{card}(A_\alpha)) + \lambda^+$, and define $\lambda_\alpha = \text{card}(A_\alpha)$.

We now define $\mathcal{M}(\Sigma, \Sigma_0)$ to be $\text{H}(\nu) \mid d_\alpha, E, R, P >_{\alpha < \lambda^+, P \in \mathcal{L}}$ such that $d_\alpha$ is the name of $\alpha < \lambda^+$ and $E$ is membership, $R$ is a binary predicate not in $L$ and the range of
$R$ is $\lambda$. We arrange it such that for each $\alpha < \lambda$ the set $R_\alpha = \{ z \in H(\nu) : (\alpha, z) \in R \}$ has cardinality $\lambda_\alpha$ and such that $\langle R_\alpha, P \rangle_{P \in L^\omega \Delta_\alpha}$. In other words put all the models $A_\alpha$ into $\mathbb{M}(\Sigma, \Sigma_1)$ in way, that when we now apply (ii) we shall get a model for $\Sigma \cup \Sigma_1$. More precisely, we observe that for each formula $\varphi \in \Sigma$:

(a) $M(\Sigma, \Sigma_1) \models \text{card}(c) < d_\mu \rightarrow \varphi^F$ 

and for each $\beta < \lambda$ and for $\Sigma_1 = \{ \varphi_i : i < \lambda \}$ an enumeration of $\Sigma_1$ we have

(b) $M(\Sigma, \Sigma_1) \models (d_\beta E c \land \text{card}(c) < d_\mu) \rightarrow \varphi^{F \beta}$. 

Now let $N, \alpha \in N$ be as in the conclusion of (ii) $\prod_{M(\Sigma, \Sigma_1)} / F$. 

Claim: $\langle R_\alpha, P \rangle_{P \in L \models \Sigma \cup \Sigma_1}$.

This follows from the definition of (a) and (b).

(iii) $\rightarrow$ (i): So assume $L$ is $[\lambda, \mu]$-compact but no $[\lambda, \mu]$-regular ultrafilter on $\lambda^\mu$ $F$ is related to $L$. So for every such $\varphi$ there is a $L^\varphi$-structure $A_\varphi$ exemplifying this.

We now proceed to construct an ultrafilter $F_0$ on $\lambda^\mu$ which contradicts the choice of the $A_\varphi$'s. For this we construct first a rich enough structure $M$ such that

(1) For each $A_\varphi$ there is a unary predicate $P_\varphi$ in $M$ with $\langle P_\varphi, P \rangle_{P \in L^\omega A_\varphi}$.

(2) $M$ is a model of enough set theory to carry out the argument and 

(3) $M$ is an expansion of $\langle H(\lambda^+), \in \rangle$.

Let $\mathbb{M}^\#$ be the complete expansion of $M$ and put $\Sigma =\text{Th}_{L^\varphi}(\mathbb{M}^\#)$, the first order theory of $\mathbb{M}^\#$ where $L^\#$ is the vocabulary of $\mathbb{M}^\#$. Furthermore put $\Sigma_1 = \{ c \in d_\lambda \land \text{card}(c) < d_\mu \land d_\alpha \in c : \alpha < \lambda \}$. Clearly $\Sigma$ and $\Sigma_1$ satisfy the hypothesis of $[\lambda, \mu]$-compactness using the model $\mathbb{M}^\#$. So $\Sigma \cup \Sigma_1$ has a model $N$. We want to use $N$ to construct our filter $F_0$. First we observe that $\mathbb{M}^\# \subseteq N$. Let $a_\varphi$ be the interpretation of $c$ in $N$. We define $F_0$ on $P_{<\mu}(\lambda)$ by $F_0 = \{ R \in P_{<\mu}(\lambda) : N \models a_\varphi \in R^{\langle N \rangle} \}$. This makes sense, since $\mathbb{M}^\#$ is a complete expansion and hence every subset of $\lambda$ of cardinality $< \mu$ corresponds to a predicate in $\mathbb{M}^\#$ (remember $\langle H(\lambda^+), \in \rangle$ is present in $\mathbb{M}^\#$).
To complete the proof we have to verify several claims:

**Claim 1**: \( F_0 \) is ultrafilter.

Obvious.

**Claim 2**: \( F_0 \) is \((\lambda, \mu)\)-regular.

Let \( X_a = \{ t \in P_{\nu_0}(\lambda) : \alpha \in \ell \} \) for \( \alpha < \lambda \). Now \( X_a \in F_0 \) if and only if \( X_a \) corresponds to \( R_a \) then \( N = a_c \in R_a \iff N = d_a \in a_c \), which is true for all \( \alpha < \lambda \) by the definition of \( a_c \).

Now let \( \{ X_{a_i} : i < \mu \} \) be a subfamily of the \( X_a \)'s. Clearly, \( \bigcap_{i<\mu} X_{a_i} \neq \emptyset \), since each \( t \) in some \( X_a \) has cardinality \( < \mu \).

Now consider the product \( \prod M^\ell / F_0 = N_0 \). If \( g \) is an element of \( N_0 \) then \( g \) is a map \( g : P_{\nu_0}(\lambda) \to M^\ell \).

so \( g \) corresponds to a function \( g^N \) in \( M^\ell \) with name \( g \), (since \( M^\ell \) is the complete expansion) and \( a_c \in \text{Dom}(\nu^N) \). So we define an embedding \( f : N_0 \to N \) by \( g / F_0 \to g^N(a_c) \).

**Claim 3**: \( f \) is well-defined and 1-1.

Let \( g / F_0 = g' / F_0 \). We want to show that this is equivalent to \( N = g(\alpha) = g'(\alpha) \), iff \( Y = \{ t \in P_{\nu_0} : g(t) = g'(t) \} \subseteq F_0 \). But the latter is true iff \( a_c \in \nu^N \) which is equivalent to \( g(\alpha) = g'(\alpha) \).

So we have shown that \( f \) is an embedding of \( N_0 \) into \( N \). Now let \( \bar{g} = \{ g / F_0 : t \in \alpha \} \) be in \( N_0 \).

**Claim 4**: For every \( L \)-formula \( \phi \), we have \( N = \phi(\bar{g}) \) iff:

\[ Y = \{ t \in P_{\nu_0} M : \phi(g_1(t), g_2(t), \ldots) \} \subseteq F_0 \]

Now \( Y \in F_0 \) iff \( \nu^N \) contains \( a_c \) iff \( N = \phi(g_1(a_c), g_2(a_c), \ldots) \).

Now look at \( A_{\nu_0} \). By assumption there is no \( N' \) extending \( \prod A_{\nu_0} / F_0 \) satisfying claim 4. But \( <P_{\nu_0}, P>_{R \in L_{\nu_0}} \) is such an \( N' \) by construction.

(iv) \( \rightarrow \) (v): This follows from the previous.

(v) \( \rightarrow \) (iv): The proof is similar to (iii) \( \rightarrow \) (i), but instead of \( F_0 \), we construct \( F_1 \) on \( \lambda \) by
Positive Results

$\mathcal{E}_0 = \{R \subset \lambda: R \text{ is an initial segment of } \lambda \text{ and } N \models \forall \alpha \in R^N \}$. To get $\mathcal{F}_1$ uniform we use lemma 3.4.

(vi) For this we just observe that in the proof of (i) $\rightarrow$ (ii) we did not use the special form of the set $I$, as lemma 3.3 does not depend on it.

This proves theorem 3.9. QED.

3.4 The Compactness Spectrum.

Let $\text{INC}(L) = \{ \lambda \in \text{card} \text{.} \langle L \text{ is not } [\lambda, \lambda] \text{-compact} \}$ and $\text{RCOMP}(L) = \text{reg} \cdot \text{INC}(L)$, where $\text{reg}$ is the class of regular cardinals.

**Theorem 3.10:** $\text{INC}(L)$ is closed under successor ($\lambda \in \text{INC}(L) \rightarrow \lambda^+ \in \text{INC}(L)$) and cofinality ($\lambda \in \text{INC}(L) \rightarrow \text{cf} (\lambda) \in \text{INC}(L)$).

**Proof:** Successor: Assume for contradiction $L$ is $[\lambda, \lambda]$-compact. So by 3.9(v) there is uniform ultrafilter $\mathcal{F}$ on $\lambda^+$ with $\mathcal{F} \in \text{UF}(L)$. Now we have two cases:

- $\lambda$ regular: By 3.2(ii) $\mathcal{F}$ is $\langle \lambda, \lambda \rangle$-regular, hence by 3.9 $L$ is $[\lambda, \lambda]$-compact.
- $\lambda$ singular: By lemma 3.2.1(i) $\mathcal{F}$ is $\langle \lambda^+, \lambda \rangle$-regular and by lemma 3.1(i) $\langle \lambda, \lambda \rangle$-regular. So we can apply 3.9(vi).

Cofinality: This is lemma 3.1 (iv). QED.

**Theorem 3.11:**

(i) $L$ is $[\lambda, \lambda]$-compact for each regular $\lambda \in [\mu_1, \omega]$ iff $L$ is $[\mu_1, \omega]$-compact.

(ii) $L$ is compact iff $L$ is $[\lambda, \lambda]$-compact for each regular $\lambda$.

(iii) Assume there are no uncountable-measurable cardinals. Then $L$ is $[\lambda, \lambda]$-compact for each regular $\lambda \in [\mu_1, \mu_2]$ iff $L$ is $[\mu_1, \omega]$-compact.

**Proof:** Apply lemma 3.1 and theorem 3.9 for (i) and (ii). To get (iii) use (i) together with theorem 2.6. QED.

The last result which we would like to state here concerns the structure of $\text{RCOMP}(L)$. The following was proven in [MS1, lemma 6.4]:

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Lemma 3.12: Let $\lambda > \mu$ be two regular cardinals and $L$ be a logic such that $\lambda \in \text{RCOMP}(L)$ but $\mu \notin \text{RCOMP}(L)$. Then there is a $\mu$-descendingly complete ultrafilter on $\lambda$.

From this we obtain immediately:

Theorem 3.13: Assume $A(\omega)$ holds (cf. section 2.5). Then $\text{RCOMP}(L)$ is an initial segment of the regular cardinals, i.e. $\lambda \in \text{RCOMP}(L)$ and $\mu < \lambda$ implies that $\mu \in \text{RCOMP}(L)$.

Note that the hypothesis $A(\omega)$ implies that there are no uncountable measurable cardinals (cf. Theorem 2.15(iii)).
4. The Occurrence Number.

4.1 Introduction.

This chapter is entirely devoted to the notion of the occurrence number of a
logic $L$. In section 4.2 we give the precise definitions and review previous
results. We also state the main theorem (4.3), the Finite Occurrence Theorem.
In section 4.3 we prove a sequence of lemmas (4.5 - 4.9) to establish a connection
between the existence of certain ultrafilters and the occurrence number.
Section 4.4 is purely set theoretic, establishing some new results about certain
ultrafilters and measurable cardinals. Finally, in section 4.5 we show that our
results are best possible. We study there an example in detail.

It should be noticed that there is a side theme evolving here: The role of the
existence of uncountable measurable cardinals in abstract model theory. We
have seen this already in section 2.2 and 3.4 and we shall see it again in section
5.2.

4.2 The Occurrence Number: Definition and Main Theorem.

In his unpublished notes Friedman [Fr] proved the following easy, but
fundamental theorem (cf. [F1]).

Theorem 4.1: Let $L$ be a logic which is compact. Then each $L(L)$-sentence $\varphi$
depends only on finitely many non-logical symbols of $L$.

Let us define the notions involved in the theorem precisely: Let $L$ be a logic, $L$ a
vocabulary and $\varphi$ an $L(L)$-sentence. Let $L_0 \subseteq L$. We say that $\varphi$ depends on $L_0$
only if for any two $L$-structures $A$ and $B$ such that $A \models L_0 \models B \models L_0$ we have $A \models \varphi$ iff
$B \models \varphi$.

We say that $\varphi$ depends on $L_0$ (properly) iff there are $L$-structures $A, B$ such that
$A \models L - L_0 \models B \models L - L_0$ but $A \models \varphi$ and $B \models \neg \varphi$. 
We define the occurrence number $OC(L)$ of a logic $L$ to be the smallest cardinal $\mu$ such that for every vocabulary $L$ (which is a set) and every $L(L)$-sentence $\varphi$ there is a subvocabulary $L_0 \subset L$ of cardinality less than $\mu$ such that $\varphi$ depends on $L_0$ only.

The occurrence number $OC(L)$ need not exist. In this case we stipulate $OC(L) = \infty$, e.g. $L_{\omega, \omega}$ has no occurrence number. The next example illustrates an other point:

Example 4.1: Let $\kappa$ be a cardinal and $\mathcal{F}$ an ultrafilter on $\kappa$. We define a logic $L_{\mathcal{F}, \kappa}$ by adding to first order logic $L_{\omega, \omega}$ the following formation rule: If $\{\varphi_i : i < \kappa\}$ are sentences of $L_{\mathcal{F}, \kappa}$ so is $\Lambda_{\mathcal{F}} \{\varphi_i : i < \kappa\}$ and for a structure $A$ we define $A \models L_{\mathcal{F}, \kappa} \Lambda_{\mathcal{F}} \{\varphi_i : i < \kappa\}$ to hold iff $\{i \in \kappa : A \models \varphi_i\} \in \mathcal{F}$.

Obviously, $OC(L_{\mathcal{F}, \kappa}) \leq \kappa^+$, but if $L_0 \subset L, \text{card}(L) = \kappa, \text{card}(L_0) < \kappa$ and we change in an $L$-structure $A$ all the interpretation of symbols in $L_0$, then validity in $A$ for $L_{\mathcal{F}, \kappa}$ is not changed. In other words a $L_{\mathcal{F}, \kappa}(L)$-sentence $\varphi$, depends on $L - L_0$ only, but $\varphi$ may be chosen such that whenever $\varphi$ depends on $L_1 \subset L$ only, then $\text{card}(L_1) = \kappa$.

Hence the occurrence number is $\kappa^+$. We conclude that in $L_{\mathcal{F}, \kappa}$ there is no semantical version of the occurrence axiom, i.e. given a $L_{\mathcal{F}, \kappa}(L)$-sentence $\varphi$, it is not necessarily true that there is a smallest $L_0 \subset L$ such that $\varphi$ depends on $L_0$ only. Nevertheless, if we are concerned with "physical" occurrence such an $L_0$ does exist.

Here is a generalization of theorem 4.1. In section 4.5 we shall see that theorem 4.2 is best possible.

Theorem 4.2: If $L$ is $(\lambda, \kappa)$-compact and $OC(L) \leq \lambda^+$ then $OC(L) \leq \kappa$.

Proof: Let $L$ be a vocabulary, $\kappa \leq \text{card}(L) \leq \lambda$ and $\varphi$ be a $L(L)$-sentence. Let $L_1$ and $L_2$ be two disjoint copies of $L$, and denote for each constant symbol $c$, function symbol $f$, and relation symbol $R$ in $L$ the corresponding symbols by $c_1, f_1, R_1, (i = 1, 2)$ respectively. Put $L_3 = L_1 \cup L_2$. Assume for contradiction that $\varphi$
depends on at least $\kappa$ many symbols of $L$. Then for every $L_0 \subset L$ with $\text{card}(L_0) < \kappa$, there are $L$-structures $A, B$ such that

1. $A \models \varphi$, $B \models \neg \varphi$ and $A \force_\varphi B \models L_0$.

Look at the following set of sentences:

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{\varphi_1, \neg \varphi_2\}$$

with

$$\Sigma_1 = \{c_1 = c_2 : c \text{ a constant symbol of } L\}$$

$$\Sigma_2 = \{\forall \bar{x} (f_1(\bar{x}) = f_2(\bar{x})) : f \text{ a function symbol of } L\}$$

$$\Sigma_3 = \{\forall \bar{x} (R_1(\bar{x}) \leftrightarrow R_2(\bar{x})) : R \text{ a relation symbol of } L\}$$

and $\varphi_1$ is the $L(L_0)$-sentence obtained from $\varphi$ by substituting $L_0$ for $L$.

Clearly, $\text{card} (\Sigma) \leq \lambda$. Now let $\Sigma_0 \subset \Sigma$, $\text{card}(\Sigma_0) < \kappa$. By (1) $\Sigma_0$ has a model. But any model of $\Sigma$ would violate the isomorphism axiom. QED.

Our main theorem in this section is the Finite Occurrence Theorem:

**Theorem 4.3** (Finite Occurrence Theorem)

Assume a logic $L$ is $[\omega, \omega]$-compact and $\text{OC}(L) \leq \mu_0$, where $\mu_0$ is the first uncountable measurable cardinal. (or $\omega$ if no such cardinal exists). Then $\text{OC}(L) = \omega$.

As an immediate corollary from Theorem 2.3 we get

**Corollary 4.4**: Assume a logic $L$ is $[\lambda, \lambda]$-compact for $\lambda < \mu_0$ and $\text{OC}(L) \leq \mu_0$. Then $\text{OC}(L) = \omega$.

The proof of theorem 4.3 is rather involved. We need a series of lemmas manipulating the occurrence number and a main lemma on the existence of measurable cardinals.
4.3 The Lemmas:

Let us fix a \([\lambda, \lambda]\) compact logic \(L\), a vocabulary \(L\) and a sentence \(\varphi \in L(L)\). We want to study subsets of \(L\) on which \(\varphi\) does not depend.

**Lemma 4.5:**

(i) For every \(L_1 \subset L\) with \(\text{card}(L_1) \leq \lambda\) there is a \(L_0 \subset L_1\) with \(\text{card}(L_0) < \lambda\) such that \(\varphi\) does not depend on \(L_1 - L_0\).

(ii) There is a \(\mu < \lambda\) such that for every \(L_1 \subset L\) with \(\text{card}(L_1) \leq \lambda\) there is a \(L_0 \subset L_1\) with \(\text{card}(L_0) < \mu\) such that \(\varphi\) does not depend on \(L_1 - L_0\).

**Proof:**

(i) For \(\text{card}(L) = \lambda\) this follows from theorem 4.2, otherwise put \(\Sigma_{L_1}\) to be \(\Sigma\) of the proof of theorem 4.2 and

\[
\Sigma_{L_1}=\Sigma_{L_1} \cup \Sigma_2 \cup \Sigma_3
\]

for \(L_1 - L_1\) instead of \(L_1\). Now use \([\lambda, \lambda]\)-compactness.

(ii) If \(\lambda\) is a successor this is equivalent to (i). So let \(\lambda\) be a limit cardinal. Assume for contradiction that for every \(\mu < \lambda\) there is \(L^\mu \subset L\) with \(\text{card}(L^\mu) \leq \lambda\) making the lemma false. Put \(L_1 = \bigcup L^\mu\), so \(\text{card}(L_1) \leq \lambda\). Now by the previous lemma there is \(L_1 \subset L_1\) with \(\text{card}(L_1) < \lambda\) such that \(\varphi\) does not depend on \(L_1 - L_0\).

But then for every \(\mu < \lambda\) \(\varphi\) does not depend on \(L^\mu-(L_0 \cap L^\mu)\) by the definition of \(L_1\). But this contradicts the definition of \(L^\mu\). QED.

The next lemma is one of the three lemmas used in the proof of Theorem 4.3.

**Lemma 4.6:** There is a \(L \subset L\) with \(\text{card}(L_1) \leq \lambda\) such that for every \(L_0 \subset L - L_1\) with \(\text{card}(L_0) < \lambda\) does not depend on \(L_0\).

**Proof:** Let \(\mu\) be as in lemma 4.5(ii). We define by induction on \(\alpha \leq \mu\) vocabularies

\(L^\alpha \subset L\) with \(\text{card}(L^\alpha) \leq \lambda\):

\(L^0 = \varnothing\) and if \(\delta\) is limit \(L^\delta = \bigcup_{\beta < \delta} L^\beta\).

For \(\beta = \alpha + 1\) we first apply lemma 4.5(ii) to \(L^\alpha\). So there is \(L^\lambda \subset L^\alpha\) with
\[ \text{card}(L^\mu) \leq \mu \text{ and } \varphi \text{ does not depend on } L^\mu. \]

Assume for contradiction that the lemma is false, so in particular it is false for \( L_1^\mu \). Hence there is \( L_\emptyset^\mu \subseteq L^\mu \) with \( \text{card}(L_\emptyset^\mu) \leq \lambda \) such that \( \varphi \) depends properly on \( L_\emptyset^\mu \). Clearly \( L_1^\mu \cap L_\emptyset^\mu = \emptyset \) as \( \varphi \) does not depend on \( L^\mu \) w.l.o.g. \( L^\mu = \emptyset \). So we put \( L_\emptyset^\mu = L^\mu \cup L_\emptyset^\mu \). Since \( \mu^+ \leq \lambda \) and \( \text{card}(L^\mu) \leq \lambda \), we can apply lemma 4.5(ii) to \( L_\emptyset^\mu \). Hence, there is \( L^\mu \subseteq L^\mu \) with \( \text{card}(L^\mu) \leq \mu \) and \( \varphi \) does not depend on \( L_\emptyset^\mu \). But \( \forall (\mu^+) \supseteq \text{card}(L^\mu) \) so for some \( \alpha < \mu^+ \) we have \( L^\mu \subseteq L^\mu \) and

\[ L_\emptyset^\mu \subseteq L^\mu \subseteq L^\mu \]

so \( \varphi \) does not depend on \( L_\emptyset^\mu \), a contradiction. QED.

The second lemma used in the proof of theorem 4.3 gives us the connection to ultrafilters. Here we use some material from section 3.3, in particular the definition of \( UF(L) \).

**Lemma 4.7:** Let \( L \) be a logic and \( \varphi \) a \( L(L) \)-sentence which depends on \( L \subseteq L \) but for each \( L_1 \subseteq L \) with \( \text{card}(L_1) \leq \lambda \), \( \varphi \) does not depend on \( L_1 \). Then there is a function \( f: P(L_2) \to \{0,1\} \) such that:

1. \( f \) is non-constant;
2. For every \( K_1, K_2 \subseteq L_1 \) with \( \text{card}(K_1 \Delta K_2) \leq \lambda \) we have \( f(K_1) = f(K_2) \) and
3. For every ultrafilter \( F \in UF(L) \) (on \( \mu \)) \( f \) is \( F \)-continuous.

Recall that if \( F \) is an ultrafilter on \( \mu \), \( \{K_i: i < \mu\} \) are subsets of \( L \) then \( \lim_{\mu} K_i = K \) iff for every \( P \subseteq L \) the set \( I_P = \{i < \mu: P \subseteq K_i \} \subseteq F \) and \( f \) is \( F \)-continuous iff \( K = \lim_{\mu} K_i \) implies that \( f(K) = \lim_{\mu} f(K_i) \).

**Proof of Lemma 4.7:** Let \( L_2 = \{P_i: i < \kappa\} \) where \( P_i \) are predicate symbols only for notational simplicity. Since \( \varphi \) depends on \( L_2 \) there are \( L_2 \)-structures \( A, B \) on the same universe \( A, A = A(R_i : i < \kappa), B = B(R_i : i < \kappa) \) with \( A \models \varphi \) and \( B \models \neg \varphi \). Let \( H \) be a structure rich enough to contain \( H(A^+), A \) and \( B \). Let \( Q_i \) \( (i < \kappa) \) be new relation symbols with one place more than \( P_i \). We interpret \( Q_i \) on \( H \) such that the new
variables range over $P(\kappa)$ and put for each $X \subseteq \kappa$:

$H \models Q_i(X, \bar{a})$ iff $i \in X$ and $\bar{a} \in R_i$ or $i$ not in $X$ and $\bar{a} \in S_i$ $X$ serves as a new parameter. We put $\varphi^X$ to be the result of substituting $Q_i(X,-)$ for $P_i(-)$ in $\varphi$. We are now ready to define $f$. For $X \subseteq \kappa$ we put $f(X)$ to be the truth value of $\varphi^X$ in $H$.

**Claim 1:** $f$ is not constant.

Clear, since $\varphi$ depends on $L_2$ and therefore $f(\varphi) = 0$ and $f(\kappa) = 1$.

**Claim 2:** If $X_1, X_2 \subseteq \kappa$ and $\text{card}(X_1 \Delta X_2) \leq \lambda$ then $f(X_1) = f(X_2)$.

This is true, since $\varphi$ does not depend on any $L_0 \subseteq L_1$ with $\text{card}(L_0) \leq \lambda$.

**Claim 3:** $f$ is $F$-continuous for any $F \in UF(L)$.

Let $\bar{X} = \lim_{F} X_i$ for some ultrafilter $F$ on $\mu$ in $UF(L)$. We have to show that $f(X) = \lim_{F} f(X_i)$. For this we use the definition of $UF(L)$. Let $N$ be the extension of $\prod_{H/F}$ as required for $F \in UF(L)$. Since $X$ is an element of $H$, so it is of $N$; and $H \models \varphi^X$ iff $N \models \varphi^X$, and similarly for each $X_i$ ($i < \mu$). Put $\bar{X} = (X_0, X_1, \cdots)_{i < \lambda}$.

Now we have for each $\alpha \in \kappa$ that

$(\ast) \ N \models \alpha \in X$ iff $N \models \alpha \in X$.

since $X = \lim_{F} X_i$. Next we observe that

$(\ast \ast) \ N \models \varphi^X$ iff $N \models \varphi^X$

because $\varphi^X$ is obtained from $\varphi$ by substituting $Q_i(X,-)$ for $P_i(-)$ and similarly for $\varphi^X$ and for every $\alpha \in \kappa$ we have $N \models Q_i(X,-)$ iff $N \models Q_i(X,-)$ by $(\ast)$.

Finally we have that

$(\ast \ast \ast) \ N \models \varphi^X$ iff $\{i \in \mu; H \models \varphi^X\} \in F$

by the property $F \in UF(L)$.

Now clearly $\lim_{F} f(X_i) = 1$ iff $N \models \varphi^X$ (by $(\ast \ast \ast)$) and $f(X) = 1$ iff $N \models \varphi^X$ by the definition of $f$, so $(\ast \ast)$ gives the claim. QED.

The third lemma, used in the proof of theorem 4.3., gives us the connection to measurable cardinals:
Lemma 4.8: If $F$ is a uniform ultrafilter on $\omega$ and $f: P(\omega) \to \{0, 1\}$ satisfies (i) - (iii) of the previous lemma, then there is an measurable cardinal $\mu_0$ such that $\omega < \mu_0 \leq \kappa$.

Lemma 4.8 is a special case of proposition 4.9 in the next section. But we are now ready to prove theorem 4.3.

Proof of theorem 4.3: Assume $L$ is $[\omega, \omega]$-compact and $OC(L) > \omega$. Then there is an $L(L)$-sentence $\varphi$ which does not depend only on a finite subset of $L$. So $\text{card}(L) \geq \omega$, and if $\text{card}(L) = \omega$ we are done by Theorem 4.2. So $\text{card}(L) > \omega$. By lemma 4.6 (for $\lambda = \omega$) we can assume that $\varphi$ does not depend on any countable subset of $L$. Now we apply lemma 4.7 to construct the function $f$ and by 3.9 and 4.6 we know that $f$ is $F$-continuous for some uniform ultrafilter on $\omega$. So by lemma 4.8 we know that $\text{card}(L) \geq \mu_0$, the first uncountable measurable cardinal. But this shows that $OC(L) > \mu_0$. QED.

4.4 Some set theory

Let us now prove the needed facts about measurable cardinals:

Proposition 4.10: Let $I$ be a set and $S \cup T = P(I)$ a partition of the power set of $I$ such that

(i) $\emptyset \in S$, $I \in T$;

(ii) If $A \in S$ and $\text{card}(A \Delta A') \leq \omega$ then $A' \in S$;

(iii) $S$ and $T$ are closed under $\omega$-limits (monotone unions and intersections).

Then $\text{card}(I)$ is measurable.

Proof of lemma 4.9: $f$ gives rise to a partition by $f(A) = 0$ iff $A \in S$ and $f(A) = 1$ iff $A \in T$. Now (i) and (ii) of the hypothesis of proposition 10 are trivially satisfied by the hypothesis on $f$, and (iii) is weaker then $F$-continuity. So the result follows. QED.
Proof of proposition 4.10:

Claim 1: There is $A^* \in T$ such that for each $B \subseteq C \subseteq A^* C \subseteq S$ implies that $B \subseteq S$.

For assume we have $\{A_n, n \in \omega\}$ such that $A_{2n} \in T$ and $A_{2n+1} \subseteq A_{2n}$. If the sequence is proper, we get a contradiction since by (iii) $\bigcap_{n \in \omega} A_{2n} \in T$ and $\bigcap_{n \in \omega} A_{2n+1} \subseteq \bigcap_{n \in \omega} A_{2n}$. So for some $n \in \omega$ the sequence stops.

Let $A^*$ be the last $A_{2n} \in T$. Obviously $A^*$ has the required property. Now, w.l.o.g. we can assume that $A^* \neq I$, for if we put

$$S^* = \{A \cap A^* : A \in S\}, \quad T^* = \mathcal{P}(I) - S^*,$$

properties (i) - (iii) are preserved.

Now let $W \subseteq S$ be a maximal ideal. Clearly $W$ is closed under countable unions, by (iii) and the maximality of $W$. We now need a sublemma:

Sublemma 4.11: If there is no measurable cardinal $\mu$, $\omega < \mu \leq \text{card}(I)$, then there are $\{B_n \subseteq I : n \in \omega\}, \{C_n \subseteq I : n \in \omega\}$ such that

(i) $B_n \in \mathcal{P}(I) - W, \quad C_n \in \mathcal{P}\{\omega\}$

(ii) $B_{n+1} \subseteq B_n, \quad C_{n+1} \subseteq C_n$

(iii) $B_n \cup C_n \in \mathcal{T}, \quad B_{n+1} \cup C_{n+1} \in S$.

Proposition 4.10 now follows immediately, for if there is no such measurable cardinal, then $A = \cap (B_n \cup C_n) = \cap (B_{n+1} \cup C_n)$ so $A \subseteq S \cap T^* = \emptyset$, a contradiction.

QED.

Proof of sublemma 4.11: We define $B_n, C_n (n \in \omega)$ by induction on $n \in \omega$. For $n = 0$ we put $B_0 = I, C_0 = \emptyset$.

For $n + 1$ we first define $B_{n+1}$. Assume no $B_{n+1}$ exists satisfying conditions (i) - (iii). Then for all $X \subseteq B_n$ we have

$$\langle x \rangle = \begin{cases} \text{if } X \not\subseteq W \text{ then } X \cup C_n \in T \\ \text{if } X \subseteq W \text{ then } X \cup C_n \subseteq S \end{cases}$$

(Here we use that $W$ is an ideal and $\mathcal{N} \subseteq S$.)
Claim 1: The quotient space \( P(\mathbb{R})/\{P(\mathbb{R}) \cap W\} \) is infinite.

For otherwise we get a countably complete ultrafilter on a subset of \( I \), contrary to our assumptions.

So by Claim 1, there are \( D_k \subset B_n \) (\( k \in \omega \), pairwise disjoint, \( D_k \not\in W \). Now put \( E_k = \bigcup D_k \), so \( C_n = \bigcap \{ E_k \cup C_n \} \), hence, as \( C_n \in S \) and \( T \) is closed under \( \omega \)-limits, for some \( k_0 \in \omega \) \( E_{k_0} \cup C_n \in \mathcal{S} \). But \( E_{k_0} \not\in W \), so by (*) \( E_{k_0} \cup C_n \in T \), a contradiction. We conclude that there is \( B_{n+1} \) satisfying (i) - (iii), in fact, we put \( B_{n+1} = E_{k_0} \).

Now we define \( C_{n+1} \). Since \( W \) is a maximal ideal in \( S \), there is \( E \in W \) and \( D \subset B_{n+1} \) such that \( D \cup E \in T \). So we put \( C_{n+1} = C_n \cup E \). Clearly the conditions (i) - (iii) are satisfied. QED.

We finish this section with three propositions which improve some of our lemmas and may be of independent interest. (Similar results had been announced by Hajnal and Prikry, but we could not find the appropriate reference.)

Proposition 4.12: Let \( I \) be a set, \( \text{card}(I) < \mu_0 \), the first uncountable measurable cardinal, \( f : P(I) \to \{0,1\} \) a map which is non-trivial and commutes with \( \omega \)-limits. Then there is a finite \( J \subset I \) such that for all \( A \subset I \) \( f(A) = f(A \cap J) \).

Proposition 4.13: If \( f \) is as in proposition 4.12, then \( f \) is \( F \)-continuous for any ultrafilter \( F \) on \( \mathcal{S} \).

Proposition 4.14: If \( f : P(I) \to \{0,1\} \) is \( F \)-continuous for \( F \) on \( \kappa \), \( F \) \( \kappa \)-complete (hence \( \kappa \) is measurable). Then there is \( \alpha \ll \kappa \) and a partition \( I_\alpha \) (\( \alpha \ll \alpha \)) of \( I \) and there are ultrafilters \( F_i \) on \( I_\alpha \) (\( \alpha \ll \alpha \)) such that

(i) \( f(A) \) depends only on the truth value of \( A \cap I \in F_i \) (\( \alpha \ll \alpha \));

(ii) The characteristic function of \( F_i \) on \( P(I_\alpha) \) is \( F \)-continuous for all \( i \ll \alpha \).

Remark: Proposition 4.12 is proved in a similar way as the lemmas in section 4.3. Proposition 4.13 follows easily from 4.12.
Let $\mathcal{F}$ be an ultrafilter on $\alpha<\mu_0$ and $A_i \ (i<\alpha)$ be a monotone sequence of subsets of $I$. We define an equivalence relation $E$ on $I$ by $(x,y) \in E$ iff for all $i<\alpha \ x \in A_i \Rightarrow y \in A_i$. Now apply 4.12 to $I/E$.

4.14 is similar to 4.10, but uses more machinery from combinatorics (Ramsey's theorem).

4.5 An example.

In this section we return to example 4.1 in section 4.2. But we assume that each $L_{\varphi,\omega}$-formula has less than $\mu$ free variables.

**Theorem 4.15:** Let $\mathcal{F}$ be a $\mu$-complete ultrafilter on $\mu$. Then $L_{\varphi,\omega}$ is $[\lambda,\lambda]$-compact for every $\lambda<\mu$.

To prove 4.15 we first prove a lemma:

**Lemma 4.16:** Let $\mathcal{D}$ be any ultrafilter on $\lambda<\mu$, and $\{ A_i : i<\lambda \}$ a family of $L$-structures. Let $\varphi$ be a $L_{\varphi,\omega}(L)$-formula and $f_i \ (j<\nu<\mu)$ be functions in $\prod A_i$.

Then the following are equivalent:

1. $\prod [A_i / \mathcal{D}] = \varphi[f_1, f_2, \ldots, f_j, \ldots, f_\mu]_{\mathcal{D}}$
2. $\{ i, \lambda : A_i = \varphi[f_1(i), f_2(i), \ldots, f_j(i), \ldots, f_\mu] \} \in \mathcal{D}$

**Proof:** This is like the proof for first order logic by induction over the complexity of $\varphi$. Here we only show the case for $\varphi = A_\psi$. We use $[\mathcal{F}]$ and $[\mathcal{F}(i)]$ as the obvious abbreviations.

Assume (ii), so:

$$X_\varphi = \{ i < \lambda : A_i = \varphi[\mathcal{F}(i)] \} \in \mathcal{D}$$

Put

$$F_i = \{ j < \mu : A_i = \psi_j[\mathcal{F}(i)] \}$$

So, for each $i \in X_\varphi$ we have $F_i \in \mathcal{F}$ by the definition of $\varphi$, hence $F = \bigcap_{i \in X_\varphi} F_i \in \mathcal{F}$ since $\mathcal{F}$ is $\mu$-complete and $\lambda < \mu$. So for each $j \in F$ $\prod [A_i / \mathcal{D}] = \psi_j(\mathcal{F})$ by the
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induction hypothesis and since \( \bar{p} \in \mathbf{F} \) we have \( \mathbf{F} \), which proves (i).

Now assume (ii) is false, hence \( X_\varphi \notin \mathbf{D} \) and, since \( \mathbf{D} \) is an ultrafilter, \( \lambda - X_\varphi \in \mathbf{D} \).

But

\[ \lambda - X_\varphi = X \cdot \varphi = \{ i < \lambda : \mathbf{A}_i | \varphi[\bar{f}] \} \]

Put

\[ \bar{F}_i = \{ i < \lambda : \mathbf{A}_i \models \varphi(\bar{f}_i) \} \]

Since \( \mathbf{F} \) is an ultrafilter, for each \( i \in X_\varphi \) \( \bar{F}_i \in \mathbf{F} \) and therefore \( \bar{F} = \bigcap_{i \in X_\varphi} \bar{F}_i \in \mathbf{F} \). So

for each \( j \in \bar{F} \) \( \mathbf{A}_j | \varphi[\bar{f}] \) by induction hypothesis, hence \( \bigcap_{i \in X_\varphi} \mathbf{A}_i | \varphi[\bar{f}] \) since \( \bar{F} \in \mathbf{F} \) and \( \mathbf{F} \) is an ultrafilter. This proves that (i) is false, QED.

To prove theorem 4.15 we now apply theorem 3.9. QED.

Theorem 4.15 shows that neither theorem 4.2 nor theorem 4.3 can be improved.
5. Amalgamation and Compactness.

5.1. An easy theorem.

In this section we prove several simple propositions and discuss some examples. The definitions were stated in chapter 1 and 2.

**Proposition 5.1:**

(i) Let $L$ be a logic. Then $(A) \rightarrow (B)$, $(B) \rightarrow (C)$ and $(D) \rightarrow (B)$, where

(A) $L$ is compact;
(B) $L$ has JEP;
(C) $L$ has AP;
(D) $L$ has RCL.

(ii) (Mundici [Mu]) (B) implies that the class of cardinals $\lambda$ such that $L$ is not $[\lambda, \lambda] \cdot$-compact is a set. More precisely, if for every countable vocabulary $L$ card$(L(L)) \leq \lambda_0$ then there are at most $2^{\lambda_0}$ such cardinals.

(iii) If we assume $A(\infty)$, as defined in section 2.5, then (B) implies compactness.

**Proof:** (A) $\rightarrow$ (B). Let $L$ be compact and $B$ and $C$ be two $L$-structures with $B \equiv C(L(L))$. Let us assume that $B \cap C = \emptyset$ and that $L_B \cap L_C = L$. Put $T = Th_L(B, B) \cup Th_L(C, C)$. It suffices to show that $T$ has a model, so by compactness that every finite subset $T_0 \subset T$ has a model. By Theorem 4.2 every $L$-formula depends only on some finite subset of $L_B \cup L_C$, in particular only on some finitely many constants $b_1, \ldots, b_n, c_1, \ldots, c_m$ from $B \cup C$. W.l.o.g. $T_0$ is $\varphi(\vec{b}) \land \psi(\vec{c})$. But $B \models \varphi(\vec{b})$ and $C \models \psi(\vec{c})$, so, since $B \equiv C(L)$, $B \models \exists \vec{y} (\varphi(\vec{x}) \land \psi(\vec{y}))$, hence $B$ can be expanded to a model of $T_0$. Note that we used closure under finite conjunction and existential quantification of $L$.

(B) $\rightarrow$ (C): Let $A \triangleleft B(L), A \triangleleft C(L)$ be three $L$-structures. By our hypothesis $<A, A> = <B, A> = <C, A> (L(L))$. So we can apply JEP to $<B, A>$ and $<C, A>$.

(D) $\rightarrow$ (B): For $A = B(L(L))$ put $T = Th_L(A) = Th_L(B)$ and $T_1 = Th_L(A, A), T_2 = Th_L(B, B)$ with $L_A \cap L_B = L$. By RCL $T_1 \cup T_2$ has a model which proves
(i) We use proposition 2.1. First we observe that if $M(\lambda)$ and $M(\mu)$ are two $L$ structures characterizing cofinally $\lambda$ and $\mu$ then there is no joint embedding. Next we observe that, w.l.o.g. we can assume that there is a countable vocabulary $L$ such that, whenever $L$ is not $[\lambda, \lambda]$-compact, $\lambda$ regular, then $\lambda$ is cofinally characterizable in some $L$-structure $M(\lambda)$. We simply use constants to code many $n$-ary relation symbols by one $(n+1)$-ary relation symbol. To finish the proof we observe that there are only $2^{2^\omega}$ many $L$-theories in $L(L)$.

(iii) This now follows from (ii) and theorem 3.13.

QED.

Note that (ii) and, since $A(\omega)$ implies that there are no uncountable measurable cardinals, (iii) follow immediately from our theorem 5.2 below.

5.2 The Main Theorem.

Our main theorem is:

**Theorem 5.2:** Let $L$ be a logic with $OC(L)=\lambda_0<\mu_0$ (where $\mu_0$ is the first uncountable measurable cardinal), satisfying $AP$. Then $L$ is compact. In particular, if there are no uncountable measurable cardinals, and $L$ is a logic with some occurrence number $OC(L)=\lambda_0$, satisfying $AP$, then $L$ is compact.

The theorem will follow from the apparently weaker theorem 5.3, which we prove in the next section. For a logic $L$ with finite occurrence, i.e. $OC(L)=\omega$, compactness follows without theorem 2.8 and 4.3.

**Theorem 5.3:** Let $L$ be a logic with $OC(L)=\lambda$, $\lambda$ regular, satisfying $AP$. Then $L$ is $[\lambda, \lambda]$-compact.

**Proof of Theorem 5.2:** By Theorem 5.3, $L$ is $[\lambda_0, \lambda_0]$-compact. Since there are no uncountable measurable cardinals, by Theorem 2.8, $L$ is $[\omega, \omega]$-compact, so by theorem 4.3 $OC(L)=\omega$. Now we apply theorem 5.3 again to get that $L$ is $[\lambda, \lambda]$-
compact for every regular \( \lambda \geq \omega \), hence, by Théoréme 3.12, \( L \) is compact. QED.

**Corollary 5.4:** Assume there are no uncountable measurable cardinals and \( L \) satisfies \( RCL \) and \( OC(L) \) exists. Then \( L \) is compact.

This improves theorem 6.4 of [MS1] and theorem 5.1(iii).

**Corollary 5.5:** The logics \( L^\omega = L_{\omega, \omega}(Q_\omega) \) with the quantifier "There exist at least \( \kappa \) many", \( \kappa \) infinite, do not satisfy \( AP \).

**Proof:** \( L^\omega \) is not \([\kappa, \kappa]\)-compact; QED.

This solves problem 1 in [MR, p. 155].

Incidentally, problem 3 [ibid] was already solved in [MS8], which gives another example of how abstract model theory can be used to solve concrete problems.

There are logics \( L \) which are \((\omega, \omega)\)-compact, not \((\omega_1, \omega)\)-compact and such that \( L^{\omega_1} \) is not contained in \( L \). Let \( Qxy \) any binary quantifier definable in \( L^{\omega_1} \) but not in \( L^{\omega_1} \), which is \((\omega_1, \omega)\)-incompact, e.g. \( Qxy(\varphi(xy)) \) which says: " \( \varphi \) is a linear \( \varphi \)-like ordering". \( L = L_{\omega, \omega}(Q) \) is \((\omega, \omega)\)-compact, since \( \Delta(L^{\omega_1}) \) is \((\omega, \omega)\)-compact.

Problem 2 [ibid] asks whether in the counterexample to \( AP \) for \( L^\omega \) all the structures can be chosen to be isomorphic. As will follow from the proof in the next section, the answer is yes.

Let us conclude this section with three examples which show that some assumptions on the occurrence number \( OC(L) \) are necessary for theorem 5.2.

**Example 5.6:** Look at the logic \( L_{\omega, \omega} \), which has no occurrence number. Since in \( L_{\omega, \omega} \) every complete theory is categorical, one easily verifies that \( L_{\omega, \omega} \) satisfies \( RCL \), hence has \( AP \). But \( L_{\omega, \omega} \) is not \([\lambda, \lambda]\)-compact for any \( \lambda \).

**Example 5.7:** Let \( L^2_{\omega, \lambda} \) be the infinitary logic with conjunctions of length \( \kappa \) and both existential quantification of elements and relations of length \( \lambda \). Magidor [Mg1] has shown that \( L^2_{\omega, \lambda} \) is \((\omega_1, \lambda)\)-compact iff \( \kappa \) is extendible. Obviously the occurrence number \( OC(L^2_{\omega, \lambda}) = \max(\kappa, \lambda) \).
Proposition 5.8: Assume $L^2_{\kappa,\kappa}$ is $(\kappa,\kappa)$-compact. Then $L^2_{\kappa,\kappa}$ satisfies $RCL$.

Proof: Let $T_i$ (i=0,1,2) be theories satisfying the hypothesis of $RCL$. We have to show that every subset $S \subseteq T = \bigcup T_i$ of cardinality less than $\kappa$ has a model. But such a $S$ is equivalent to a single formula and the predicates and constants not in $T_0$ can be quantified away. So any model of $T_0$ can be expanded to satisfy $S$ since $T_0$ is complete. QED.

Proposition 5.9: Let $\kappa$ be extendible. Then $L^2_{\kappa,\kappa}$ is $(\kappa,\kappa)$-compact.

Proof: By [Mg1] $L^2_{\kappa,\kappa}$ is $(\kappa,\kappa)$-compact. We propose to show that every $L^2_{\kappa,\kappa}$-formula is equivalent to an $L^2_{\kappa,\kappa}$-formula using additional predicates. The idea is to replace an existential quantification over less than $\kappa$ variables (or relation symbols) by a single quantification over finitely many relation symbols coding the $\kappa$ many. For this we need that in $L^2_{\kappa,\kappa}$ the power set operation is absolute and that we have conjunctions over less than $\kappa$ many formulas. Also, every ordinal $<\kappa$ is describable in $L^2_{\kappa,\kappa}$ by a single sentence, which will give us the parameters. The details are left to the reader. From this $(\kappa,\kappa)$-compactness follows immediately. QED.

Example 5.7 shows that some large cardinal assumption on the occurrence number of $L$ is needed in Theorem 5.2.

Unfortunately extendible cardinals are even bigger than the first supercompact cardinal (cf. [Mg1]) but Vopenka's principle implies that extendible cardinals are stationary. For more results related to compactness of infinitary logics the reader may consult Stavi [St] and [Ma3]. In the latter it is shown that Vopenka's principle is equivalent to the assumption that every logic which is finitely generated has some $(\kappa,\kappa)$-compactness.

Example 5.8: The logic $L_{\omega,\omega}$ satisfies AP trivially, since $A<_{\omega,\omega}B$ implies that $A=B$, but does not satisfy $RCL$ (cf. [MS1]). $L_{\omega,\omega}$ has no occurrence number and $L_{\omega,\omega}$ is
5.3. Proof of the Main Theorem.

We give first an outline of the proof, to help the reader. We assume for contradiction that $\lambda$ is regular and $L$ is not $[\lambda, \lambda]$-compact. Using theorem 2.1 we construct a class $K$ of linear orderings with additional predicates in which points of cofinality $\lambda$ are absolute. Inside $K$ we show the existence of some sufficiently homogeneous structure $N$. In $N$ we shall find $M_i$ ($i=0, 1, 2$) being a counterexample to $AP$ for $L$. The occurrence number and the isomorphism axiom will be needed to show that $M_0 <_L M_i$ ($i=1, 2$) and the absoluteness of "cofinality $\lambda$" to show that there is no amalgamating structure.

The counterexample to amalgamation is patterned after the following example:

Let $K$ be the class of dense linear orderings with an additional unary predicate $\text{Red}$ such that both $\text{Red}$ and its complement are dense. Let $A <_K B$ hold if $A$ is an elementary substructure of $B$ and the universe of $A$ is a dense subset of the universe of $B$. We shall show that $K$ with this notion of substructure $<_K$ does not allow amalgamation. For this, let $A_0$ be the rationals properly coloured, and let $A_i$ ($i=1, 2$) the rationals augmented by one element (say $\pi$) coloured $\text{Red}$ in $A_i$ and not coloured in $A_0$. Clearly, $A_0 <_K A_i$ ($i=1, 2$), but no amalgamating structure exists, since otherwise $\pi$ is simultaneously coloured and not coloured.

Now, let $\lambda \geq OC(L)$ be regular and $L$ not $[\lambda, \lambda]$-compact. By theorem 2.1, $\lambda$ is cofinally characterizable in $L$ in a structure $M$. We need some more information on $M$.

Let $\Sigma_i = \{ \exists x, \alpha < \lambda \}$ be the counter-example to $[\lambda, \lambda]$-compactness. Put $\Sigma_0^a = \{ \exists x, \beta < \alpha \}$ and $M_a = M_i \cup \Sigma_0^a$. W.l.o.g. the $M_a$'s are structures of some countable language $L$ (coding more predicates with parameters), and have the same power $\mu = \lambda$, $M_a = \langle M_a, Q_n (n \in \omega) \rangle$. 

not compact.
We want to code all the $\mathbf{M}_n$'s into one structure. So we let $\mathbf{M}$ to be such that:

1. $\mathbf{M} = \langle M, <, \bar{Q}_n, c_j \mid n \in \omega, j \in \lambda \rangle$
2. $\langle M, < \rangle$ is a linear order of cofinality $\lambda$ such that every initial segment has power $\mu$ (of order type $\mu^\kappa + \lambda$, e.g.).
3. $\{c_j : j < \lambda \} \subset M$ is increasing and unbounded.
4. If $x \leq c_j$ but $x > c_i$ for every $i < j$ then $\langle \{y \in M : y < x \}, \bar{Q}_n(x, -, -\ldots, -) \rangle \cong \mathbf{M}_n$.

Let $T = Th_L(\mathbf{M})$ for some fixed $\mathbf{M}$ as described above.

**Claim:** Then $T$ cofinally characterizes $\lambda$.

This is proved like theorem 2.1.

For the rest of this section $\mathbf{M}$ is fixed. We now define a class of structures $K(\mathbf{M})$:

The vocabulary of $K(\mathbf{M})$ is that of $\mathbf{M}$ without the constant symbols for $c_j$ but with two additional unary predicate symbols $P$ and $R$ and one additional binary predicate symbol $E$. Actually our main focus is on the order together with $P, R$, and $E$ is used to code copies of $\mathbf{M}$, which we need to guarantee the absoluteness of cofinality $\lambda$.

A model in $K(\mathbf{M})$ is of the form $A = \langle A, <, \bar{Q}_n, P, R, E \rangle$ with the requirements:

1. If $x \in P$ then the cofinality of $x$ in $\langle A, < \rangle$ is $\lambda$ with a witnessing sequence $\{c_j(x) : j < \lambda \}$.
2. $(a, x) \in E$ implies that $a < x$.
3. $(a, x) \in E$ implies that $x \in P$ and $a \not\in P$.
4. $P(x)$ implies that $E(c_j(x), x)$ for every $j \in \lambda$.

Put $J^A_\lambda = \{a \in A : (a, x) \in E \}$ and $J^A_\lambda$ be the substructure of $\langle A, <, \bar{Q}_n \rangle$ induced by $J^A_\lambda$.

1. The structure $\langle J^A_\lambda, c_j(x) \rangle$ is isomorphic to $\mathbf{M}$.
2. $R \subset P$.

We call a structure in $K(\mathbf{M})$ pure if additionally

1. $\bar{Q}_n$ is false where not defined by the previous requirements.
Comments:

Note that if $A \in K(M)$ is pure and $P$ in $A$ is empty, then $A$ is just a linear ordering, i.e. all the other relations are empty, too, by (K7).

If we add to $M$ one point at the end, say $x$ and let $P=\{x\}$, we get a structure in $K(M)$. We denote this structure by $M^{+1}$.

In general the structures in $K(M)$ are linearly ordered structures where every point in $P$ has a copy of $M$ attached to it, in a way that different points have almost disjoint copies of $M$, and $M$ cofinally reaches its point in $P$. The choice of $R$ can be any subset of $P$. More precisely:

**Fact 1:** For every $A \in K(M)$ and every $a, a' \in A$. $J_a^A \cap J_{a'}^A$ is bounded below both $a,a'$.

This is proved using the fact that $M$ is of order type $\omega^*+\lambda$. Note that this is first order expressible and could have been stated also as an axiom among (K1-K7).

**Fact 2:** If $A \in K(M)$ and $a \in P^d$ and we form $A'$ by changing the truth value of $a \in R^d$, but leaving everything else fixed, then $A' \in K(M)$.

Next we define the notion of $K$-substructure, $A \subset K B$ for $A,B \in K(M)$ by:

(K8) $A \subset B$

(K9) If $x \in P^d$ then $J_x^B \subset A$.

(K10) If $x \in P^B - P^A$ then $\{a \in A : a < x\}$ is bounded below $x$ in $B$, i.e. there is $b_x \in B$ such that $b_x < x$ and for each $a \in A$ with $a < x$ we have $a < b_x$.

The idea behind this is that in $B$ points are added to $P^B$ in a way that they are not limits of points from $A$, and that points in $A$ are of cofinality $\lambda$ also in $B$ with the same copy of $M$ ensuring this as in $A$.

This ends the definition of $K(M)$ and of $K$-substructures.

Before we proceed with the proof of theorem 5.3 we collect some more facts:

**Definition:** If $A_1,A_2 \in K(M)$ we define $A_1 + A_2$ to be the disjoint union of $A_1,A_2$ with the linear ordering of $A_1$ and $A_2$ for their elements and $a_1 < a_2$ for every
\[ a_1 \in A_1, a_2 \in A_2. \text{ For the other relations we just take their unions.} \]

**Fact 3:** If \( A_1, A_2 \in K(M) \) so \( A_1 + A_2 \in K(M) \) and \( A_i \subseteq K A_i + A_2 \) \((i=1,2)\).

This is clear from the definitions.

**Definition:**

Denote by \( l_I^A = \{ a \in A : a < x \} \) and by \( I_I^A \) the structure \( A \upharpoonright I_I^A \):

If \( B \in K(M) \) and \( A \subseteq B \), we define a substructure \( C(A) \) of \( B \) by

\[
C(A) = B \upharpoonright \bigcup_{a \in A} J_B^a \cup A.
\]

This makes sense by fact 1 and ensures that:

**Fact 4:** For every \( B \in K(M), A \subseteq B \), \( C(A) \subseteq KB \) but in general \( C(A) \) is not pure. Furthermore, if \( A \) is bounded in \( B \) by \( b \), i.e. there is \( b \in B \) with \( A \subseteq J_B^b \), so \( C(A) \subseteq J_B^b \) and \( C(I_B^b) = I_B^b \).

**Fact 5:** If \( A \in K(M) \) and \( d \in P \) then \( A \upharpoonright I_I^A \subseteq KA \).

**Fact 6:** If \( \{ A_i : i < \alpha \} \) is a sequence of structures in \( K(M) \) such that \( A_i \subseteq KA_{i+1} \) then

\[
A = \bigcup_{i \in \alpha} A_i \subseteq K(M) \text{ and } A_i \subseteq KA \text{ for each } i < \alpha.
\]

**Definition:** If \( A_1, A_2 \in K(M), B_i \subseteq KA_i \) \((i=1,2)\) and \( f : B_1 \cong B_2 \) is an isomorphism, we define \( A_1 + f A_2 \) in the following way: Form the disjoint union of \( A_1 \) and \( A_2 \) modulo \( f \) (i.e. identify elements only via \( f \)). This makes it into a partially ordered structure where \( a_i \in A_i \) \((i=1,2)\) are comparable only if one of them is in the range or domain of \( f \), or there is \( b \) between \( a_1, a_2 \) which has been identified. For incomparable \( a_1, a_2 \) we extend the order on \( A_1 + f A_2 \) setting \( a_1 < a_2 \).

**Fact 7:** If \( A_1, A_2 \in K(M) \) and \( f : B_1 \cong B_2 \), \( B_i \subseteq KA_i \) \((i=1,2)\) then \( A_1 + f A_2 \in K(M) \) and \( A_i \subseteq KA_1 + f A_2 \).

The proofs of the facts are left to the reader.

The next lemma is crucial for our construction:

**Lemma 5.9:** If \( A \in K(M) \) and \( B \) is a \( L \)-extension of \( A \) and \( \{ d_j : j < \lambda \} \) is cofinal in \( J_B^a \) for \( a \in P \), then \( \{ d_j : j < \lambda \} \) cofinal in \( J_B^a \)
Proof: Let \( a \in P^A \), so \( J \models \mathcal{M} \) by (K5) and by our assumption on \( L \) and \( \mathcal{M} \), \( L \) cofinally characterizes \( \lambda \) in \( \mathcal{M} \). Using relativization of \( L \) the structure \( J \models \mathcal{M} \) is an \( L \)-extension of \( \mathcal{M} \) so \( \mathcal{M} \) is cofinal in \( J \), hence \( \{ d_j : j < \lambda \} \) is cofinal in \( J \) which proves the lemma. QED.

The next lemma is proved in the following section. But from it we can now complete the proof of theorem 5.3.

Lemma 5.10: There is a structure \( N \) in \( K(\mathcal{M}) \) and \( d_1 < d_2 < d_3 \) in \( N \) with \( d_i \in P^N \) \((i = 1, 2, 3)\), \( d_1 \in R^N \), \( d_2 \not\in R^N \) such that

(i) \( N \upharpoonright J^d_i \cong N \upharpoonright J^d_2 \cong N \upharpoonright J^d_3 \) and

(ii) If \( A \subset R^N \) \((i = 1, 2)\) is bounded in \( N \upharpoonright J^d_i \), then \( N \upharpoonright J^d_1 \cong N \upharpoonright J^d_2 \) over \( A \) \((i = 1, 2)\).

Proof of Theorem 5.3: Put \( M_i = N \upharpoonright J^d_i \) \((i = 1, 2, 3)\). We have to verify some claims:

Claim 1: \( M_1 \subset L \mathcal{M}_3 \) \((1 \neq 1, 2)\).

Proof: Let \( \varphi \) be an \( L(M_i) \)-sentence. Since the occurrence number \( OC(L) \leq \lambda \), \( \varphi \) depends on \( \lambda \) many constants, hence there is \( a \in \mathcal{M}_i \) and all the constants of \( \varphi \) are in \( J^d_i \). So by fact \( 4 \) \( \mathcal{M}_i \upharpoonright J^d_i \) is a bounded \( K \)-substructure of both \( \mathcal{M}_i \) and \( \mathcal{M}_3 \).

So by lemma 5.10(ii) \( \langle \mathcal{M}_i, J^d_i \rangle \) is isomorphic to \( \langle \mathcal{M}_3, J^d_3 \rangle \) hence by the basic isomorphism axiom, \( \langle \mathcal{M}_i, J^d_i \rangle \models \varphi \) if and only if \( \langle \mathcal{M}_3, J^d_3 \rangle \models \varphi \).

Now let \( f : M_1 \cong M_2 \) be the isomorphism from lemma 5.10(i). Then \( g_i : \mathcal{M}_i \rightarrow \mathcal{M}_3 \) \((i = 1, 2)\) the \( L \)-embeddings from the claim.

Since \( L \) has \( AP \), let \( A \) be the amalgamation for \( g_1 : \mathcal{M}_1 \rightarrow \mathcal{M}_3 \), \( g_2 : \mathcal{M}_2 \rightarrow \mathcal{M}_3 \) \((i = 1, 2)\).

Claim 2: \( A \models d_1 = d_2 \).

Proof: \( d_i \in P^{M_3} \) \((i = 1, 2)\) are both of cofinality \( \lambda \) and \( g_i(M_1) \) is cofinal in \( \mathcal{M}_3 \upharpoonright J^d_{i1} \), and \( g_2 f(M_1) \) is cofinal in \( \mathcal{M}_3 \upharpoonright J^d_{i2} \), so by lemma 5.9 also in \( A \upharpoonright J^d_{i1} \) and \( A \upharpoonright J^d_{i2} \), hence \( A \models d_1 = d_2 \).

But claim 2 contradicts our assumption of lemma 5.10 that \( d_1 \in R^A \) and \( d_2 \not\in R^A \).

This completes the proof of 5.3. QED.
5.4 Proof of Lemma 5.10:

The proof 5.10 is similar to the classical proof of the existence of homogeneous structures. The first step is the amalgamation lemma for $K$-extensions:

Lemma 5.11: If $A \in K(M)$, $d_1,d_2 \in P^A$, $B_i \subset K A \uparrow I^A_i$ $(i=1,2)$ and $f : B_1 \cong B_2$ and both $B_i$ are either bounded or unbounded in $A \uparrow I^A_i$ then there are $A' \in K(M)$ and $B_1', B_2'$ and $f'$ such that

(i) $A \subset K A'$

(ii) $B_1 \subset K I^A_1$ unbounded,

(iii) $B_2 = I^A_2 \subset K I^A_2$ unbounded,

(iv) $f \subset f'$ with $f' : B_1' \cong B_2'$.

Proof: There are two cases:

Case 1: $d_1 < d_2$.

Put $A_i = I^A_i$ $(i=1,2)$ and form $A_0 + A_2$ and $f_A$. There are natural embeddings $\pi_1 : A_0 \rightarrow A_0$ depending if $A_2$ is identified with the first or second copy of $A_2$ in $A_0$. Let $g : \pi_1(A_0) \cong A_0 \subset K A$ and form $A_0 + f A = A'$. There is a natural embedding $h : A \subset K A'$ which determines the value of $d_1, d_2$ in $A$. We have to define $f'$. The range of $f'$ is $h(A_0)$ and on $h(B_0)$ $f'$ is $f$, otherwise it is $g^{-1}$. So we have proved (i) and (iv). To prove (ii) and (iii) we have only to show that $B_2 = h(A_0)$ is unbounded in $I^A_2$ and similarly for $B_1 = (f')^{-1}(B_2)$. In the first case this is trivial and in the second case this comes from $f'$ if $B_1$ is unbounded and from $g^{-1}$ if $B_1$ is bounded.

Case 2: $d_2 < d_1$. The proof is similar, but put $A_i = A_2 + f A_i$. In both cases we use heavily facts 4 and 8. QED.

Lemma 5.12: Under the same hypotheses as in Lemma 5.11, we can even have $B_1 = I^A_1$ have $B_1 = I^A_1$ and $B_2 = I^A_2$. 
Proof: Use lemma 5.11 countably many times with alternating roles of \( d_1, d_2 \) and take unions of chains using fact 5. QED.

Proof of Lemma 5.10: Let \( A \) be in \( K(\tilde{M}) \) such that \( P^A \) is cofinal in \( A \) and both \( (P-R)^A \) and \( R^A \) are cofinal in \( P^A \). For instance take \( M \) and put \( M^{+1} \) and for \( x \) the new element set \( P(x) \) in the obvious way. We have freedom to choose both \( R(x) \) and \( \neg R(x) \). So put \( M_i = M^{+1} \) with \( x \in (P-R) \) if \( i \) is even and \( x \in R \) if \( i \) is odd. Then form \( A_{n+1} = A_n + M_{n+1} \) and \( A_0 = M_0 \). Now put \( A = \bigcup_{n \in \omega} A_n \).

Now choose any \( d_1, d_2, d_3 \) with \( d_1 < d_2 < d_3 \) in \( P^A \) with \( d_1 \in R^A \), \( d_2 \in (P-R)^A \). Let \( \{c_f(d_i) : f < \lambda \} (i = 1, 2, 3) \) be cofinal witnessing sequences for \( d_i \) respectively and assume w.l.o.g. \( c_0(d_1) = c_0(d_2) = c_0(d_3) \). By lemma 5.12 we can assume that there is \( B_0, A \subset \chi B_0 \) with \( B_0(d_i) = c_0(d_i) \). We can achieve this using an iteration, alternating the roles of the \( d_i \)'s in pairs. Iterating this we construct \( B_{n+1} \) extending \( B_n \) such that \( I_{B_{n+1}}(d_i) \) are all isomorphic over \( I_{B_n}(d_i) \) for \( i = 1, 2, 3 \). For a limit we take unions. Clear \( B_\alpha \) is the required model. QED.
References.


Positive Results


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