BOUND S ON PATH CONNECTIVITY

by

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Abstract

The path-connectivity of a graph $G$ is the maximal $k$ for which between any $k$ pairs of vertices there are $k$ edge-disjoint paths (one between each pair). An upper bound for the path-connectivity of $n^q$ ($q < 1$) separable graphs [Lei80] is shown to exist.

If the edge connectivity of a graph is $\kappa_E$, then between any two pairs of vertices and for every $t \leq \kappa_E$ there exists a $t \leq t' \leq t+1$ such that there are $t'$ paths between the first pair and $\kappa_E - t'$ between the second pair. All paths are edge-disjoint. Also if the edge-connectivity of a graph is $\geq 4$ then between any three pairs of vertices there are edge-disjoint paths.
1. Introduction

The connectivity properties of VLSI circuits play a major role in establishing bounds on circuit embeddings [CC80, Lei80, Val81, Leo70]. Here we consider some connectivity properties of separable graphs [Lei80] which are more general graphs than VLSI. Given several subgraphs in a graph $G$ we study how many edge-disjoint paths exist between them (this is the path connectivity). The problem of finding two paths between two distinct pairs of vertices was treated in [PS78, Tho80]. This result is generalized to show that if the edge-connectivity of a graph is $k$, approximately $\frac{k}{2}$ edge-disjoint paths exists between each pair of vertices. Upper and lower bounds are given for the path connectivity and these bounds are related to the edge connectivity of the graph and its separability constants.

These results are useful also for studying networks of processors and embeddings of semantic networks in host graphs.

2. Path connectivity definitions

A chain $C$ in a graph $G$ is a path in $G$. A set $F$ of chains in a graph $G$ is called full if $\bigcup_{C \in F} V(S) = V(G)$. Let $F$ be a set of chains in a graph $G$. A path between two chains in a graph $G$ is a path between any of their endpoints and whose edges are disjoint of any other chains of $F$ embedded in $G$.

Definition 2.1

Let $F$ be a set of edge-disjoint chains in a graph $G$. The path connectivity of $F$ is the maximum $k$ such that between any $k$ disjoint pairs of chains there are $k$ edge-disjoint paths.
We shall denote the path connectivity of a set of chains $F$ by $dpc(F)$. The path connectivity of a graph $G$ is the path connectivity of a full set of chains of length zero in $G$. The following examples show that the path connectivity is not directly related to the edge connectivity.

Example 1.

![Diagram of a graph with vertices $p_1, p_2, p_3, p_4$ and edges between them.]

The edge connectivity of $G$ is 2 but its path connectivity is 1 since there are no two edge-disjoint paths between $p_1p_3$ and $p_2p_4$.

Example 2.

Consider the graph:

![Diagram of a graph with vertex $p_0$ and a circle $G'_k$.]

$G_k = \quad G'_k$

Where $G'_k$ is a $k$-1 clique. The edge-connectivity of $G_k$ is 2 but the path connectivity of $G_k$ is $\frac{k}{2}$. 
Let $F$ be a set of chains in a graph $G$. Given a graph $A \subseteq G$, $CF(A)$ denotes the number of chains in $F$ having both of their endpoints in $A$. Then the following weak relationship between edge-connectivity and path-connectivity holds.

**Theorem 2.1**

Let $F$ be a set of chains in a graph $G$. If $G$ has an edge-separating set $S$ which separates $G$ into $G_1, G_2$ such that $|CF(G_1)| > |S|$, $|CF(G_2)| > |S|$ then the path connectivity of $F$ is less than $|S|$.

**Proof**

There are at most $|S|$ edge-disjoint paths between chains in $G_1$ and chains in $G_2$.

Theorem 2.1 gives bounds on the path connectivity of graphs. Therefore some separation theorems are wanted.

3. Graph separation

Following [Lei80] let $g$ be a real function $c > 0$ and $\frac{1}{2} \geq c > 0$, a graph $G$ is (vertex) separable by $(a,c,g)$ if for every subgraph $G_1 \subseteq G$ there is a separation set $S_{G_1}$ of (vertices) edges which separates $G_1$ into $G_1', G_1''$ such that

1. $cg(|V(G_1)|) \geq |S_{G_1}|$

2. $|V(G_1')| \geq \alpha |V(G_1)| \quad |V(G_1'')| \geq \alpha |V(G_1)|$

It is easily seen that if a graph $G$ is separable by $(a,c,g)$ then it is vertex separable by $(a,2c,g)$ (take the endpoints of the edge separating set), and if $G$ has degree $\leq d$ and is vertex separable by $(a,c,g)$ then it is separable by $(a,dc,g)$ (by taking the edges incident to the vertex separation set).

We shall now give a method for obtaining upper bounds for path connectivity.
Assume $G$ is a graph separable by $(\alpha, c, g)$. Separate $G$ by $S_1$ into $G_1, G_1'$ and set $n = |V(G)|$. Separate $G_1$ by $S_2'$ into $G_2, G_2'$. Assume that the number of edges of $S_1$ incident with $G_2$ is $\leq \frac{1}{2}|S_1|$ (otherwise select $G_1'$). $G_2$ is separated from $G - G_2'$ by a set of edges $S_{G_2'}$ such that

$$|S_{G_2'}| \leq |S_2'| + \frac{1}{2}|S_1| \leq \frac{1}{2}cg(n) + cg(|V(G_1)|) \cdot |V(G_1)| \leq (1 - \alpha)|V(G)|$$

Repeating this process $p + 1$ times we get a subgraph $G_p$, separated from $G - G_p$ by a set of edges $S_{G_p}$ such that

$$|S_{G_p}| \leq \sum_{i=0}^{p} \frac{c}{2^i} g((1 - \alpha)^{p-i} n)$$

where $n = |V(G)|$ and $|V(G_p)| \geq \alpha^{p+1} n$. From this we get the following result on path connectivity.

**Lemma 3.1**

Let $G$ be a graph separable by $(\alpha, c, g)$ and $|V(G)| = n$.

Define

$$A(m) = \sum_{i=0}^{m} \frac{c}{2^i} g((1 - \alpha)^{m-i} n)$$

If there are $m, k > 0$ such that

$$A(m) < k < \alpha^{m+1} n$$

then the path connectivity of $G$ is less than $k$.

**Proof**

Repeatedly separate $G$ and use Theorem 2.1.

4. **Upper bounds for separable graphs**

Assume $G$ is separable by $(\alpha, c, n^q)$, $c > 0$, $\frac{1}{2} \leq \alpha > 0$, $g < 1$

$$A(m) = \sum_{i=0}^{m} \frac{c}{2^i} ((1 - \alpha)^{m-i} n)^q = c (1 - \alpha)^{m} n^q \sum_{i=0}^{m} [2(1 - \alpha)^{i}]^{-i}$$
Since $\alpha \leq \frac{1}{2}$, $q < 1$, $(1-\alpha)^{q} > \frac{1}{2}$. We have:

$$A(m) < c \frac{(1-\alpha)^{q}n^{q}}{1-2(1-\alpha)^{q}}$$

Using the premise of Lemma 3.1

$$\frac{c(1-\alpha)^{q}n^{q}}{1-2(1-\alpha)^{q}} \leq \alpha^{m+1}n$$

and

$$\frac{c}{\alpha(1-2(1-\alpha)^{q})} \left[ \frac{(1-\alpha)^{q}}{\alpha} \right]^{m} \leq n^{1-q}$$

therefore

$$m \leq \frac{\{1-q\} \log n}{\log \left( \frac{\alpha(1-2(1-\alpha)^{q})}{c} \right)} = a \log bn = m_{0}$$

For $a = a(\alpha,q)$ and $b = b(\alpha,q)$.

Since $\log n$ tends to infinity, for almost all $n$ there exists an integral $m$, for which the inequality holds. Select $m = m_{0} - \varepsilon$ then

$$A_{\varepsilon}(m) = A(m_{0} - \varepsilon) < \frac{c}{1-2(1-\alpha)^{q}}(1-\alpha)^{q}m_{0}n^{q}(1-\alpha)^{-q\varepsilon}$$

Which is of the form ($u$ is a constant)

$$A_{\varepsilon}(m) < u n^{-q}n^{q}(1-\alpha)^{-q\varepsilon} = u(1-\alpha)^{-q\varepsilon} \quad (a \leq 1)$$

Assume now that in $G$ there is a set $F$ of chains of bounded length $l$. $m+1$ separations yield:

$$|V(G_{m})| \geq \alpha^{m+1} |V(G)|$$

$$|S_{m}| \leq u(1-\alpha)^{-q\varepsilon}$$

For some constant $u_{1}$

$$\alpha^{m+1} |V(G)| \geq u_{1}n^{-q}n^{q} = u_{1} \alpha^{-\varepsilon} \quad (\alpha < 1, m \leq \log bn)$$

Each edge of $S_{m}$ may belong to a different chain therefore

$$|C_{F}(G_{m})| \geq \frac{|V(G_{m})|-1-|S_{m}|}{l+1}$$
and to get a bound for the path connectivity it is required that

\[ |C_F(G_m)| > |S_m| \]

which is satisfied if

\[
\frac{u_1 \alpha^{-z} - l u (1-\alpha)^{-q} \zeta}{l + 1} > u(1-\alpha)^{-q} \\
u_1 \alpha^{-z} > (2l+1)u(1-\alpha)^{-q} \\
\left[ \frac{(1-\alpha)^q}{\alpha} \right] > \frac{u_1}{u_1 (2l+1)} \\
e > \frac{u(2l+1)}{u_1 \log \frac{(1-\alpha)^q}{\alpha}} = \varepsilon_0 > 0
\]

Therefore for sufficiently large \( |V(G)| \) the equations of Lemma 3.1 can be satisfied yielding:

**Theorem 4.1**

Let \( G \) be the set of all graphs separable by \( (\alpha, c, n, q) \), \( q < 1, c > 0, 0 < \alpha \leq \frac{1}{2} \)

let \( \bar{F} \) be the set of all full sets of chain of bounded length \( l \geq 0 \) in graphs from \( \bar{G} \). Then there exists \( k = k(l, \alpha, q, c) \) such that the path connectivity of all sets from \( \bar{F} \) is less than \( k \).

**Proof**

If \( q < 0 \) then the conditions of Lemma 3.1 hold trivially. If \( q \geq 0 \) it has been shown that there is a \( n_0 = n_0(l, \alpha, q, c) \) such that for every \( g \in \bar{G} \) with \( |V(G)| \geq n_0 \) all the full sets of chains in \( G \) are of bounded path connectivity. The remaining case is when \( |V(G)| < n_0 \), however since there is a finite number of full sets of chains in graphs for which \( |V(G)| < n_0 \) and we can select the maximal path connectivity.

**Corollary 4.1.1**

All full sets of chains of bounded length in planar graphs with bounded
degree have bounded path connectivity.

*Proof*

By [LT77] planar graphs are vertex separable by $(1/2 \cdot c, n)$ where $c > 0$ and if their degree is bounded by $d$ then they are separable by $(1/2 \cdot dc, n)$, and the result follows from Theorem 4.1.

**Corollary 4.1.2**

Planar graphs of bounded degree are of bounded path connectivity.

*Proof*

Take a full set of chains of length 0 and the result follows from Corollary 4.1.1.

5. **2-Path connectivity**

In this section we relate the edge-connectivity and the path connectivity. Let $G(V,E)$ be a graph. A path $\pi = (v_0, v_1, \ldots, v_i)$ is a $v_0 - v_i$ path. Define $\pi[v_0, v_i] = (v_0, v_1, \ldots, v_i)$. $F_E(\pi) = (v_0, v_i)$, $L_E = (v_{i-1}, v_i)$ to be the first and last edges of $\pi$ (similarly $F_V, L_V$ for vertices). If $\pi$ is a $v_0 - v_i$ path and $\sigma$ is a $v_i - v_r$ path then $\pi \sigma$ is the $v_0 - v_r$ path obtained from the concatenation of $\pi$ and $\sigma$.

**Theorem 5.1**

Let $p_1, p_2, q_1, q_2$ be four distinct vertices of a graph $G$, and $P(Q)$ a set of $k$ edge-disjoint $p_1 - p_2$ paths ($q_1 - q_2$ paths). Then for all $0 \leq m \leq k$ there exist $s = s(m) \in \{m, m+1\}$ a set $P_s$ of $s$ edge-disjoint $p_1 - p_2$ paths and a set $Q_s \subseteq Q$ such that:

1. $|Q_s| = k - s$

2. $P_s \cup Q_s$ are edge-disjoint.
Proof

The theorem is proved by induction on s.

Base

\[ s = 0 \quad P_0 = \phi \quad Q_0 = Q. \]

Induction step

Assume we have constructed \( P_s \) and \( Q_s \) then we have to construct \( P_{s+1} \), \( Q_{s+1} \) or \( P_{s+2} \), \( Q_{s+2} \). Let \( A = \bigcup_{\pi \in P_s} E(\pi) \). For each path \( \pi \in P \) such that \( F_E(\pi) \not\subseteq A \) let \( h_\pi \) be the first vertex of its first intersection with a path of \( Q_s \) by an edge (if \( \pi \) is edge-disjoint of \( Q_s \) take \( h_\pi = p_2 \)) then:

\[ H_s = \{ \pi[p_1, h_\pi] : \pi \in P, F_E(\pi) \not\subseteq A \} \]

Similarly for \( \pi \in P \) such that \( L(\pi) \not\subseteq A \) \( h_\pi \) is the last vertex of the last edge intersection of \( \pi \) with \( Q_s \) then:

\[ T_s = \{ \pi[t_\pi, p_2] : \pi \in P, L_E(\pi) \not\subseteq A \} \]

We shall show by induction that for all \( \tau \in P_s \) one of the following holds:

1. \( \tau = \pi \)
2. \( \tau = \pi[p_1, h_\pi] \sigma[h_\pi, t_\pi] \pi'[t_\pi, p_2] \)
3. \( \tau = \pi[p_1, h_\pi] \sigma[h_\pi, q_1] \sigma'[q_1, t_\pi] \pi'[t_\pi, p_2] \)

For \( \pi, \pi' \in P \) and \( \sigma, \sigma' \in Q \).

This assumption is trivially true for \( s = 0 \). Assume \( E(P_s) \cap (E(H_s) \cup E(T_s)) \neq \phi \).

For each path \( \tau \in P_s \) let \( l(\tau) \) be the length of its \( Q \) portion \( l_s = \sum_{\tau \in P_s} l(\tau) \). Let \( \pi[p_1, h_\pi] \) be a path of \( H_s \) which edge-intersects \( \tau \) of \( P_s \). \( \tau = \omega[p_1, h_\pi] \omega'[t, p_2] \).

Since the paths of \( P \) are edge-disjoint, \( \pi \) intersects \( \tau \) in its \( Q \) portion \( \sigma \). Let \( v \) be the first vertex of the first edge in the intersection of \( \pi \) and \( \tau \).
FIGURE 3

Replace \( \tau \) in \( P_d \) by \( \tau' = \pi[p_1, v][u, t][w][t, p_d] \). \( P_s \) will remain edge-disjoint.
Similarly proceed with paths of \( T_s \) and recompute \( H_s \) and \( T_s \). Since \( l(\tau') < l(\tau) \), \( l_s \) has decreased. Therefore, this process terminates with \( l_s = 0 \).
Henceforth it is assumed that \( E(P_s) \cap (E(H_s) \cup E(T_s)) = \phi \). We are now ready
to construct \( Q_{s+1}, P_{s+1} \) or \( Q_{s+2}, P_{s+2} \).

Case 1

There is a path \( \pi \in H_s \cap T_s \).

Then this is a \( p_1-p_2 \) path which is edge-disjoint of \( Q_s \). Choose any \( \sigma \in Q_s \) and define:

\[
P_{s+1} = P_s \cup \{ \pi \}
\]

\[
Q_{s+1} = Q_s - \{ \sigma \}
\]

Case 2

There exists a path \( \sigma \in Q_s \) such that for \( \pi \in H_s \), \( h_\pi = L_\pi(\pi) \in \sigma \) and for \( \pi' \in T_s \), \( t_\pi' = F_\pi(\pi') \in \sigma \) then
Let $B$ be a set of paths then let $F_Y(B)$ denote $\{F_Y(\pi) | \pi \in B\}$. $L_Y(B)$ is defined similarly.

Case 3

$H_s \cap T_s = \emptyset$ and for all $\sigma \in Q_s$ either $V(\sigma) \cap L_Y(H_s) = \emptyset$ or $V(\sigma) \cap F_Y(T_s) = \emptyset$.

Since $H_s \cap T_s = \emptyset$ for all paths $\pi \in H_s$, $L_Y(\pi) \in V(Q_s)$ and for all $\pi' \in T_s$, $F_Y(\pi') \in V(Q_s)$ and as

$$|Q_s| = |H_s| = |L_Y(H_s)| = |F_Y(T_s)| = k - s$$

There exist paths $\sigma^H$, $\sigma^T \in Q_s$ such that:

$$|V(\sigma^H) \cap L_Y(H_s)| \geq 2$$

$$|V(\sigma^T) \cap F_Y(T_s)| \geq 2$$

Let $h_1, h_2 \in V(\sigma^H) \cap L_Y(H_s)$, $t_1, t_2 \in V(\sigma^T) \cap F_Y(T_s)$ be all distinct vertices of $V(G)$ and $\pi_i[p_1, h_i] \in H_s$, $\pi'_i[t_i, p_2] \in T_s$ ($i = 1, 2$)

Set:

$$P_{s+1} = P_s \cup \bigcup_{i=1}^{2}\{\pi_i[p_1, h_i] \sigma^H[h_i, q_i] \sigma^T[q_i, t_i] \pi'_i[t_i, p_2]\}$$

$$Q_{s+1} = Q_s - \{\sigma^H, \sigma^T\}$$

Let $P, Q$ be two sets of edge-disjoint paths. Denote by $P \nmid Q$ the set of all paths in $P$ intersecting paths in $Q$. $P \parallel Q = P - P \nmid Q$. Using this notation and a proof similar to Theorem 5.1 we have the following generalizations of Theorem 5.1.

Corollary 5.1.1

Let $p_1, p_2, q_1, q_2$ be four distinct vertices of a graph $G$ and $P (Q)$ a set of edge-disjoint $p_1-p_2$ paths ($q_1-q_2$) paths. Let
\( k = \min \{|P \not\in Q|, |Q \not\in P|\} \). Then for all \(0 \leq m \leq k\) there exist \(s = s(m) \in \{m,m+1\}\), a set \(P_s\) of edge-disjoint \(p_1-p_2\) paths and a set \(Q_s \subseteq Q\) such that:

1. \(|P_s| = s + |P-(P \not\in Q)|\)
2. \(|Q_s| = k-s + |Q-(Q \not\in P)|\)
3. \(P_s \cup Q_s\) is a set of edge-disjoint paths.

We have also proved:

**Corollary 5.1.2**

Under the same assumptions and notations of Corollary 5.1.1 \(Q_s \subseteq Q\) and all paths of \(P_s\) are of the form \(\pi_1\sigma \pi_2\) where \(\pi_1, \pi_2\) are subpaths of two \(p_1-p_2\) paths and \(\sigma\) is a \(q_1-q_2\) subpath or a concatenation of two \(q_1-q_2\) subpaths.

Another generalization concerns the size of the set \(P\) and \(Q\) in Theorem 5.1.

**Theorem 5.2**

Let \(p_1, p_2, q_1, q_2\) be four distinct vertices of a graph \(G\) and \(P/Q\) two sets of edge-disjoint \(p_1-p_2\) paths (\(q_1-q_2\) paths), such that \(|Q| \geq |P| > \frac{2}{3}Q + \frac{1}{3}k\) then for all \(0 \leq m \leq k+1\) there exist \(s = s(m) \in \{m,m+1\}\), a set \(P_s\) of \(s\) edge-disjoint \(p_1-p_2\) paths and a set \(Q_s\) of \(|Q| - s\cdot q_1-q_2\) edge-disjoint paths such that \(P_s \cup Q_s\) are edge-disjoint.

**Proof**

Following the proof of Theorem 5.1 the result is trivial for \(m = 0\). Assume we have \(P_s\) and \(Q_s\) construct \(H_s\) and \(T_s\) as in Theorem 5.1 using the same notation.

Assume there is a path \(\sigma \in Q_s\) and a set of paths \(\pi_1 \cdots \pi_k \in H_s\) such that \(h_{\pi_j} = L_P(\pi_j) \in \sigma\).
If $k > 2$ assume $p_{n_1}$ is the first vertex on $\sigma$ and $p_{n_k}$ the last one. Let $r = \sigma[q_1, p_{n_1}] \pi_1 h_{n_k}[h_{n_k}, q_2]$. Define $Q_s = (Q_s - \{\sigma\}) \cup \{r\}$. Therefore, we may assume that for each path $\sigma$ of $Q_s$ there are at most two paths $\pi_1, \pi_2$ of $H_s$ (similarly $T_s$) such that $L_V(\pi_1), L_V(\pi_2) \in \sigma$. Now follow the proof of Theorem 5.1. If none of the cases 1, 2, 3 holds assume that $0 < t \leq \frac{1}{2} |H_s|$ paths of $Q_s$ contain the endpoints of two paths of $H_s$. Then all paths of $Q_s$ contain only one endpoint of a path of $T_s$ and no path of $Q_s$ contain an endpoint of both $H_s$ and $T_s$. Therefore $|T_s| $ paths of $Q_s$ contain an endpoint of $T_s$ and $t + (|H_s| - 2t)$ paths of $Q_s$ containing an endpoint of $H_s$. Consequently:

$$|H_s| = |T_s| = |P| - |P_s|$$

$$|Q| - |P_s| \geq |T_s| + t + (|H_s| - 2t) = |T_s| + |H_s| - t \geq |T_s| + |H_s| = \frac{3}{2} |P| - \frac{3}{2} |P_s|$$

$$\frac{2}{3} |Q| + \frac{1}{3} |P_s| \geq |P| = \frac{1}{3} |P_s| \leq k$$

Contradiction.

We were not able to prove as in Theorem 5.1 that $Q_s \subset Q$. However we can have a corollary analogous to Corollary 5.1.1.
Corollary 5.2.1

Let \( P_1, P_2, q_1, q_2 \) be four distinct vertices of a graph \( G \) and \( P(Q) \) a set of edge-disjoint paths \((q_1-q_2)\) paths.

If \( |Q \setminus P| \geq |P \setminus Q| > \frac{2}{3} |Q \setminus P| + \frac{1}{3} k \) \( k > 0 \) then, for all \( 0 \leq m \leq k \) there exists \( s = s(m) \in \{m, m+1\} \) a set \( P_s \) of edge-disjoint \( P_1 - P_2 \) paths and a set \( Q_s \) of \( q_1 - q_2 \) edge-disjoint paths, such that:

1. \( P_s = s + |P \setminus Q| \)
2. \( Q_s = |Q| - s \)
3. \( P_s \cup Q_s \) are edge-disjoint paths.

The proof is similar to Corollary 5.1.1. If we have only \( |Q| \geq |P| \) then in order to construct a \( P_s \) path we use a \( Q_s \) path containing an endpoint of \( H_s \) and a \( Q_s \) path containing an endpoint of \( T_s \). Yielding:

Theorem 5.3

Let \( P_1, P_2, q_1, q_2 \) be four distinct vertices of a graph \( G \) and \( P(Q) \) a set of edge-disjoint \( P_1 - P_2 \) paths \((q_1-q_2)\) paths. Let \( k = \min \{|P|, \frac{1}{2} |Q|\} \), then for any \( 0 \leq s \leq k \) there is a set \( P_s \) of edge-disjoint paths and a set \( Q_s \subseteq Q \) such that:

1. \( |Q_s| = |Q| - 2s \)
2. \( Q_s \cup P_s \) are edge-disjoint.

6. k-Path connectivity

The difficulty of generalizing Theorem 5.1 from two pairs of vertices to \( k \) is that in the construction of Theorem 5.1 we change the paths of \( P_s \) such that...
$E(P_a) \cap (E(H_s) \cup E(T_s)) \neq \emptyset$ holds. Consequently, $P_a$ need not be contained in $P$. Thus as a result of changing $P_{11} - P_{12}$ paths the paths between previous pairs of vertices might not be edge-disjoint. Theorem 5.2 can be proved without changing the paths, however in this case $Q_s$ is not necessarily contained in $Q$.

By this method we can prove the following:

Theorem 6.1

Let $\{(P_{11}, P_{12})\}_{i=1}^n$ be a sequence of pairs of distinct vertices in a graph $G$.

Let $P_1 \cdots P_n$ be sets of $P_{11}-P_{12} : \cdots : P_{n1}-P_{n2}$ paths respectively, such that the paths of $\bigcup_{j=1}^n P_j$ are edge-disjoint, and $|P_i| = k$. Let $(q_1, q_2)$ be a pair of vertices distinct from $P_{ij} q_1 \neq q_2$ and $Q$ a set of $k$ edge-disjoint $q_1-q_2$ paths. Let $t = (t_1, \ldots, t_n)$ ($t_i \leq k$) be a sequence of non-negative integers. $\sum_{i=1}^n t_i \leq n$. Then there exists a sequence $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_n)$ satisfying $|t_i - \bar{t}_i| \leq 1$, $\sum_{i=1}^n (t_i - \bar{t}_i) \leq 1$ and sets $P'_{11}, \ldots, P'_{n1}$ of $P_{11}-P_{12}$ paths and a set $Q'$ of $q_1-q_2$ paths such that:

1. $\bigcup_{i=1}^n P'_i \cup Q'$ is edge-disjoint

2. $|P'_{ij}| = \bar{t}_i \leq k$

3. $|Q'| = k - \sum_{i=1}^n \bar{t}_i$

Let us define:

Definition 6.1

The $k$-path connectivity of a graph $G$, is the maximal integer $pc_k(G)$, such that for every $r$-sequence of $m$ ($m \leq k$) distinct pair of vertices $(P_{11}, \ldots, P_{r1}) \ldots (P_{1r}, \ldots, P_{rr})$ in which each vertex appears at most $k$ times there are edge-disjoint paths $P_1, \ldots, P_r$ where $P_j$ is a $P_{ij}-P_{j2}$ path.
We get as an immediate result of Theorem 6.1.

**Corollary 6.1.1**

For any graph $G$

$$c_n = \max_k \{ k \mid \text{pc}_k(G) \leq n+1 \} \geq \frac{1}{n+1} \max_k \{ k \mid \text{pc}_k(G) \leq n \} - 1$$

The solution of the recursive equation of Corollary 6.1.1 gives us only exponential relation between the path connectivity and the edge-connectivity. It follows from [Nas60, Nas61, Edm73, Tut61] that if the edge-connectivity is $\kappa_E$ then there are $\frac{\kappa_E}{2}$ spanning trees. Taking a single path from each tree enables us to prove that $c_k \geq \frac{\kappa_E}{2}$. For $k = 1$ we have $\text{dpc}(G) \geq \frac{\kappa_E(G)}{2}$ which is better than the bounds implied by Corollary 6.1.1. Cypher [Cyp80] proved that if $\kappa_E(G) \geq k+2$ for $k \leq 5$ then $\text{dpc}(G) \geq k$. The more refined methods used in this paper can be used to get better results. For example:

**Proposition 6.1**

If $\kappa_E(G) \geq 4$ then $\text{dpc}(G) \geq 3$

**Proof**

The proof is given briefly since it does not use any new ideas. Assume we have three pairs of distinct vertices $(p_{i1}, p_{i2})$ $(i = 1,2,3)$, and sets $P_i$ $(i = 1,2,3)$ paths, such that $P_i$ consists of four edge-disjoint $p_{i1} - p_{i2}$ paths. Following the proof of Theorem 5.1 there are two cases:

**Case 1**

There exists a $p_{11} - p_{12}$ path $\pi_1$ and three $p_{21} - p_{22}$ paths $\sigma_1$, $\sigma_2$, $\sigma_3$ such that $\{\pi_1, \sigma_1, \sigma_2, \sigma_3\}$ are edge-disjoint. Then the proof is as in [Cyp80] with $\pi_1$ as the barrier and $\{\sigma_1, \sigma_2, \sigma_3\}$ as the skeleton for the $p_{31} - p_{32}$ paths.

**Case 2**
There exists a set $P'_1$ of two $p_{11}$-$p_{12}$ paths and a set $P'_2$ of two $p_{21}$-$p_{22}$ paths such that $P'_1 \cup P'_2$ is edge-disjoint. In this case, we have from Theorem 5.1 that one path of $P'_2$ passes through $p_{11}$ and the second through $p_{12}$.

![Diagram](https://example.com/diagram.png)

**FIGURE 5**

Now consider $P_3$. If it has a head on a $P'_1$ path and a tail on a $P'_2$ path then we can construct the required $p_{31}$-$p_{32}$ path and delete these two paths. Therefore all heads and tails are on $P'_1$ paths or $P'_2$ paths. Assume they are on $P'_1$. Furthermore, assume that all heads are on $\pi_1$ and all tails are on $\pi_2$. (otherwise as in Theorem 5.1 a $p_{31}$-$p_{32}$ path can be constructed using only one $p_{11}$-$p_{12}$ path). If all heads are on $\pi_1$ use the heads closest to $p_{11}$ and $p_{12}$ and construct a $p_{11}$-$p_{12}$ path $\pi'_1$ edge-disjoint of $\pi_2$, $P'_2$ and the remaining two $p_{31}$-$p_{32}$ paths (denote the remaining two paths by $P'_3$). Reconstruct the heads of $P'_3$. Following the proof of Theorem 5.1 we may assume that the new heads are edge-disjoint of $\pi'_1$. As all the tails are on $\pi_2$ we can construct the required $p_{31}$-$p_{32}$ path by using only $\pi_2$ (if it has a head) or an additional $p_{21}$-$p_{22}$ path $\omega$ (if the head is on $\omega$).

7. Conclusions

Cypher [Cyp80] conjectured that if the edge-connectivity of a graph is $\kappa_E = 2k - 1$ then between any $\kappa_E$ pairs of vertices in the graph there are edge-
disjoint paths. From this conjecture follows that for even \( t + 1 \) edge-connectivity implies path-connectivity \( \geq t \).

This conjecture is strengthened by the following:

In this paper and in [PS78] the conjecture is proved for \( t = 2 \). Furthermore in [1Z] this conjecture is proved for \( t = 3,4 \). The case \( t = 3 \) is stronger than the result in this paper.

For an even number of pairs of vertices the following example shows that at least \( t + 1 \) edge-connectivity is needed to ensure \( t \) path-connectivity.

**Example**

Let \( V(G) = \{x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}, v_1, v_2\} \) be the vertices of a graph \( G \). The graphs \( K_{2n} = G[x_1, \ldots, x_{2n}] \) and \( K'_{2n} = G[y_1, \ldots, y_{2n}] \) are cliques. The vertex \( v_1 \) is connected in \( G \) to the vertices \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) and the vertex \( v_2 \) is connected to \( \{x_{n+1}, \ldots, x_{2n}, y_{n+1}, \ldots, y_{2n}\} \).

It is easy to see that the edge-connectivity of \( G \) is \( 2n \). But the path-connectivity of \( G \) is less than \( 2n \) since for the \( 2n \) pairs:

\[
(v_1, v_2), (x_i, y_i), i = 1, \ldots, 2n-1
\]

\( n \) \( x_i - y_i \) paths pass through either \( v_1 \) or \( v_2 \) using up all its \( 2n \) edges, leaving no free edges for an edge-disjoint \( v_1 - v_2 \) path.
8. References

[CC80]

[Cyp80]

[Edm73]

[I2]

[Lei80]

[Leo70]

[LT77]

[Nas60]
[Nas61]

[PS78]

[Val81]

[Tho80]

[Tut61]

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