SUPER-NETS, AND THEIR HIERARCHY

by

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ABSTRACT

In this paper we introduce the concept of Super-Net, which subsumes various extensions of Petri nets, proposed earlier in the literature. We consider languages associated with both non-labeled as well as labeled Super-Nets, and use these languages to establish a hierarchy between various types of nets. In the last section we apply this hierarchy to show that Petri net languages are not closed under Kleene star.
1. Introduction

An extensive literature is presently available demonstrating the suitability of Petri nets to the concise and precise modeling of systems involving concurrency (see e.g. [2,3,4,10]). In order to further expand the modeling power of Petri nets, numerous modifications and extensions have been proposed. In particular, we mention the introduction of inhibitor arcs [1,5,10], OR-logic transitions [1], Boolean-type places [13,14,15], and places with finite capacity [7].

In this paper we introduce the concept of Super-Net which subsumes all the above extensions. Furthermore, Super-Nets also include "emptying" arcs, which are intended to model the reset-to-zero facility of counting devices.

In modeling power of various net families can be precisely compared by means of formal languages associated with such nets [5,6,10,16].

In this connection the transitions of a net are frequently considered to be labeled by letters from some finite alphabet $\Sigma$ [5,6,10].

In this paper we consider languages associated with both non-labeled as well as labeled Super-Nets. We define various special types of Super-Nets and use their associated languages in order to establish a hierarchy between them. Some of our theorems reformulate known results. However, we consider most of our results to be new.
(1) $P$ and $T$ are finite sets of places and transitions, respectively.

(2) $P \cap T = \emptyset$, $P \cup T \neq \emptyset$.

(3) $V$ is a function,

$$ V: (P \times T) \cup (T \times P) \rightarrow \omega \cup \{I, E, L\} $$

Note $I, E, L$ are symbols indicating "Inhibiting", "Emptying", and "Logical" arcs, respectively.

(4) $V(T \times P) \subseteq \omega$

(5) $K$ is a function

$$ K: P \rightarrow \{\infty\} \cup \omega \times \{A, R\} $$

Note $A, R$ are symbols indicating "Absorbing", and "Restricting" places.

If $k(p) = \infty$, we say that the place $p$ has infinite capacity. If $k(p) = (k, A)$ or $k(p) = (k, R)$ we say that the place $p$ has finite capacity $k \in \omega$. We denote by $k(p)$ the capacity ($\infty$ or $k \in \omega$) of the place $p$.

**Definition 2.2.** A marked SUP-Net is a pair $S = (N, M)$, where $N$ is a SUP-Net and $M$ is a marking of $N$, i.e. a function $M: P \rightarrow \omega$, satisfying the condition

$$(\forall p \in P) [k(p) \in \omega \rightarrow M(p) \leq k(p) ]$$

A marked SUP Net $S = (P, T, V, K, M)$ is represented graphically as follows:

1. Places are represented by circles ($\bigcirc$)
2. Each place $p$ is labeled by $p/K(p)$. 
5. The transition \( t \in T \) is connected by a directed arc to the place \( p \in P \), iff \( V(t,p) > 0 \). The arc is labeled by \( V(t,p) \).

6. The integer \( m = M(p) \) is written inside the circle representing \( p \).

Usually, one does not write 0 inside the circle.

An example of a marked SUP-Net is shown in Fig. 1.

![Diagram of a marked SUP-Net]

Fig. 1. - Example of marked SUP-Net

In the sequel we need the following:

**Definition 2.3.** Let \( S = (P,T,V,K,M) \) be a marked SUP-Net. We define a function \( W: P \times T \to \omega \) as follows:

\[
W(p,t) = \begin{cases} 
V(p,t) & \text{if } V(p,t) \in \omega \\
0 & \text{otherwise}
\end{cases}
\]
Definition 2.4. Let $S = (P, T, V, K, M)$ be a marked SUP-Net. A transition $t \in T$ is enabled iff the following conditions are satisfied:

1. $(\forall p \in P)[V(p,t) \in \omega \rightarrow M(p) \geq V(p,t)]$.
2. $(\forall p \in P)[V(p,t) = I \rightarrow M(p) = 0]$.
3. $(\forall p \in P)[V(p,t) = E \rightarrow M(p) \geq 0]$.
4. $(\exists p \in P) V(p,t) = L \rightarrow (\exists p \in P)[V(p,t) = L \land M(p) > 0]$.
5. $(\forall p \in P)[K(p) \in (\omega, R) \rightarrow M(p) + V(t,p) - W(p,t) \leq K(p)]$.

Definition 2.5. Let $S = (P, T, V, K, M)$ be a marked SUP-Net and $t \in T$ an enabled transition of $S$. We define the marking $M'$ of $N = (P, T, V, K)$ as follows:

$$(\forall p \in P)M'(p) = \min[M(p) + V(t,p) - W(p,t), k(p)].$$

We say that $M'$ is obtained from $M$ by firing $t$ (notation: $M[t] > M'$).

Frequently, it is convenient to represent a marking $M$ by the vector $(M(p_1), M(p_2), \ldots, M(p_n))$, where $P = \{p_1, \ldots, p_n\}$.

For the example of Fig.1 we obtain the following "firing sequence".

$$(0,3,1) [t_2] > (0,5,1)$$
$$(0,5,1) [t_2] > (0,6,1)$$
$$(0,6,1) [t_5] > (0,6,0)$$
$$(0,6,0) [t_4] > (0,2,1)$$
$$(0,2,1) [t_1] > (1,2,1)$$
$$(1,2,1) [t_1] > (2,2,1)$$
$$(2,2,1) [t_1] > (3,2,1)$$
$$(3,2,1) [t_5] > (2,2,0)$$
$$(2,2,0) [t_3] > (0,2,0)$$
Definition 2.6. Let $N = (P, T, V, K)$ be a SUP-Net. For every $t \in T$, we set

$$t^* \triangleq \{p \in P | V(t, p) \neq 0\}.$$

$$t^* \triangleq \{p \in P | V(p, t) \neq 0\}.$$

We call every $p \in t^*$ an output place of $t$, and every $p \in t^*$ an input place of $t$.

3. Classification of Super-Nets

In this section we represent various types of nets as special cases of SUP-Nets.

Definition 3.1. The following table defines various types of nets as special cases of SUP-Nets, by restricting $\text{range}(V)$ and $\text{range}(K)$.

<table>
<thead>
<tr>
<th>TYPE OF NET</th>
<th>RANGE($V$)</th>
<th>RANGE($K$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GP-Net</td>
<td>$\omega$</td>
<td>${\infty}$</td>
</tr>
<tr>
<td>P-Net</td>
<td>${0,1}$</td>
<td>${\infty}$</td>
</tr>
<tr>
<td>I-Net</td>
<td>${0,1} \cup {I}$</td>
<td>${\infty}$</td>
</tr>
<tr>
<td>E-Net</td>
<td>${0,1} \cup {E}$</td>
<td>${\infty}$</td>
</tr>
<tr>
<td>L-Net</td>
<td>${0,1} \cup {L}$</td>
<td>${\infty}$</td>
</tr>
<tr>
<td>A-Net</td>
<td>${0,1}$</td>
<td>$\omega \times {A}$</td>
</tr>
<tr>
<td>R-Net</td>
<td>${0,1}$</td>
<td>$\omega \times {R}$</td>
</tr>
</tbody>
</table>

The above defined I-Nets coincide with $TN_{com}$ in [1]. They are called inhibitor nets in [5] and "Petri nets with inhibitor arcs" in [10]. The above L-Nets coincide with $TN_{log}$ in [1]. The Boolean-type Petri nets, introduced in [14] coincide with A-Nets, where $\text{range}(K) = \{(1, A)\}$.

The above-defined types of Super-Nets can be combined into more complex classes. E.g., the mixed-type nets discussed in [15] are
Similarly, the place-transition (P/T) nets of [7] are combinations of GP-type and R-type Super-Nets.

4. Super-Net Languages

With a given marked SUP-Net $S = (P,T;V,K,M)$ we associate a language $L(S)$ over the alphabet $T$.

**Definition 4.1.** Let $N = (P,T;V,K)$ be a SUP-Net, and let $w \in T^+$, i.e. $w$ is a finite string of transitions $W = t_1 t_2 \ldots t_r$. $w$ is called a firing sequence of the marked SUP-Net $S = (N,M)$ iff there exist marking $M_1, \ldots, M_r$ such that

$$M[t_1>M_1, M_1[t_2>M_2, \ldots, M_{r-1}[t_r>M_r.$$ 

In this case we write $M[w>M_r$, and say that $M_r$ is reachable from $M$. We also write $M[\lambda>M$ for every marking $M$, where $\lambda$ denotes the empty sequence.

**Definition 4.2.** Let $S = (P,T;V,K,M)$ be a marked SUP-Net. We define its language $L(S)$ as follows:

$$L(S) \triangleq \{ x \in T^* \mid (\exists M') M[x>M'] \}.$$ 

In this paper we are also concerned with labeled Super-Nets and their languages.

**Definition 4.3.** A labeled SUP-Net is a triple $r = (S, \Sigma, \eta)$, where $S = (P,T;V,K,M)$ is a marked SUP-Net, $\Sigma$ is a finite alphabet and $\eta$ is a mapping $\eta: T \to \Sigma \cup \{ \lambda \}$. $r$ is called $\lambda$-free iff $\eta(T) \subseteq \Sigma$. The language of $r$ is defined by
If $M[n(w)] > M'$ where $w \in \mathcal{T}^*$ we also write $M[n(w)] > M'$, and say that $M'$ is obtained from $M$ by "firing $n(w)$". Clearly every marked SUP-Net $S$ is also a labeled $\lambda$-free SUP-Net $\Gamma = (S; \Sigma, \eta)$ where $\Sigma = \mathcal{T}$ and $(\forall t \in \mathcal{T}) \eta(t) = t$.

**Definition 4.4.** Let $L$ be a language over some finite alphabet $\Sigma$. We say that $L$ is $GP$-realizable iff $L = L(S)$ for some marked GP-Net $S = (P, \Gamma, V, K, M)$ with $\Gamma = \Sigma$. $L$ is $LGP$-realizable iff $L = L(\Gamma)$ for some labeled $\lambda$-free GP-Net $\Gamma = (S; \Sigma, \eta)$. $L$ is $L^\lambda GP$-realizable iff there exists an arbitrary labeled GP-Net $\Gamma$ with $L = L(\Gamma)$. We denote by $GPL$, $LGPL$, $L^\lambda GPL$ the sets of all GP-realizable, LGP-realizable, and $L^\lambda GP$-realizable languages, respectively.

In a similar way we associate sets of languages with the other types of Super-Nets defined above (Definition 3.1).

Evidently, all the languages defined above are prefix languages. Our language families $GPL$, $LGPL$, and $L GPL$ correspond to the classes $P^f$, $P$, and $P^\lambda$ in [10, p.157], respectively.

5. **The Hierarchy of Super-Nets (without labeling)**

In this section we study the hierarchy of the following language families: $GPL$, $PL$, $IL$, $EL$, $EL$, $AL$, and $RL$.

**Lemma 5.1.** The languages in $AL$, $LAL$, $L^\lambda AL$, $RL$, $LRL$, $L^\lambda RL$ are regular.

**Proof.** This result follows immediately, since the corresponding SUP-Nets all have finite sets of reachable markings. Hence each such SUP-Net may be viewed as finite automaton with the set of reachable markings as its
The following lemma is an immediate consequence of a result derived in [11, 12].

Lemma 5.2. Let \( L \) be a language over \( \Sigma \) in either \( RL \) or \( PL \) or \( IL \) or \( GPL \). Assume \( x\sigma \in L \), where \( x \in \Sigma^+ \) and \( \sigma \in \Sigma \). Furthermore, let \( y \) be a permutation of \( x \), with \( y \in L \). Then \( y\sigma \in L \).

For two sets \( A \) and \( B \), we write \( A \subseteq B \) to state that \( A \) is a subset of \( B \), and \( A \subset B \) to state that \( A \subseteq B \) and \( A \neq B \).

The following theorem is proven in [12].

Theorem 5.1.
(a) \( PL \subseteq IL \)
(b) \( PL \subseteq LL \)
(c) \( IL \) and \( LL \) are not comparable, i.e. neither \( IL \subseteq LL \) nor \( LL \subseteq IL \) holds.

Theorem 5.2.
(a) \( RL \subseteq PL \)
(b) \( RL \subseteq AL \)
(c) \( AL \) and \( PL \) are not comparable.

Proof. (a) Let \( S \) be a marked \( R \)-Net. One easily verifies that there exists a marked \( P \)-Net \( S' \), which is equivalent to \( S \), i.e. \( L(S) = L(S') \). This is illustrated in Fig. 2. (henceforth, we omit the label '1' of arcs). Thus, \( RL \subseteq PL \).

On the other hand, there exist languages in \( PL \) which are not regular, e.g. the language of the marked \( P \)-Net \( S_3 \) of Fig. 3.
Fig. 2.- (a) Example of a marked $R$-Net $S_1$.
(b) An equivalent marked $P$-Net $S_2$, i.e. $L(S_1) = L(S_2)$.

Fig. 3.- Example of a marked $P$-Net $S_3$.

Thus, by Lemma 5.1, we have $RL \subseteq PL$.

(b) With every marked $R$-Net one easily associates an equivalent marked $A$-Net (see Fig. 4). Hence $RL \subseteq AL$. 
To show that $RL \subseteq AL$, we consider the marked $A$-Net $S_5$ of Fig. 5.

Clearly $aab \in L(S_5)$ and $aab \notin L(S_5)$, but $aabb \notin L(S_5)$. Hence, by Lemma 5.2, $L(S_5) \notin RL$. Consequently, $RL \subseteq AL$.

(c) Consider the marked $P$-Net $S_3$ of Fig. 5. Since $L(S_3)$ is not regular, $PL \subseteq AL$ does not hold, in view of Lemma 5.1. The above argument in the proof of (b) about $S_5$, also shows that $L(S_5) \notin PL$. Hence $AL \subseteq PL$ does not hold. It follows that $AL$ and $PL$ are not comparable.

Theorem 5.3. $PL \subseteq EL$.

Proof. Clearly, $PL \subseteq EL$, by Definition 3.1. Consider now the marked $E$-Net $S_6$ of Fig. 6.

We have $L(S_6) \neq L(S_5)$. Hence $L(S_6) \notin PL$. Thus, $PL \subseteq EL$.

Lemma 5.3. Let $L$ be a language over $\Sigma$ in either $AL$ or $PL$ or $EL$ or $LL$. Assume $\sigma_1 \sigma_2 \in L$, where $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$. Then $\sigma_1 \sigma_2 \in L$. 
Proof. Let $L = L(S)$, where $S$ is the relevant SUP-Net. Assume $\sigma_1 \sigma_2 \notin L$, but $\sigma_1 \sigma_1 \sigma_2 \in L$. After firing $\sigma_1$ in $S$ for the first time, there exists a set of input places of $\sigma_2$ which are not marked, preventing $\sigma_2$ from firing. But this set of input places of $\sigma_2$ remains not marked, after $\sigma_1$ is fired a second time, in contradicting with our assumption that $\sigma_1 \sigma_1 \sigma_2 \in L$. Hence $\sigma_1 \sigma_2 \in L$.

Theorem 5.4. $PL \subseteq GPL$

Proof. Evidently, $PL \subseteq GPL$, by Definition 3.1. We now consider the marked GP-Net $S_7$ of Fig. 7.

![Fig. 7. Example of a marked GP-Net $S_7$.](image)

We have $ab \in L(S_7)$, but $aab \notin L(S_7)$. Hence, by Lemma 5.3, $L(S_7) \notin PL$.

It follows that $PL \subseteq GPL$.

Let $S = (P,T,V,K,M)$ be a marked L-Net. For any $t \in T$ we define an equivalence relation $E_t$ on $\cdot t$ as follows:

$$E_t \triangleq \{(p,p') | p \in \cdot t \land p' \in \cdot t \land [p = p' \lor V(p,t) = V(p',t) = L]\}.$$ 

We denote by $Q_t$ the partition $\cdot t/E_t$ of $\cdot t$. For any $q \in Q_t$, we set

$$M(q) = \max\{M(p) | p \in q\}.$$ 

We say that $q \in Q_t$ is an output of $t$, iff $(\exists p \in q) V(t,p) = 1$. 

Technion - Computer Science Department - Technical Report CS0240 - 1982
Theorem 5.5. The language types \( EL \) and \( LL \) are not comparable.

**Proof.** (a) Consider the marked \( E \)-Net \( S_8 \) of Fig. 8.

\[
egin{array}{c}
\text{a} \\
\downarrow \\
p \\
\downarrow \\
b \\
\text{E} \\
\downarrow \\
c
\end{array}
\]

*Fig. 8.* - Example of a marked \( E \)-Net \( S_8 \).

We wish to show that \( L(S_8) \notin LL \).

Assume that \( L(S_8) = L(\hat{S}) \), where \( S = (N,M) \) is a marked \( L \)-Net.

We have \( a^4 b^4 \in L(S) \). Let \( M[a^4] \succ M' \in S \). Then, for every \( q \) in \( Q_b \), which is not an output of \( b \), we must have \( M'(q) \geq 4 \). Now, \( a^4 c a b^2 \in L(S) \). Let \( M'[ca b^2] \succ M'' \in S \). Then, for every \( q \) in \( Q_b \), which is not an output of \( b \), \( M''(q) \geq 1 \). Also, since \( M'' \) was obtained by firing \( b \), we must have \( M''(q) \geq 1 \) for every \( q \) in \( Q_b \), which is an output of \( b \). Hence \( b \) is enabled by \( M'' \). It follows that \( a^4 c a b^2 \notin L(S) \), but \( a^4 c a b^2 \notin L(S_8) \), contradicting our assumption that \( L(S_8) = L(S) \).

Consequently, \( L(S_8) \notin LL \), i.e. \( EL \notin LL \).

(b) To show that \( LL \notin EL \), we consider the marked \( L \)-Net \( S_9 \) of Fig. 9.

\[
egin{array}{c}
a \\
\downarrow \\
p_1 \\
\downarrow \\
L \\
\downarrow \\
b \\
\downarrow \\
p_2 \\
\downarrow \\
\text{L}
\end{array}
\]

*Fig. 9.* - Example of a marked \( L \)-Net \( S_9 \).

Assume \( L(S_9) = L(\hat{S}) \), where \( S = (N,M) \) is marked \( E \)-Net. Since \( b \in L(S) \) and \( b^2 \notin L(S) \), we must have \( \cdot b \cdot b \cdot \neq 0 \).
Let \( k = \max \{ M(p) \mid p \in 'b-b' \} \).

Since \( b \in L(S) \), we have \( k \geq 1 \) as well as \( (\forall p \in 'b') M(p) \geq 1 \).

Now, \( a^{k+1}b^{k+1} \in L(S) \). Therefore, \( (\forall p \in 'b-b')(p \in a^*a') \).

Let \( M[a^{k+1}] > M' \). It follows that \( (\forall p \in 'b-b')[M'(p) \geq k+2] \).

Since \( a^{k+1}b^2 \in L(S) \), we must have \( (\forall p \in 'b-b') V(p,b) \neq E \).

Let \( M'[b^{k+1}] > M'' \). Then \( (\forall p \in 'b-b') M''(p) \geq 1 \). Since \( M'' \) was obtained by firing \( b \), we must also have \( (\forall p \in 'b-b') M''(p) \geq 1 \).

It follows that \( b \) is enabled in \( M'' \), hence \( a^{k+1}b^{k+2} \in L(S) \). But \( a^{k+1}b^{k+2} \in L(S_g) \). Consequently, \( L(S_g) \in EL \). Thus, \( LL \neq EL \).

Since \( LL \neq EL \) and \( EL \neq LL \), Theorem 5.5 is proven.

\[ \square \]

**Theorem 5.6.** \( AL \) is not comparable with either \(GPL\) or \(IL\) or \(LL\) or \(EL\).

**Proof.** (a) Consider the marked P-Net \( S_3 \) of Fig. 3. \( S_3 \) is also a marked GP-Net, I-Net, L-Net and E-Net. Since \( L(S_3) \) is not regular, neither \( GPL \subseteq AL \) nor \( IL \subseteq AL \) nor \( LL \subseteq AL \) nor \( EL \subseteq AL \) can hold, in view of Lemma 5.1.

(b) Consider the marked A-Net \( S_5 \) of Fig. 5. The argument in the proof of Theorem 5.2(b) about \( S_5 \), also shows that \( L(S_5) \in GPL \) and \( L(S_5) \in IL \). Consequently neither \( AL \subseteq GPL \) nor \( AL \subseteq IL \) can hold.

(c) Consider the marked A-Net \( S_{10} \) of Fig. 10.

![Diagram of S_{10}](image-url)
Assume that $L(S_{10}) = L(S)$, where $S = (N,M)$ is a marked $L$-Net. We have $a^3b^2 \in L(S)$. Let $M[a^3>M'$ and $M'[b^2>M''$ in $S$. Since $a^3b^3 \in L(S)$, there exists an element $q$ of $Q_b$ which is not an output of $b$, such that $M'(q) = 2$ and $M''(q) = 0$. Since $M'(q) = 2$, we have $(\forall p \in q)[V(a,p) = 1 \rightarrow V(p,a) \neq 0]$. We also have $a^3b^2a^2 \in L(S)$. Let $M''[a^2>M'''$. Thus, we have $(\forall p \in q)[M'''(p) \leq 1]$. It follows that $a^3b^2a^2b^2 \in L(S)$, but $a^3b^2a^2b^2 \in L(S_{10})$, contradicting our assumption that $L(S_{10}) = L(S)$. Consequently, $A \subseteq L$ does not hold.

(d) Consider the marked $A$-Net $S_{10}$ of Fig. 10. Assume that $L(S_{10}) = L(S)$, where $S = (N,M)$ is a marked $E$-Net. We have $a^3b^2 \in L(S)$. Let $M[a^3>M'$ and $M'[b^2>M''$ in $S$. Since $a^3b^3 \in L(S)$ we have $(\exists p \in b-b^-)[V(p,b) \neq E \wedge M'(p) = 2 \wedge M''(p) = 0]$. Let $p$ satisfy this condition. Since $M'(p) = 2$, $V(a,p) = 1$ implies $V(p,a) = 1$.

We distinguish between two cases:

Case I: $V(a,p) = 1 \wedge V(p,a) = 1$.

Since $M''(p) = 0$ it follows that $a^3b^2a \in L(S)$.

Case II: $V(a,p) = 0$.

We have $a^3b^2a \in L(S)$. Let $M''[a>M'''$ in $S$. Hence, $M'''(p) = 0$.

It follows that $a^3b^2ab \in L(S)$.

But, since $a^3b^2ab \in L(S_{10})$, both cases I and II contradict our assumption that $L(S_{10}) = L(S)$. Consequently, $A \subseteq L$ does not hold.

Parts (a), (b), (c) and (d) complete the proof of Theorem 5.6.

\[ \square \]

Lemma 5.4. Let $L$ be a language over $\Sigma$ in either $EL$ or $GPL$.

Let $S = (N,M)$ be a marked $E$-Net or a marked $GP$-Net.

Let $a \in L(S)$ and $M[V_a>M_1, M[V_a>M_2$ in $S$. If $M_2 \geq M_1$ and $M_1 \geq M_2$, then $a \in L(S)$. 

- End of proof.
Proof. Since \( M_2 \geq M_1 \) and \( y_3 \) is firable in \( M_1 \), \( y_3 \) is also firable in \( M_2 \). □

A similar lemma appears in [12].

**Theorem 5.7.** \( IL \) is not comparable with either \( EL \) or \( GPL \).

**Proof.** (a) Consider the marked I-Net \( S_{11} \) of Fig. 11.

\[
\begin{array}{c}
\text{Fig. 11: Example of a marked I-Net } S_{11}.
\end{array}
\]

Using Lemma 5.4 one easily shows that \( L(S_{11}) \not\in GPL \) (cf. [1,9,12]) and that \( L(S_{11}) \in EL \).

(b) Consider the marked E-Net \( S_6 \) of Fig. 6. Clearly \( abab \in L(S_6) \) and \( aab \in L(S_6) \), but \( aabb \notin L(S_6) \). Hence, by Lemma 5.2, \( L(S_6) \notin IL \).

(c) Consider the marked GP-Net \( S_7 \) of Fig. 7. Assume that \( L(S_7) = L(S) \), where \( S = (N,M) \) is a marked I-Net. We have \( a \in L(S) \) but \( ab \notin L(S) \).

Let \( M[a > M'] \). We have one of the following two cases:

Case I: \( (\exists p \in \cdot b) \left[ V(p,b) = 1 \land M'(p) = 0 \right] \). Let \( p \) denote a place satisfying this condition. Now, \( a^2 \in L(S) \), and let \( M'[a > M'' \). We still have \( M''(p) = 0 \), hence \( a^2 b \in L(S) \), but \( a^2 b \in L(S_7) \), contradicting our assumption that \( L(S_7) = L(S) \).

Case II: \( (\exists p \in \cdot b) \left[ V(p,b) = 1 \land M'(p) > 0 \right] \). Let again \( p \) denote a place satisfying this condition and \( M'[a > M'' \). Since \( a^2 b \in L(S) \) we have \( p \in \cdot a, V(p,a) = 1 \) and \( M''(p) = 0 \). Hence \( a \) is not enabled in \( M'' \), but \( a^3 \in L(S_7) \), contradicting our assumption that \( L(S_7) = L(S) \).
Theorem 5.8. \( \text{GPL} \) is not comparable with either \( \text{IL} \) or \( \text{EL} \).

Proof. (a) Consider the marked \( \text{GP-Net} S_7 \) of Fig. 7. We have \( ab \in L(S_7) \), but \( aab \notin L(S_7) \). Hence, by Lemma 5.3 \( L(S_7) \notin \text{IL} \) and \( L(S_7) \notin \text{EL} \).

(b) Consider the marked \( \text{L-Net} S_9 \) of Fig. 9. Clearly \( bab \in L(S_9) \) and \( ab \in L(S_9) \), but \( abb \notin L(S_9) \). Hence, by Lemma 5.2 \( L(S_9) \notin \text{GPL} \).

(c) Consider the marked \( \text{E-Net} S_6 \) of Fig. 6. Clearly, \( abab \in L(S_6) \) and \( aab \in L(S_6) \), but \( aabb \notin L(S_6) \). Hence, by Lemma 5.2 \( L(S_6) \notin \text{GPL} \).

Parts (a), (b), and (c) complete the proof of Theorem 5.8.

The theorems of Section 5 yield the hierarchy of SUP-Nets illustrated in Fig. 12, where \( XL \rightarrowYL \) denotes \( XL \subset YL \), and \( XL \rightarrow \rightarrow YL \) indicates that \( XL \) and \( YL \) are not comparable.

![Hierarchy of SUP-Nets](image)

Fig. 12. - Hierarchy of SUP-Nets.

6. Some Closure Properties of \( \text{GPL} \)

By means of Lemma 5.2, solutions to some open problems mentioned in [10, p.186] are easily provided.
Theorem 6.1. GPL is not closed under union.

Proof. Consider the marked GP-Nets $S_{12}$ and $S_{13}$ of Fig. 13.

We have $aabb \in L(S_{12}) \cup L(S_{13})$ and $ba \in L(S_{12}) \cup L(S_{13})$ but $bab \notin L(S_{12}) \cup L(S_{13})$. Hence, by Lemma 5.2 $L(S_{12}) \cup L(S_{13}) \notin GPL$. Consequently, GPL is not closed under union. \hfill \Box

Theorem 6.2. GPL is not closed under concatenation.

Proof. Consider the marked GP-Net $S_{14}$ of Fig. 14.

We have $abab \in L(S_{14}) \cdot L(S_{14})$, where \cdot denotes concatenation, and $aab \in L(S_{14}) \cdot L(S_{14})$ but $aabb \notin L(S_{14}) \cdot L(S_{14})$. Hence, by Lemma 5.2 $L(S_{14}) \cdot L(S_{14}) \notin GPL$. Consequently, GPL is not closed under concatenation. \hfill \Box
Theorem 6.3. GPL is not closed under the concurrency (shuffle) operator $\parallel$ (see [10]).

Proof. Consider the marked GP-Nets $S_{15}$ and $S_{16}$ of Fig. 15.

Clearly $abc \in L(S_{15}) \parallel L(S_{16})$ and $ba \in L(S_{15}) \parallel L(S_{16})$, but $bac \notin L(S_{15}) \parallel L(S_{16})$. Hence, by Lemma 5.2 $L(S_{15}) \parallel L(S_{16}) \notin GPL$. Consequently, GPL is not closed under the concurrency operator. $\square$

Theorem 6.4. GPL is not closed under prefix regular substitution (see [10]).

Proof. Consider the marked GP-Net $S_{17}$ of Fig. 16.

Consider the prefix regular substitution $f(a) = \{\lambda, a, ab, abb, b, ba\}$. 
Theorem 6.5. GPL is not closed under Kleene star (iteration).

Proof. Consider the marked GP-Net $S_{14}$ of Fig. 14. We have $abab \in (L(S_{14}))^*$ and $aab \in (L(S_{14}))^*$ but $aabb \notin (L(S_{14}))^*$. Hence, by Lemma 5.2 $(L(S_{14}))^* \notin GPL$. Consequently, GPL is not closed under Kleene star.

In [10] Petri nets with final markings and the corresponding languages are discussed. One easily verifies that the results and proofs of Theorems 6.2, 6.3 and 6.5 also apply to the family of languages denoted by $L^f$ in [10, p.186]. Similar results concerning $L^f$ are obtained (by a different approach) in [12].

7. Hierarchy of Labeled $\lambda$-free Super-Nets

In this section we study the hierarchy of the various types of labeled $\lambda$-free SUP-Nets, defined in Section 3.

Theorem 7.1. We have $LRL = LAL = L^{\lambda}RL = L^{\lambda}AL$. Each of these sets coincides with the set of all the prefix regular languages.

Proof. (a) By Lemma 5.1, the languages in sets $LRL$, $LAL$, $L^{\lambda}RL$, $L^{\lambda}AL$ are all prefix regular.

(b) Let $L$ be a prefix regular language. Clearly there exists a finite, deterministic automaton $A$ such that $L = L(A)$. By applying a construction similar to that in [5, p.35], one obtains a labeled A-Net $\Gamma$ as
In the sequel we let $\Gamma_1$ denote the labeled $\lambda$-free SUP-Nets corresponding to the marked SUP-Nets $S_1$ defined in Section 5.

**Theorem 7.2.** $LRL \subseteq LPL$.

**Proof.** Similar to proof of Theorem 5.2(a).

M. Hack [5, p.64] has shown that any labeled $\lambda$-free GP-Net $\Gamma$ can be transformed into a labeled $\lambda$-free P-Net $\Gamma'$ such that $L(\Gamma) = L(\Gamma')$.

A stronger result is the following.

**Theorem 7.3.** $LGPL = LPL$.

**Proof.** Clearly $LPL \subseteq LGPL$.

Consider the labeled $\lambda$-free GP-Net $\Gamma = (P, T, V, K, M, S, n)$. We construct a labeled $\lambda$-free P-Net $\Gamma' = (P', T', V', K', M', S, n')$ where

$$P' = \{[p, i] | p \in P, \; 0 \leq i \leq n_p - 1\}$$

and $n_p = \max(\max_{t \in T} V(p, t), \max_{t \in T} V(t, p))$

$$T' = \{[t, i] | t \in T, \; 1 \leq i \leq n_t\}$$

and $n_t = \prod_{p \in P} n_p$

$$V'(p, t) = 0 \lor V(p, t) = n_p$$

$V'$ has to satisfy the following three conditions, where $+$ indicates addition mod $n_p$. 
I. Assume \( V(p,t) = m \)

1. \( m = 0 \) \((\forall i \in \{1,2,\ldots,n_t\})(\forall j \in \{0,1,\ldots,n_p-1\})V'([p,i],[t,i]) = 0 \)

2. \( m = n_p \) \((\forall i \in \{1,2,\ldots,n_t\})(\forall j \in \{0,1,\ldots,n_p-1\})V'([p,j],[t,i]) = 1 \)

3. \( 0 < m < n_p \) \((\forall i \in \{1,2,\ldots,n_t\})(\exists j(p,t,i) \in \{0,1,\ldots,n_p-1\}) \)

\[
V'([p,j(p,t,i)],[t,i]) = 1, V'([p,j(p,t,i)+1],[t,i]) = 1, \ldots, V'([p,j(p,t,i)+m-1],[t,i]) = 1, V'([p,j(p,t,i)+m],[t,i]) = 0, \ldots, V'([p,j(p,t,i)-1],[t,i]) = 0 .
\]

II. Assume \( V(t,p) = k \)

1. \( k = 0 \) \((\forall i \in \{1,2,\ldots,n_t\})(\forall j \in \{0,1,\ldots,n_p-1\})V'([t,i],[p,j]) = 0 \)

2. \( k = n_p \) \((\forall i \in \{1,2,\ldots,n_t\})(\forall j \in \{0,1,\ldots,n_p-1\})V'([t,i],[p,j]) = 1 \)

3. \( 0 < V(p,t) < n_p \) \((\forall i \in \{1,2,\ldots,n_t\})[V'([t,i],[p,j(p,t,i)]) = 1, \ldots, V'([t,i],[p,j(p,t,i)+k-1]) = 1, V'([t,i],[p,j(p,t,i)+k]) = 0, \ldots, V'([t,i],[p,j(p,t,i)-1]) = 0 .\]

4. \( V(p,t) = 0 \lor V(p,t) = n_p \)

\((\forall i \in \{0,1,\ldots,n_t\})(\exists j(p,t,i) \in \{0,1,\ldots,n_p-1\}) \)

\[
V'([t,i],[p,j(p,t,i)]) = 1, V'([t,i],[p,j(p,t,i)+1]) = 1, \ldots, V'([t,i],[p,j(p,t,i)+k-1]) = 1, V'([t,i],[p,j(p,t,i)+k]) = 0, \ldots, V'([t,i],[p,j(p,t,i)-1]) = 0 .\]

III. \( (\exists t \in T)(\exists i,j, i \neq j, 1 \leq i,j \leq n_t)(\forall p \in P') \)

\[
[V'([t,i],p) = V'([t,j],p) \land V'(p,[t,i]) = V'(p,[t,j])] .
\]

Furthermore,

\((\forall p \in P')(K'(p) = \infty)\)
One easily verifies that the validity of the condition (*) can be preserved for any sequence of firings. Hence we have \( L(r') = L(r) \).

Thus, \( \text{LPL} = \text{ LGPL} \).

\( \square \)
The following lemma recalls a well-known result.

**Lemma 7.1.** Let \( \{a_i\}_{i=1}^{\infty} \) be an infinite sequence of vectors over \( \omega^N \). Then there exists an infinite increasing sequence of integers \( \{i_j\}_{j=1}^{\infty} \) such that

\[
(\forall m, 1 \leq m \leq n) \left[ (\forall j, k)(a_{i_j}(m) < a_{i_k}(m)) \lor (\forall j, k)(a_{i_j}(m) = a_{i_k}(m)) \right].
\]

**Proof.** Construct an infinite subsequence of \( \{a_i\}_{i=1}^{\infty} \) increasing or non-changing in the first coordinate; construct from this subsequence an infinite subsequence increasing or non-changing in the second coordinate and so forth.

The following lemma is an extension of Lemma 5.4.

**Lemma 7.2.** Let \( L \) be a language over \( \Sigma \) in \( L^\lambda PL \). Let \( \Gamma = (S, \xi, \eta) \) be a labeled P-Net where \( L(\Gamma) = L \). Let \( y_1, y_2 \in L(\Gamma) \) and \( M[y_1] > M_1 \), \( M[y_2] > M_2 \) in \( \Gamma \). If \( M_2 > M_1, y_1y_3 \in L(\Gamma) \) and \( M_1[y_3] > M_3 \), then there exists a firing sequence \( x \in L(S) \) such that \( n(x) = y_2y_3 \), \( M_2[y_3] > M_3 \) and \( M_3 = M_3 + M_2 - M_1 \).

**Proof.** Since \( M_2 > M_1 \) and \( y_3 \) is fireable in \( M_1 \), \( y_3 \) is also fireable in \( M_2 \), and \( M_3 = M_3 + M_2 - M_1 \).

**Lemma 7.3.** (König's Infinity Lemma \([8]\)).

Let \( T \) be an infinite directed tree. If the out-degree of every vertex in \( T \) is finite then there exists an infinite directed path in \( T \).

In the sequel we shall need the following concept.
1. The root is labeled by \((\lambda, M)\).

2. Let \(V_1\) be a vertex labeled by \((\check{x}, M_1)\), where \(M[\check{x}] > M_1\) in \(\Gamma\). If \(M_1[\check{y}] > M_2\), where \(\check{y} \in \Sigma \cup \{\lambda\}\) in \(\Gamma\), then there is a directed edge from \(V_1\) to another vertex \(V_2\) labeled by \((\check{x}\check{y}, M_2)\).

Clearly the out-degree of each vertex in \(\text{Tr}(\Gamma)\) is finite. Indeed, from any marking we can obtain no more than \(|T|\) new markings, by firing a single transition.

**Lemma 7.4.** Let \(L\) be a language over \(\Sigma\). Let \(\Gamma = (S, \Sigma, \eta)\) be an \(\lambda\)-free SUP-Net where \(L(\Gamma) = L\). Let \(\{y_i\}_{i=0}^{\infty}\) be an infinite sequence of finite words defined as follows:

\[
y_0 = \lambda \in L(\Gamma) \quad y_j = y_{j-1}\sigma_j \in L(\Gamma) \quad \sigma_j \in \Sigma \quad j = 1, 2, \ldots.
\]

Then there exists an infinite directed path in \(\text{TR}(\Gamma)\) the vertices of which represent the words of the sequence \(\{y_i\}_{i=0}^{\infty}\).

**Proof.** Let us concentrate only on vertices in \(\text{Tr}(\Gamma)\) which represent words in the sequence \(\{y_i\}_{i=0}^{\infty}\). The corresponding subtree of \(\text{Tr}(\Gamma)\) is an infinite tree because the sequence \(\{y_i\}_{i=0}^{\infty}\) is infinite. Since the subtree is infinite and the out-degree of each vertex in the tree is finite there exists an infinite directed path in \(\text{Tr}(\Gamma)\), passing through some of those vertices by Lemma 7.3. Each directed step along an edge in this path represents a firing of some \(\sigma \in \Sigma\) since \(\Gamma\) is \(\lambda\)-free. Furthermore, all prefixes of any word \(y_j \in \{y_i\}_{i=0}^{\infty}\) belong
Theorem 7.4. \( LPL \subseteq LLL \).

Proof. Clearly \( LPL \subseteq LLL \), by definition. Consider the labeled \( \lambda \)-free L-Net \( \Gamma_{20} \) of Fig. 18.

Assume that \( L(\Gamma_{20}) = L(\Gamma) \) where \( \Gamma \) is a labeled \( \lambda \)-free P-Net. By Lemma 7.4 there exists an infinite directed path in \( Tr(\Gamma) \) the vertices of which represent the prefixes of the infinite word

\[
acbcacbc \cdots a^i b^i c^i \cdots .
\]

We concentrate on markings of this path. Let \( \{M_i\}_{i=2}^\infty \) be the infinite sequence where \( M[acbc \cdots b^{i-1} c^{i-1} a^i] > M_i \). Let \( \{M_i\}_{i=2}^\infty \) be the infinite sequence where \( M[acbc \cdots a^{i-1} c^{i-1} b^{i-1} c^{i-1}] > M_i \).

By Lemma 7.1 we construct an infinite increasing sequence \( \{i_j\}_{j=1}^\infty \) such that \( (\forall j, k; j < k)[M_{i_j} \leq M_{i_k}] \).

Now, let us look at the subsequence \( \{M_{i_j}\}_{j=1}^\infty \). By Lemma 7.1 we can find two integers \( s \) and \( k \) such that \( s < k \) and \( M_s \leq M_{i_k} \). Let \( s = i_\ell, r = i_k \). Then, \( M_s \leq M_r \) and \( M_s \leq M_r \).

Now, by Lemma 7.2 the following markings can be obtained in \( \Gamma \):

\[
M[acbc \cdots b^{r-1} c^{r-1} a] > M_r.
\]
\[ M_x[a^s] \Rightarrow M_y \]
\[ M_y = M_s + (\bar{M}^r + M_r - M_s) - \bar{M}^s = M_r = \bar{M}^r - \bar{M}_s = M_r \]
\[ M_y[c^r] \Rightarrow M_z \]

But \( acbc...b^{r-1}c^{-1}a^{-1}c^{-1}b^{r-1}c^{-1}a^{-1}c^{-1} \in L(\Gamma_{20}) \) since \( \max(r,2s) + s < 2s + r \), contradicting our assumption that \( L(\Gamma_{20}) = L(r) \).

Hence, \( LPL \subseteq LLL \).

The following theorem compares \( LPL \) and \( LEL \). A stronger result requiring a rather complex proof, will be obtained in Section 8 (Theorem 8.2).

**Theorem 7.5.** \( LPL \subseteq LEL \).

**Proof.** The proof is similar to that of Theorem 7.4, with \( \Gamma_{20} \) replaced by \( \Gamma_8 \). The infinite word of interest becomes

\[ aca^2bca^3b^2c...a^ib^{-1}c... \]

The infinite sequences \( \{M_i\}_{i=2}^\infty \) and \( \{\bar{M}_i\}_{i=2}^\infty \) are defined by

\[ M[ac...a^{-1}b^{-2}ca^i] \Rightarrow M_i \]
\[ M[ac...a^{-1}b^{-2}c] \Rightarrow \bar{M}_i, \]

respectively.

**Lemma 7.5.** There exists a language in \( IL \) which is neither in \( L^{\lambda}LL \) nor in \( L^{\lambda}EL \).

**Proof.** Consider the marked \( IL \)-Net, \( S \), of Fig. 11. By an argument
Theorem 7.6. LLL \subseteq LIL.

Proof. By a construction, similar to that of [1] one easily proves that LLL \subseteq LIL. This construction is illustrated in Fig. 19. In view of Lemma 7.5, we thus have LLL \subseteq LIL.

Fig. 19. (a) Example of a labeled \lambda-free L-Net \Gamma_{21},
(b) An equivalent labeled \lambda-free I-Net \Gamma_{22},
i.e. L(\Gamma_{22}) = L(\Gamma_{21}).

The theorems of Section 7 yield the hierarchy of labeled \lambda-free SUP-Nets illustrated in Fig. 20 where XL \rightarrow YL denotes XL \subseteq YL, and XL \leftrightarrow YL indicates that XL = YL.
8. Hierarchy of Labeled Super-Nets

In this section we study the hierarchy of the various types of labeled SUP-Nets (including λ-labels).

Theorem 8.1.  

a) \( L^{\lambda}RL \subseteq L^{\lambda}PL \)  
b) \( L^{\lambda}PL = L^{\lambda}GPL \).

Proof. The proofs are similar to those of Theorem 7.2 and Theorem 7.3, respectively.

Theorem 8.2. \( L^{\lambda}PL \subseteq L^{\lambda}EL \).

Proof. Clearly \( L^{\lambda}PL \subseteq L^{\lambda}EL \) by definition. Consider the labeled E-Net \( \Gamma_8 \) of Fig. 8. Assume that \( L(\Gamma_8) = L(\Gamma) \) where \( \Gamma = (P,T,V,K,M,\Sigma,\eta) \) is a labeled P-Net. Let \( |P| = n \). We consider the infinite sequence \( \{A_i\}_{i=1}^{\infty} \) of words:

\[
A_i = a^i b^{i-1} c a^{i+1} b^i c ... a^{i+n-1} b^i c \in L(\Gamma).
\]

Now, we construct \( 2n \) infinite sequences of marking in \( \Gamma \) from the words in \( \{A_i\}_{i=1}^{\infty} \):

\[
\{M^j_i\}_{i=1}^{\infty} \quad j = 1,2,...,2n
\]

\[
M[a^i b^{i-1} c ... a^{i+j-2} b^i c a^{i+j-1} \gg M^j_{i,j} \quad j = 1,2,...,n
\]

\[
M[a^i b^{i-1} c ... a^{i+j-1} b^i c a^{i+j-2} c \gg M^2 j \quad j = 1,2,...,n
\]

Next, we extract from \( \{M^j_i\}_{i=1}^{\infty} \) an infinite subsequence \( \{M^j_{i,j}\}_{i,j=1}^{\infty} \) according to Lemma 7.1. From \( \{M^2 j\}_{j=1}^{\infty} \) we extract another infinite
Now, we have an infinite, increasing subsequence \( \{i_j\}_{j=1}^{\infty} \) such that
\[
(\forall \varepsilon, 1 \leq \varepsilon \leq 2n)(\forall q, 1 \leq q \leq n)[(\forall j; j < k)(M_{i_j}^\varepsilon(q) < M_{i_k}^\varepsilon(q)) \land (\forall j, k)(M_{i_j}^\varepsilon(q) = M_{i_k}^\varepsilon(q))] .
\]

Claim 1. \( (\forall j, k; j < k)(\forall \varepsilon, 1 \leq \varepsilon \leq n) [M_{i_j}^{2^\varepsilon - 1} < M_{i_k}^{2^\varepsilon - 1}] . \)

Proof. Assume that for some \( j < k \) and \( 1 \leq \varepsilon \leq n \) \( M_{i_j}^{2^\varepsilon - 1} = M_{i_k}^{2^\varepsilon - 1} \).

Thus, by Lemma 7.2,
\[
x = a_j b_j c_1 a_{j+1} b_{j+1} c_{j+1} a_{j+2} b_{j+2} c_{j+2} ... a_i b_i c_i .
\]

But, \( x \in L(\Gamma_0) \), since \( i_j + \varepsilon - 1 < i_k + \varepsilon - 2 \), contradicting our assumption.

Claim 2. There exists an infinite increasing sequence \( \{t_j\}_{j=1}^{\infty} \), with
the properties of \( \{i_j\}_{j=1}^{\infty} \) and furthermore \( (\forall \varepsilon_1, 1 \leq \varepsilon_1 \leq n)(\forall q, 1 \leq q \leq n)
\]
\[
(\forall j, k)(M_{t_j}^{2^{\varepsilon_1} - 1}(q) = M_{t_k}^{2^{\varepsilon_1} - 1}(q)) \land (\forall j, j > 1)(\forall \varepsilon_2, 1 \leq \varepsilon_2 \leq n)(\forall q, j > 1)
\]
\[
(2^{\varepsilon_2 - 1}M_{t_j}^{2^{\varepsilon_2} - 1}(q) - M_{t_{j+1}}^{2^{\varepsilon_2} - 1}(q) \leq M_{t_j}^{2^{\varepsilon_1} - 1}(q) - M_{t_{j+1}}^{2^{\varepsilon_1} - 1}(q)) .
\]

Proof. We can construct such a sequence as a subsequence of \( \{i_j\}_{j=1}^{\infty} \), in view of the above properties of \( \{t_j\}_{j=1}^{\infty} \), as follows:

\[
t_1 = i_1
\]
\[
t_2 = i_2
\]
\[
f_3 = 2 + \max_{1 \leq q \leq n} (M_{t_2}^{2^{\varepsilon_1} - 1}(q) - M_{t_1}^{2^{\varepsilon_1} - 1}(q))
\]

Now, by Lemma 7.2 the following reachable markings can be obtained in \( r \):

\[
\begin{align*}
M_{t_n} & > M_{t_n}^1, \\
M_{t_n}^1 [b^{t_n-1}] & > M_{t_n-1}^1, \\
M_{t_n}^1 - M_{t_n-1}^1 > M_{t_n-1}^2 & > M_{t_n-2}^2.
\end{align*}
\]

Let \( P_1 = \{ p | M_{t_n}^1 (p) - M_{t_n-1}^1 (p) \neq 0 \} \) and \( Q = \{ p | M_{t_n}^3 (p) - M_{t_n-2}^3 (p) \neq 0 \} \).

Clearly, by Claim 1, \( 1 \leq |P_1| \leq n \).

We now distinguish between two cases:

Case I) \( Q \subseteq P_1 \)

By Lemma 7.2, the following reachable markings can be obtained in \( r \):

\[
\begin{align*}
M_{t_n}^{t_n-2+1} & > M_{t_n-2}^2, \\
M_{t_n-2}^2 & = M_{t_n-2}^3 + M_{t_n-2}^1 - M_{t_n-2}^2 = M_{t_n-2}^3 + M_{t_n-2}^1 - M_{t_n-1}^1 - M_{t_n-2}^2. \\
& > M_{t_n-2}^3 + M_{t_n-1}^1 - M_{t_n-1}^1.
\end{align*}
\]

By Claim 2 and the fact that \( Q \subseteq P_1 \) we have

\[
M_{t_n}^1 - M_{t_n-1}^1 > M_{t_n-1}^3 - M_{t_n-2}^3 = M_{t_n-2}^3 + M_{t_n-1}^1 - M_{t_n-1}^1 > M_{t_n-1}^3.
\]
contradicting our assumption.

Case II) $Q \notin P_1$.

By Lemma 7.2, the following reachable markings can be obtained
in $\Gamma$:

$$M_1^4 [b \rightarrow M_{n-2}^5] < M_{n-1}^3$$

Let

$$P_2 = \{ p | M_{t_n}^1(p) - M_{t_{n-1}}^1(p) + M_{t_n}^3(p) - M_{t_{n-2}}^3(p) \neq 0 \}$$

and

$$Q = \{ p | M_{t_{n-2}}^5(p) - M_{t_{n-3}}^5(p) \neq 0 \}.$$ 

Clearly $P_2 \supset P_1$; hence, by Claim 1 $2 \leq |P_2| \leq n$.

Again, one of the following cases holds, Case I) $Q \subseteq P_2$

Case II) $Q \notin P_2$. We handle both cases as before.

After $n$ steps we have $n \leq |P_n| \leq n$. Therefore, every transition is
enabled. Hence, we have $x_{cc} \in L(\Gamma)$, where $x \in \Sigma^*$. But
$x_{cc} \notin L(\Gamma_0)$,
contradicting our assumption that $L(\Gamma) = L(\Gamma_0)$.

Thus, $L^\lambda P_L \subseteq L^\lambda E L$.

Theorem 8.3. $L^\lambda L_L \subseteq L^\lambda L$. 

\[ \Box \]
Theorem 8.4. \( \mathcal{L}^\Lambda \mathcal{E} \mathcal{L} \subseteq \mathcal{L}^\Lambda \mathcal{L} \).

Proof. Consider the labeled \( E \)-Net \( \Gamma = (P, T, V, K, M, \Sigma, n) \).

We construct a marked labeled \( I \)-Net \( \Gamma' = (P', T', V', K', M', \Sigma, n') \) where

\[
P' = P \cup \{ p' \} \cup \{ p_t | (t \in T) \land (\exists p \in P) V(p, t) = E \}
\]

\[
T' = T \cup \{ t | (t \in T) \land (\exists p \in P) V(p, t') = E \} \cup \{ [t, p] | (t \in T) \land (p \in P) \land \\
\land V(p, t) = E \}.
\]

The function \( V' \) is defined as follows:

I. for all \( p \in P \)

  do

  for all \( t \in T \)

    do

      if \( V(p, t) \neq 0 \) then \( V'(p, t) := 1 \)

      if \( \neg (\exists p' \in P) V(p', t) = E \) then \( V'(t, p) := V(t, p) \)

      else

        do

          \( V'(t, p) := V(t, p) \);

          \( V'(p, t) := 1 \);

          \( V'(p_t, t) := 1 \);

        end

      end

    end

  if \( V(p, t) = E \) then

    do

      \( V'(p, t) := 1 \)

      \( V'(p, [t, p]) := 1 \)

      \( V'(p_t, [t, p]) := 1 \)

      \( V'(t, p_t) := 1 \)

    end

  end
Fig. 21 illustrates this construction.

Clearly $L(T') = L(T)$. Hence, $L^{\lambda}EL \subseteq L^{\lambda}IL$. By Lemma 7.5, we have $L^{\lambda}EL \subset L^{\lambda}IL$.

The theorems of Section 8 and Theorem 7.1 yield the hierarchy of labeled SUP-Nets illustrated in Fig. 22.
9. Further Results on the Hierarchy of Super-Nets

Theorem 9.1. \( \text{AL} \subseteq \text{LAL} \).

Proof. Clearly \( \text{AL} \subseteq \text{LAL} \).

Consider the labeled \( \lambda \)-free \( \text{A-Net} \) \( \Gamma_{25} \) of Fig. 23.

![Fig. 23. Example of a labeled \( \lambda \)-free \( \text{A-Net} \) \( \Gamma_{25} \).](image)

We have \( ab \notin L(\Gamma_{25}) \), but \( aab \notin L(\Gamma_{25}) \). Hence by Lemma 5.3, \( L(\Gamma_{25}) \notin \text{AL} \).

It follows that \( \text{AL} \subsetneq \text{LAL} \).

The following result has been obtained in [6].

Theorem 9.2. \( \text{LPL} \subsetneq \text{L}^3 \text{PL} \).

The following theorem has been obtained in [5].

Theorem 9.3. \( \text{IL} \subsetneq \text{L}^3 \text{IL} \).

Definition 9.1. Let \( \Gamma = (P,T,V,K,\Sigma,\eta) \) be a labeled SUP-Net. Assume \( w = t_1t_2...t_r \in T^+ \) and \( \eta(t_i) = \sigma \in \Sigma \). We say that \( i \) is a \( \sigma \)-index of \( w \). \( I(\sigma,w) \) will denote the set of all \( \sigma \)-indices of \( w \).

Lemma 9.1. Let \( S = (P,T,V,K,M) \) be a marked P-Net or a marked I-Net. Let \( x = t_1t_2...t_r \) be a firing sequence of \( S \). Let \( y \) be a permutation of \( x \). If \( y \) is also a firing sequence of \( S \), then
The proof of the following theorem uses a similar technique as the proof of Theorem 2 in [6].

**Theorem 9.4.** \( \text{LEL} \subseteq \text{LEL}^\lambda. \)

**Proof.** Clearly \( \text{LEL} \subseteq \text{LEL}^\lambda \) by definition. Now, consider the labeled \( E \)-Net \( r_{26} \) of Fig. 24. It will be shown that \( L(r_{26}) \in \text{LEL} \).

Assume that \( L(r) = L(r_{26}) \) where \( r \) is a labeled \( \lambda \)-free \( E \)-Net. Let \( L(r') = L(r) \) where \( r' = (P, T, V, K, M, \Sigma, \eta) \) is a labeled \( I \)-Net constructed as in Theorem 8.4, and \( \Sigma = \{a, b, c\} \).

We define the following sets:

\[
A = \{x \in (a+b)^* | i = \sum_{j \in I(a,x)} 2^j \}
\]

\[
B = \{x \in (a+b)^* | i > \sum_{j \in I(a,x)} 2^j \}
\]

One easily verifies that \( A \subseteq L(r_{26}) \) and \( B \cap L(r_{26}) = \emptyset \).

For any positive integer \( n \), let

\[
C_n = \{x \in (a+b)^* | \text{length of } x \text{ is } n\}
\]

\[
D_n = \{M' | M[x \rightarrow M'] \land (x \in C_n) \land (\exists i \in \omega)((xc^i \in A) \land (\exists M') (M'[c^i \rightarrow M']) \}
\]

**Claim 1.** \( |D_n| \geq |C_n| = 2^n \).

**Proof.** Let \( i < j, x_i c^i \in A, x_j c^j \in A \), where \( x_i, x_j \in C_n \). Let \( M[x_i \rightarrow M_i', M[x_j \rightarrow M_j]. \) If \( M_i = M_j \), we have \( M_i[c^i \rightarrow M'] \) for some \( M' \).

Hence \( x_i c^i \in L(r') \). But this contradicts the condition \( B \cap L(r_{26}) = \emptyset \).

Therefore, \( M_i \neq M_j \). Hence the number of markings in \( D_n \) equals to or is greater than the number of words in \( C_n \), which is \( 2^n \).
Fig. 24. - Example of a labeled E-Net, $\Gamma_{26}$
Claim 2. In order to generate a word of length \( n \) in \( H \) we have to fire at most \( m|P| + n|P| + n \) transitions.

Proof. In order to remove all tokens of the initial marking we have to fire no more than \( m|P| \lambda \)-transitions. Every place can receive no more than \( n \) tokens while the word is generated. In order to remove these tokens we have to fire at most \( n|P| \lambda \)-transitions. In order to remove the word itself, we have to fire \( n \) transitions. 

Let \( n < \max (m,|P|,3) \). Hence, \( n^3 > n + m|P| + n|P| \). The number of combinations to choose \( k \) t's such that \( t \in T \) is \( \binom{|T|+k-1}{k} \). Therefore, by Lemma 9.1 and Claim 2:

\[
|D_n| = \binom{|T|+n}{n} + \binom{|T|+n+1}{n+1} + \ldots + \binom{|T|+n^3-1}{n^3} < n^3\binom{|T|+n^3-1}{n^3}.
\]

But if we choose \( n \) big enough we will have \( n^3\binom{|T|+n^3-1}{n^3} < 2^n \) contradicting Claim 1. Hence \( L(H_26) \notin \text{LEL} \). Thus, \( L^\lambda \text{EL} \supset \text{LEL} \).

In view of the results of this section, as well as arguments used in Sections 5, 7, and 8, we obtain the hierarchy of SUP-Nets shown in Fig. 25.
The hierarchies derived in Sections 5, 7, 8, 9 are not complete. However, we list the following conjectures, by means of which the hierarchies could be completed.

Conjectures. a) \( L(r_8) \in \text{LIL} \)

b) \( L(r_8) \in \text{L}^\lambda \text{LL} \)

c) \( L(r_{20}) \in \text{L}^\lambda \text{EL} \).

10. \( \text{LGPL} \) and \( \text{L}^\lambda \text{GPL} \) are not closed under Kleene Star (Iteration)

The following theorem solves an open problem discussed in [10, p.186].

Theorem 10.1. \( \text{LGPL} \) and \( \text{L}^\lambda \text{GPL} \) are not closed under Kleene star (iteration).

Proof. Consider the marked GP-Net \( S_{27} \) of Fig. 26.

![Fig. 26 - Example of a marked GP-Net \( S_{27} \).](image)

Consider the marked E-Net \( S_8 \) of Fig. 8. One easily verifies that \( L(S_8) = (L(S_{27}))^* \). In the proof of the Theorem 8.2 we showed that there is no labeled P-Net \( \Gamma \) such that \( L(\Gamma) = L(S_8) \). Since \( \text{L}^\lambda \text{PL} = \text{L}^\lambda \text{GPL} \) by Theorem 8.1 it follows that there is no labeled GP-Net \( \Gamma' \) such that \( L(\Gamma') = L(S_8) = (L(S_{27}))^* \). Hence we proved the theorem.
REFERENCES


