ON THE DENSEST PACKING OF CIRCLES IN CONVEX FIGURES

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ABSTRACT

Let the term "unit circle" mean a circle of radius 1. The main result of this paper is:

Theorem 1: Let $C_1, ..., C_n$ be $n \geq 2$ disjoint unit circles in the plane and let $B$ be the smallest convex figure containing them. Then the area of $B$ is greater than $n\sqrt{2}$.

This generalizes a similar well known result, in which $B$ is assumed to be a convex hexagon. It is also indicated how the proof of Theorem 1 can extended to prove that the area of the smallest convex figure $B$ containing $n$ disjoint unit circles is $n\sqrt{2} + \Omega(\sqrt{n})$. 
1. INTRODUCTION

The following result had been proved by several authors (see e.g. [1, 3, 4]).

**Theorem 1':** Let $B$ be a convex hexagon containing $n$ disjoint unit circles, and let $a(B)$ be the area of $B$. Then

$$a(B) \geq n \sqrt{2}.$$

We shall show that if $n \geq 2$, the assumption on $B$ can be replaced by the weaker assumption that $B$ is any convex figure containing the $n$ circles. In particular $B$ can be the convex hull of the $n$ circles.

The paper has 5 sections, including this introductory section. In the next section we give the required definitions. In Section 3 we use ideas from Fejes Toth's proof of Theorem 1' to reduce the problem to a combinatorial problem [1, p. 164]. In Section 4 we give a geometric property of the problem, and in Section 5 we combine these results to prove Theorem 1.
2. PRELIMINARIES

Let $C_1, \ldots, C_n$ be $n$ disjoint unit circles, and let $B$ be the convex hull of $C_1, \ldots, C_n$. A "balanced partition" of $B$ is the partition of $B$ to $n$ disjoint figures $F_1, \ldots, F_n$, $F_i$ containing $C_i$, obtained by separating each pair of adjacent circles by a perpendicular bisector of the segment connecting their centers. Each $F_i$ in that partition is either a polygon or a "corner-figure" whose boundary consists of one circular arc and several straight line segments.

A "balanced graph" for $B$ is a graph $H(V,E)$ obtained from the balanced partition of $B$ as follows: The vertices $V$ of $H$ are

a) The intersection points of the segments separating the circles.

b) The intersection points of the boundary of $B$ with these segments.

c) One or more points on each circular arc on the boundary of $B$.

The edges $E$ of $H$ are those line segments of the balanced partition of $B$ which connect 2 vertices of $H$ but do not contain any vertex of $H$ as an interior point (see Figure 2.1). Note that $H$ is a planar graph with $n+1$ "faces": $n$ faces which correspond to the figures $F_1, \ldots, F_n$, and one "infinite" face.

The degree of a vertex in $H$ is the number of edges emanate from it. Note that the only vertices in $H$ whose degree is less than $3$ are the points on the circular arcs of $B$. (The fact that vertices (a) have degree $\geq 3$ follows from the fact that the perpendicular bisectors of the 3 sides of a triangle meet. For vertices (b) this fact is immediate.)

A "standard graph" is a balance graph which has exactly one
vertex on each circular arc. A "standard corner-figure" is a corner-figure which has exactly one vertex on its circular arc.

Figure 2.1 gives a standard graph for a convex hull of 3 circles.

A standard graph $H(V,E)$ of a convex hull of 3 circles.

$V = \{P_1, \ldots, P_7\}$ $E = \{(P_1,P_2),(P_2,P_3),\ldots,(P_6,P_1),(P_2,P_7),(P_4,P_7), (P_6,P_7)\}$. $H$ has 3 "standard corner figures".

For an integer $k$, let $a(k)$ denote the area of the smallest $k$-gon circumscribing the unit circle. Since this $k$-gon is a regular $k$-gon we have that

$$a(k) = k \text{tg}(\pi/k).$$

A corner-figure $F$ with $k$ vertices (and $k$ edges) on its boundary is called a "regular" if all its
Let \( H \) be a balanced graph with figures \( F_1, \ldots, F_n \), and let \( F_i \) has \( k_i \) edges on its boundary. Then:

\[
k(H) = k_1 + \ldots + k_n.
\]

3. REDUCTION OF THEOREM 1 TO A COMBINATORIAL PROBLEM

For the next lemmas, we need the following inequality known as "Jensen inequality". Let \( f \) be a real function which is convex in a domain \( D \), and let \( x_1, \ldots, x_n \in D \). Then:

\[
f(x_1) + \ldots + f(x_n) \geq nf[(x_1 + \ldots + x_n)/n].
\]

Lemma 3.1 If there exists a proper graph \( H \) for \( B \) such that \( k(H) \leq 6n \), then \( a(B) \geq n\sqrt{2} \).

Proof: Let \( F_1, \ldots, F_n \) be the figures in \( H \) containing \( C_1, \ldots, C_n \), and let \( k_i \) be the number of edges of \( F_i \). Then:

\[
a(B) = a(F_1) + \ldots + a(F_n) \geq a(k_1) + \ldots + a(k_n) \geq na[k(H)/n] \geq na(6).
\]

The first inequality above follows since \( H \) is proper, the second by Jensen inequality since \( a(x) = xtg(\pi/4) \) is a convex function at \((0, \pi/2)\), and the last by the fact that \( k(H)/n = 6 \) and \( a(x) \) is a decreasing function. The lemma now follows since \( a(6) = 6tg(\pi/6) = \sqrt{3} \).

Lemma 3.2 Let \( H(V,E) \) be a balanced graph for \( B \). Let \( b \) denote the number of vertices of degree 2 in \( H \), and let \( m \) denote the number of vertices (and edges) of \( H \) on the boundary of \( B \). If

\[
(*) \quad m + 6 \geq 2b.
\]
Proof: Let \(|V| = \nu\) and \(|E| = \epsilon\). It follows directly from the definitions that:

(1) \(k(H) = k_1 + \ldots + k_n = 2\epsilon - m\).

By Euler formula for planar graph we have:

(2) \(n = \epsilon + 1 - \nu\)

Hence, it is enough to show that \(2\epsilon - m \leq 6(\epsilon + 1 - \nu)\), which is equivalent to:

(**) \(6\nu \leq 4\epsilon + 6 + m\).

Let \(d(x)\) denote the degree of vertex \(x\). Then:

\[2\epsilon = \sum_{x \in V} d(x) \geq 3(\nu - b) + 2b = 3\nu - b\]

or, equivalently

(3) \(6\nu \leq 4\epsilon + 2b\).

By the assumption (**), \(2b \leq m + 6\). Substituting this in (3) gives (**). \(\Box\)

**Lemma 3.3** Let \(H_s\) be a standard graph for \(B\), and let \(b\) and \(m\) be the numbers of the vertices of degree 2 in \(H_s\) and of the edges of \(H_s\) on the boundary of \(B\) respectively. Then \(2b \leq m\).

Proof: In the standard partition, \(b\) is equal to the number of circular arcs on the boundary of \(B\), which is not larger than the number of the circles in \(B\) touching the boundary of \(B\). Since each pair of adjacent circles is separated by a line, each pair of consecutive circular arcs on the boundary of \(B\) must be separated by a vertex of \(H\) of degree \(\geq 3\) (namely, a vertex which corresponds to the intersection point of the boundary of \(B\) with a line separating the circles which correspond to
boundary of $B$ is at least twice the number of arcs on the boundary of $B$, and hence at least $2b$. □

**Corollary:** Let $B$ be the convex hull of the unit circles $C_1, \ldots, C_n$, and let $H_s$ be a standard graph of $B$. Then if there is a proper graph $H_p$ for $B$ which is obtained from $H_s$ by adding $\epsilon \leq 6$ vertices to the boundary of $B$, then $\alpha(B) > n/\sqrt{12}$.

**Proof** Let $m$ and $b$ be as in Lemma 3.3. Then $2b \leq m$. Let $b' = b + \epsilon$ and $m' = m + \epsilon$ be the corresponding numbers in the proper graph $H_p$ obtained from $H_s$ as described. Then $2b' \leq m' + \epsilon \leq m' + 6$. Hence, $H_p$ is a proper graph satisfying the conditions of Lemma 3.1. The corollary follows. □

4. A LOWER BOUND ON THE AREA OF CORNER-FIGURES

Let $F$ be a standard corner-figure circumscribing a unit circle $C$ with center $O$. Assume that $F$ has $k$ edges, and let $P_1, \ldots, P_k$ be the vertices of $F$, occurring in this order, where $P_1$ is the unique vertex on the circular arc of $F$. Assume further that for $i=2, \ldots, k-1$ edge $P_i P_{i+1}$ touches $C$ at $Q_i$, and let $Q_k Q_1$ be the circular arc of $C$ (see Figure 4.1). Then, by the definition of balanced partition and since $P_k P_1 P_2$ is a portion of the boundary of $B$, $\angle Q_1 P_2 Q_2 \leq \pi/2$ and $\angle Q_{k-1} P_k Q_k \leq \pi/2$. (To see this, note that $OO'$ is a portion of a segment $OO'$, where $O'$ is a center of one of the unit circles embedded in $B$. If $\angle Q_1 P_2 Q_2 > \pi/2$, then the distance of $O'$ from the boundary of $B$ is less than 1.) Define the "length of $F$"
Lemma 4.1 For an integer \( k \geq 3 \) and a real number \( \alpha, 0 \leq \alpha \leq \pi \), let \( A(k, \alpha) = \inf(a(F) | F \text{ is a standard corner figure of length } \alpha \text{ with } k \text{ sides}) \). Then

(a) \[ A(3, \alpha) \geq \frac{\alpha}{2} + 2 \tan\left(\frac{(2\pi - \alpha)}{4}\right) \]

(b) If \( \alpha \leq \frac{\pi}{2} \), then \[ A(4, \alpha) \geq \frac{\alpha}{2} + 3 \tan\left(\frac{(2\pi - \alpha)}{6}\right) \]

(c) If \( k \geq 5 \), or \( (k = 4 \text{ and } \alpha \geq \frac{\pi}{2}) \), then

\[ A(k, \alpha) \geq \frac{\alpha}{2} + 2 + (k-3) \tan\left(\frac{(\pi - \alpha)}{2(k-3)}\right) \]

Proof: Without loss of generality, assume that \( F \) circumscribes a unit circle \( C \) (otherwise replace the edges of \( F \) by parallel edges which touch \( C \)). Referring to Figure 4.1, let \( \gamma_1 = \angle Q_1 O P_2 \), \( \gamma_2 = \angle P_k O Q_k \), and for \( i = 1, \ldots, k-3 \), let \( \beta_i = \angle P_{i+1} O Q_{i+1} \). Then

\[ a(F) = \frac{\alpha}{2} + \tan\gamma_1 + \tan\gamma_2 + \tan\beta_1 + \ldots + \tan\beta_{k-3} \]
And, by the discussion above

(1) \( \gamma_1 > \pi/4 \) and \( \gamma_2 > \pi/4 \);

(2) \( \gamma_1 + \gamma_2 + \beta_1 + \ldots + \beta_{k-3} = (2\pi-\alpha)/2 \).

By the convexity of the tangent function and by Jensen inequality, \( a(F) \) attains its minimum if all the \( \gamma_j \)'s and the \( \beta_i \)'s are equal. However, by (1), this can happen only if \( k=3 \), or \( k=4 \) and \( \alpha \leq \pi/2 \), and in these cases, \( \gamma_j = \beta_i = (2\pi-\alpha)/(2(k-1)) \). This proves (a) and (b).

If \( k \geq 5 \), or \( k=4 \) and \( \alpha > \pi/2 \), \( a(F) \) attains its minimum only if \( \gamma_1 = \gamma_2 = \gamma \) and \( \beta_1 = \ldots = \beta_{k-3} = \beta \). Hence, in this case

\[
a(F) > \frac{\pi}{2} + 2 \tan \gamma + (k-3) \tan \beta = \frac{\alpha}{2} + 2 \tan \gamma + (k-3) \tan \left[ \frac{(2\pi-\alpha-4\gamma)}{2(k-3)} \right].
\]

Since (1) implies that in this case \( \gamma > \beta \), one can easily verify by computing the derivative of \( a(F) \) with respect to \( \gamma \) that \( a(F) \) increases with \( \gamma \), and hence \( a(F) \) attains its minimum value when \( \gamma \) attains its minimum value, which is \( \pi/4 \). This implies (c). \( \square \)

Note that the lower bounds on \( A(k, \alpha) \) given in the lemma above decrease with both \( k \) and \( \alpha \).

**Corollary** Let \( F \) be a corner figure. Then \( a(F) > a(6) \).

**Proof.** Assume that \( F \) is a standard corner-figure. Since \( A(k, \alpha) \) decreases with \( k \), we can assume that \( F \) has \( k \geq 5 \) sides. Hence,

\[
a(F) > A(k, \alpha) = \frac{\alpha}{2} + 2 + (k-3) \tan \left[ \frac{(\pi-\alpha)}{2(k-3)} \right] > \frac{\alpha}{2} + 2 + (k-3) \left[ \frac{\tan \left( \frac{\pi-\alpha}{2(k-3)} \right)}{2(k-3)} \right] = 2 + \frac{\pi}{2} > \sqrt{12} = a(6).
\]

The second inequality above follows from the fact that \( \tan x > x \) for \( x \in (0, \pi/2) \). \( \square \)
5. COMPLETION OF THE PROOF OF THEOREM 1

Let \( H_s \) be the standard graph of \( B \), and \( F_1, \ldots, F_c \) be the standard corner-figures of \( H_s \). Let \( \alpha_i \) be the length of \( F_i \). We start with an observation concerning the \( \alpha_i \)'s.

**Observation 5.1** \[ \alpha_1 + \ldots + \alpha_c = 2\pi. \]

**Proof.** Consider the convex polygon obtained by replacing the circular arcs on the boundary of \( B \) by the continuations of the straight line segments on the boundary of \( B \). Then one can easily verify that this polygon has \( c \) sides, and the angle \( \beta_i \) which corresponds in this polygon to the circular arc \( \alpha_i \) satisfies \( \beta_i = \pi - \alpha_i \). The observation follows since in each convex polygon with angles \( \beta_1, \ldots, \beta_c \),

\[ (\pi - \beta_1) + \ldots + (\pi - \beta_c) = 2\pi. \]

Let \( F \) be a corner-figure with \( k \) edges, and let \( \varepsilon \) be the minimal integer satisfying \( a(F) \geq a(k+\varepsilon) \). Then \( \varepsilon \) is the "deficiency of \( F \)" to be denoted by \( d(F) \) (that is, \( d(F) \) is the minimal number of vertices which should be added to the circular arc of \( F \) to make it proper).

\( d(B) \) is defined by: \( d(B) = d(F_1) + \ldots + d(F_c) \). By the corollary at the end of Section 3, Theorem 1 will follow from:

**Theorem 5.1:** Let \( C_1, \ldots, C_n \) be disjoint unit circles, and let \( B \) be the convex hull of \( C_1, \ldots, C_n \). Then \( d(B) \leq 6 \).

Theorem 5.1 will follow easily from the following lemma:

**Lemma 5.1:** Let \( F \) be a corner figure of length \( \alpha \). Then

\[ \frac{\alpha}{d(F)} > \frac{2\pi}{\sqrt{2}}. \]

(2) For convenience, we define \( \alpha/0 = \infty \) for \( \alpha > 0 \).
Proof. Let $F$ have $k$ sides. Then $k \geq 3$. On the other hand, by the corollary to Lemma 4.1, $a(F) > a(6)$. It follows that $d(F) \leq 3$.

We consider 4 cases:

1) $\alpha > 6\pi/7$. In this case, $\alpha/d(F) > (6\pi/7)/3 = 2\pi/7$.

2) $6\pi/7 \geq \alpha > 4\pi/7$. It is enough to show that in this case $d(F) \leq 2$.

This is immediate if $k \geq 4$. Thus, assume that $k=3$. Then, by Lemma 4.1, using the fact that $A(k,\alpha)$ decreases with $\alpha$, we have:

$$a(F) \geq A(3,6\pi/7) \geq 3\pi/7 + 2\tan(2\pi/7) \approx 3.85... > a(5) = 3.63...$$

3) $4\pi/7 \geq \alpha > 2\pi/7$. It is enough to show that $d(F) \leq 1$. This is immediate if $k \geq 5$. We have to show that this is true also when $k=3$ or $k=4$. Assume first that $k=3$. Then, by the same arguments as above:

$$a(F) \geq A(3,4\pi/7) \geq 2\pi/7 + 2\tan(5\pi/14) = 5.05 > a(4) = 4.$$

If $k=4$, then

$$a(F) \geq A(4,4\pi/7) \geq 2\pi/7 + 2 + \tan(3\pi/14) = 3.69... > a(5) = 3.63...$$

4) $2\pi/7 \geq \alpha$. We shall show that in this case $d(F) = 0$.

This is clear for $k \geq 6$. For $k=3$, we have:

$$a(F) \geq A(3,2\pi/7) \geq \pi/7 + 2\tan(3\pi/7) = 9.21... > a(3) = 5.19...$$

For $k=4$:

$$a(F) \geq A(4,2\pi/7) \geq \pi/7 + 3\tan(2\pi/7) = 4.22... > a(4) = 4.$$ 

For $k=5$:

$$a(F) \geq A(5,2\pi/7) \geq \pi/7 + 2 + 2\tan(5\pi/28) = 3.70... > a(5) = 3.63...$$

This completes the proofs of the lemma. □
Proof of Theorem 5.1. Let $H_s$ be the standard graph of $B$. Let $F_1,\ldots,F_c$ be the standard graphs of $H_s$, and let $a_i$ be the length of $F_i$. Then

$$2\pi = a_1 + \ldots + a_c \geq \sum_{i \mid d(F_i) > 0} a_i \geq \sum_{i \mid d(F_i) > 0} d(F_i)[a_i/d(F_i)]$$

$$> [d(F_1) + \ldots + d(F_c)]2\pi/7 = d(B)2\pi/7.$$ 

Thus, $2\pi > d(B)(2\pi/7)$, which implies that $d(B) < 7$. Hence, $d(B) \leq 6$. \hfill \Box 

Note. In Theorem 5.1, 6 cannot be replaced by a smaller number: In Figure 2.1, each of the 3 standard corner figures have 4 sides, and the area of each of them is

$$\pi/3 + 2 + g(\pi/6) = 3.624 < a(5) = 3.63.$$ 

and hence, $d(F) = 2$ for each of them, which implies that $d(B) = 6$ in this case.

Concluding Remark. By a more careful analysis, which takes into account the strictly positive differences between the areas of proper corner figures of $k$ sides and $a(k)$, and the fact that the length of the boundary of a convex figure containing $n$ disjoint unit circles is greater than $2\pi \sqrt{n}$, one can prove the following stronger version of Theorem 1:

Theorem 2. Let $S(n) = \inf\{a(B) \mid B$ is a convex containing $n$ disjoint unit circles\}. Then $S(n) - n\sqrt{2} \geq \Omega(\sqrt{n})$. 

(3) $f(n) = \Omega(g(n))$ if for some $c_1,c_2 > 0$ and for all large enough $n$, 

$$c_1 \leq f(n) \leq c_2 g(n).$$
By [2, Theorem 3.2], for each \( n \), a circle of area \( n\sqrt{2} \) contains a set of \( n \) points, the distance between any pair of which is at least 1.

Let \( r_n = (n/\pi)^{1/2} 12^{1/4} \) be the radius of such a circle. Then, clearly a circle of radius \( r_n + 1 \) contains \( n \) disjoint unit circles. The area of that larger circle is \( n\sqrt{2} + \Omega(\sqrt{n}) \) combining this with Theorem 2, we get:

\[
\frac{n}{\sqrt{2}} + \Omega(\sqrt{n}) \leq S(n) \leq \frac{n}{\sqrt{2}} + \Omega(\sqrt{n})
\]
or

\[
S(n) = \frac{n}{\sqrt{2}} + \Omega(\sqrt{n}).
\]
REFERENCES


