ON THE LENGTH OF OPTIMAL TSP CIRCUITS IN SETS OF BOUNDED DIAMETER

by

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Let \( V \) be a set of \( n \) points in \( \mathbb{R}^k \). Let \( d(V) \) denote the diameter of \( V \), and \( \ell(V) \) denote the length of the shortest circuit which passes through all the points of \( V \). (Such a circuit is an "Optimal TSP circuit"). \( \ell^k(n) \) are the extremal values of \( \ell(V) \) defined by:

\[
\ell^k(n) = \max \{ \ell(V) \mid V \in \gamma^k_n \}
\]

where

\[
\gamma^k_n = \{ V \mid V \subseteq \mathbb{R}^k, |V| = n, d(V) = 1 \}.
\]

A set \( V \in \gamma^k_n \) is "longest" if \( \ell(V) = \ell^k(n) \). In this paper, we first study some geometrical properties of longest sets in \( \mathbb{R}^2 \), which are used to obtain \( \ell^2(n) \) for small \( n \)'s, and then derive asymptotic bounds on \( \ell^k(n) \). Let \( \delta(V) \) denote the minimal distance between a pair of points in \( V \), and let:

\[
\delta^k(n) = \max \{ \delta(V) \mid V \in \gamma^k_n \}.
\]

It is easily observed that \( \delta^k(n) = o(n^{-1/k}) \). Hence,

\[
c_k = \limsup_{n \to \infty} \delta^k(n)^{1/k} \text{ exists. It is shown that for all } n:
\]

\[
c_k n^{-1/k} \leq \delta^k(n)
\]

and hence, for all \( n \), \( \ell^k(n) \geq c_k n^{-1-1/k} \). For \( k=2 \), this implies that

\[
\ell^2(n) \geq (\pi^2/12)^{1/4} n^{1/2},
\]

which generalize an observation of Fejen Toth that

\[
\lim_{n \to \infty} \ell^2(n)n^{-1/2} \geq (\pi^2/12)^{1/4}.
\]

It is also shown that:

\[
\ell^k(n) \leq [(3-\sqrt{3}k/(k-1)]n\delta^k(n) + o(n^{1-1/k}) \leq [(3-\sqrt{3}k/(k-1)]n^{1-1/k} + o(n^{1-1/k}).
\]

The above upper bound is used to improve a similar result obtained by Few for large \( k \)'s [5]. For \( k=2 \), Few's technique is used to show that:

\[
\ell^2(n) \leq (\pi n/2)^{1/2} + o(1).
\]
1. INTRODUCTION

Let $\mathbb{R}$ denote the set of the real numbers. The Euclidean Traveling Salesman Problem (TSP) in $\mathbb{R}^k$ is the following: Given $n$ points $x_1, \ldots, x_n$ in $\mathbb{R}^k$, find the shortest circuit (that is: closed curve) which passes them. Such a circuit is an "optimal TSP circuit". It is easily verified that an optimal TSP circuit is a polygonal line through $x_1, \ldots, x_n$. In some applications it is required that the distance between any 2 points in the given set is bounded by some constant $D$. (e.g., when the points represent nodes in a communication network. $D$ represents the maximal distance in which 2 nodes can communicate. An optimal TSP circuit in this case corresponds to a most efficient communication protocol [6].

The problem addressed in this paper is the following: Given $n$, $k$ and $D$, what is the maximal length of an optimal TSP circuit through $n$ points in $\mathbb{R}^k$, the distance between any pair of which is at most $D$. Denote this length by $\xi^k(n,D)$. It is not hard to verify that $\xi^k(n,D) = D \cdot \xi^k(n,1)$. Hence, we can restrict ourselves to the case where $D = 1$. For brevity, we denote $\xi^k(n,1)$ by $\xi^k(n)$. We shall be interested in both the values $\xi^k(n)$ and the properties of point sets of "maximal length" which realize these values.

It had been noted (see [5]) that this problem is closely related to the following problem of "Optimal packing": Allocate $n$ points in $\mathbb{R}^k$ such that the distance between any pair of them $\leq 1$, and the minimal distance between any pair of them is maximized. Denote this "maximal minimal distance" by $\delta^k(n)$. It is easily observed that $\delta^k(n) \geq n \cdot \xi^k(n)$. Thue and others (see [4,7]) had shown that $\delta^2(n)$ is asymptotically equal to $(n^2/12)^{1/4} n^{-1/2}$. The exact values of $\delta^2(n)$ for $n \leq 7$ (and the
The paper has 5 sections. The rest of this section includes the
necessary definitions and notations. In Section 2, some geometrical
properties of sets of maximal length in \( \mathbb{R}^2 \) are proved. These prop-
erties are then used to compute \( \lambda^2(4) \) and to give some results concern-
ing \( \lambda^2(5) \). In Section 3 we give lower bounds on \( \lambda^k(n) \), which generalize
the observation mentioned above about the connection between \( \delta^k(n) \) and
\( \lambda^k(n) \). In that section we also give a result on \( \delta^k(n) \) which seems
to be of independent interest (Theorems 3.2 and 3.2'). In Section 4
upper bounds on \( \lambda^k(n) \) are given; first we give an upper bound for
arbitrary \( k \), which improves a similar result obtained by Few in [5],
and then we use the technique of Few to give a better bound on \( \lambda^2(n) \).
In Section 5 two related results are discussed.

Notations and Definitions: Let \( V = \{x_1, \ldots, x_n\} \) be a set of \( n \) points
in \( \mathbb{R}^k \) (for some \( k \)). A path in \( V \) is a sequence \( P = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \)
of points of \( V \). For \( j = 1, \ldots, k-1 \), \( (x_{i_j}, x_{i_{j+1}}) \) is an arc of \( P \).
An arc \( (x-y) \) will be identified with the straight line segment connect-
ing \( x \) and \( y \). The length of a path \( P \) is defined by:

\[
\ell(P) = \ell(x_{i_1}, \ldots, x_{i_k}) = \sum_{j=1}^{k-1} \delta(x_{i_j}, x_{i_{j+1}})
\]

where \( \delta(x; y) \) is the Euclidean distance between \( x \) and \( y \).

An Hamiltonian circuit or a TSP circuit in \( V \) is a path
\( H = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \) in which \( i_j \neq i_k \) for \( j \neq k \). We shall
identify 2 Hamiltonian circuits if one is obtained from the other by
reversing the order of the points. Thus, for \( n \geq 3 \), there are \((n-1)/2\)
distinct Hamiltonian circuits on sets of \( n \) points.
is defined by:

\[ \ell(V) = \min \{ \ell(H) \mid H \text{ is an Hamiltonian circuit in } V \}. \]

For a set \( V \) and a point \( x \), \( d(x, V) = \max \{ \delta(x, y) \mid y \in V \} \).

The diameter of \( V \), \( d(V) \), is defined by:

\[ d(V) = \max \{ d(x, y) \mid x, y \in V \}. \]

For a positive integer \( n \) and \( k \), let:

\[ V_n^k = \{ V \mid V \subseteq \mathbb{R}^k, d(V) = 1, |V| = n \} \]

**Definition 1.2**: For a positive integer \( n \):

\[ \ell^k(n) = \max \{ \ell(V) \mid V \in V_n^k \}. \]

A set \( V^* \) is a "longest set" if:

(i) \( V^* \in V_n^k \);

(ii) \( \ell(V^*) = \ell^k(n) \).

(1) The diameter of \( V \) is sometimes denoted as "the maximal Chord length of \( V \)."

(2) The use of the term "max" (and not "sup") in the definition of \( \ell^k(n) \) is justified by the fact that \( V_n^k \) is homeomorphic to a compact subset of \( \mathbb{R}^{kn} \) and that \( \ell(V) \) is a continuous function. Similar remarks, apply to few other definitions in the paper.
Definition 1.3: Let \( V \subseteq \mathbb{R}^k \) for some \( k \). Then

\[
\delta(V) = \min\{\delta(x,y) \mid x \neq y, x, y \in V\}
\]

For positive integers \( n \) and \( k \),

\[
\delta^k(n) = \max\{\delta(V) \mid V, \mathcal{V}_n^k\}
\]

The numbers \( \delta^k(n) \) are sometimes called "packing constants" \([3]\).

Most of the proofs in the paper are given for the case \( k = 2 \), and it will be clear from the text when they generalize to arbitrary \( k \).

\( \mathcal{V}_n^2 \) will be denoted by \( V_n \), and \( \mathcal{V}_n^2 \) will denote a set in \( V_n \).

Similarly, \( \ell(n) \) and \( \delta(n) \) will denote \( \ell^2(n) \) and \( \delta^2(n) \) respectively.
2. SOME PROPERTIES OF LONGEST SETS IN $\mathbb{R}^2$

In this section we prove lemmas which provide some insight on the structure of planar longest sets. We then use these lemmas to find a maximal set $V_n^*$ of cardinality 4, and thus to compute $\lambda(4)$, and to give some results concerning $V_n^*$ and $\lambda(5)$. Note that trivially, $\lambda(1) = 0$, $\lambda(2) = 2$, and $\lambda(3) = 3$. $V_n^*$ for $n = 1, 2, 3$ are given in Figure 1.

![Figure 2.1](image)

Lemma 2.1: Let $V_n = \{x_1, \ldots, x_n\}$ be a set of $n$ points in the plane ($n \geq 4$), not all of them on the same line. Then an optimal TSP circuit in $V_n$ is a simple curve (that is: a curve which does not intersect itself).

*Proof:* Let $H = (x_1 - x_2, - \ldots - x_i, x_{i+1})$ be a TSP circuit in $V_n$. We shall show that if $H$ intersects itself, then $H$ is not optimal.

For simplicity, assume that $(i_2, \ldots, i_n) = (2, \ldots, n)$.

Suppose that for some $i$ and $j$ ($|i-j| > 1$), the arcs $(x_i, x_{i+1})$ and $(x_{j}, x_{j+1})$ intersect (see Figure 2.2).

Assume first that $x_i, x_{i+1}, x_{j}, x_{j+1}$ are not co-linear.

Then by replacing $(x_i, x_{i+1})$ and $(x_{j}, x_{j+1})$ by $(x_i, x_j)$ and $(x_{i+1}, x_{j+1})$ we obtain a TSP circuit which is shorter than $H$ (due to the triangle inequality). The proof for the case where $x_i, x_{i+1}, x_{j}, x_{j+1}$ are co-linear is also not hard and is omitted. □
Definition 2.1: Let $V$ be a set of points. Then $\text{CON}(V)$ is the boundary of the convex hull of $V$. (That is: the boundary of the smallest convex Figure which contains $V$.)

Lemma 2.2: Let $V^* \in V_n$. If $V^*$ is longest, then for each $x$ in $V^*$, $d(x, V^*) = 1$ iff $x \in \text{CON}(V^*)$.

Proof: Since $d(V^*) = 1$, if $x \in V^*$ and $d(x, V^*) = 1$, then $x$ must be in $\text{CON}(V^*)$. Hence, it is suffices to show that for every $x$ in $\text{CON}(V^*_n)$, $d(x, V^*_n) = 1$. For contradiction, assume that for some $x \in \text{CON}(V^*_n)$, $d(x, V^*_n) < 1$. We shall show that there is a $V'_n \in V_n$ such that $h(V'_n) > l(V^*_n)$: Let $L$ be a supporting line of $V^*_n$ through $x$ (that is: a line tangent to $\text{CON}(V^*_n)$ at $x$, see Figure 2.3).
\( V' \) is obtained by removing \( x \) a small distance \( h \) in a direction perpendicular to \( L \), as shown in Figure 2.3. By doing this, \( \delta(x,y) \) is increased for all \( y \in V_n^* \), and hence \( \lambda(V'_n) > \lambda(V_n^*) \). On the other hand, if \( h \) is small enough, \( d(x, V'_n) \) is less than 1 (since \( d(x, V_n^*) < 1 \)), and hence \( d(V'_n) = 1 \). Thus, \( V'_n \) is in \( V_n^* \). This completes the proof of the lemma.

**Definition 2.2:** Let \( x, y \in V \). Then the arc \((x-y)\) is **essential** if it participates in every optimal TSP circuit in \( V \). \((x-y)\) is **redundant** if it participates in no optimal TSP circuit in \( V \). \((x-y)\) is non-essential (non-redundant) if it is not essential (redundant).

A point \( x \) of \( V \) is "**internal in \( V \)**" if it is not in \( \text{CON}(V) \).

**Lemma 2.3:** Let \( V_n^* \) be a longest set, and let \( x \) be an internal point in \( V_n^* \). Then

(a) for all \( y \in V_n^* \), \((x-y)\) is non-essential;

(b) Let \( L \) be any line through \( x \). Then there are \( y, z \in V_n^* \) such that \( L \) separates \( y \) and \( z \), and both \((x-y)\) and \((x-z)\) are non-redundant.

**Proof:** (a) Assume that for some \( y \in V_n^* \), \((x-y)\) is essential. We derive a contradiction by showing that \( V_n^* \) is not longest. Suppose first that there is no \( z \) in \( V_n^* \) such that \( x \) lie on the arc \((y-z)\) as in Figure 2.4. Then, by removing \( x \) a distance \( h \) away from \( y \)
along the line containing \((x-y)\), \(\delta(x,y)\) increases by \(h\), while for any \(u \in V_n^*\), if \(\delta(x,u)\) decreases, it decreases by less than \(h\), and if \(x \notin \{u,v\}, \delta(u,v)\) remains unchanged.

It follows that the length of any TSP circuit which contains \((x-y)\) (and hence of any optimal TSP circuit) increases by some positive value. By making \(h\) small enough, \(d(x,V_n^*)\) remains smaller than 1 and the lengths of the non optimal TSP circuits remain larger than the length of the previously optimal circuits and hence \(\lambda(V_n^*)\) is increased, in contradiction with the assumption that \(V_n^*\) is longest.

The argument above does not work if there is a point \(z\) such that \(x\) lies in \((y-z)\) as in Figure 2.4, because then removing \(x\) as before does not increase the length of the TSP circuits which use the path \((y-x-z)\). In this case, \(x\) can be removed away from \(y\) in a direction which form a small but positive angle \(\alpha\) with \((x,z)\), and a similar argument does apply.

(b) For contradiction, assume that there is a line \(L\) through \(x\) as in Figure 2.5, such that for all nodes \(y\) on the left side of \(L\) \((x-y)\) is redundant. Then, by removing \(x\) in a direction perpendicular to \(L\) as shown in Figure 2.5, \(\delta(x,z)\) increases for all \(z\) which are not on the left side of \(L\), and hence, for all \(z\) such that \((x-y)\) is non redundant. Hence, similarly to the proof of (a), one can increase \(\delta(V_n^*)\) by removing \(x\) a small distance \(h\) in that direction. □

Figure 2.5
Corollary: Let \( V^*_n \) be a longest set, and let \( x \) be an internal point of \( V^*_n \). Then there are at least 3 points \( y_1, y_2, \) and \( y_3 \) in \( V^*_n \) such that \( (x-y_i) \) is non-redundant \( (i = 1, 2, 3) \).

Proof: Since every optimal TSP circuit must pass through \( x \), there are \( y_1, y_2 \) in \( V^*_n \) such that the path \( (y_1-x-y_2) \) is in an optimal TSP circuit. Hence, \( (y_1-x) \) and \( (y_2-x) \) are non-redundant. If there is no \( y_3 \) such that \( (y_3-x) \) is non-redundant, then both \( (y_1-x) \) and \( (y_2-x) \) occur in every optimal circuit, which means that \( (y_1-x) \) and \( (y_2-x) \) are essential, in contradiction to Lemma 2.3(a).

Definition 2.3: Let \( C \) be a closed curve in the plane and let \( D \) be a real number. Then \( C \) is a "curve of constant width \( D \)" if for each \( x \in C \), \( d(x,C) = 1 \).

A "figure of constant width" is a convex figure whose boundary is a curve of constant width. Examples of curves of constant width \( D \) are a circle of diameter \( D \), and a Reuleaux triangle of side length \( D \) (see Figure 2.6). For more about curves of constant width see [9]. We shall use the following lemma concerning these curves.

![Figure 2.6 - Reuleaux triangle](image-url)

Lemma 2.4: (a) Every curve of constant width \( D \) has a perimeter \( \pi D \).
(b) The area of a figure of constant width \( D \) is at most \( \pi D^2/4 \).
(a) is Barbier theorem. (b) follows from (a) by the isoperimetric theorem [2, 9].
Since every set of nodes of diameter 1 can be embedded in a figure of constant width 1 \( [9] \), Lemma 2.4 implies the following:

**Lemma 2.5:** Let \( V_n \) be in \( V_n \). Then:

(a) The perimeter of \( \text{CON}(V_n) \) is less than \( \pi \).

(b) There is a convex figure whose area is less than \( \pi/4 \) which contains \( V_n \).

**Definition 2.4:** Let \( k, n \) be given, \( 2 \leq k \leq n \). Then:

\[
V_{n,k} = \{ V_n \mid V_n \in V_n, \ |V_n \cap \text{CON}(V_n)| = k \}
\]

\[
\ell(n,k) = \sup \{ \ell(V_n) \mid V_n \in V_{n,k} \}.
\]

**Lemma 2.6:** \( \ell(4) = 2(1 + \sqrt{3}/3) = 3.1547 \ldots \)

**Proof:** Let \( V_4^* \) be a longest set of 4 points. Then \( V_4^* \) is either in \( V_{4,3} \) or in \( V_{4,4} \). Hence, \( \ell(4) = \max(\ell(4,3), \ell(4,4)) \). We shall prove first that \( \ell(4,4) \leq \pi \), and then that \( \ell(4,3) = 2(1+\sqrt{3}/3) > \pi \).

Let \( V_4 \) be in \( V_{4,4} \). Then \( \ell(V_4) \) is the perimeter of \( V_4 \), which by Lemma 2.5 is less than \( \pi \). Hence, \( \ell(4,4) \leq \pi \). (3)

Let \( V_4' = \{x,y,z,u\} \), where \( x, y \) and \( z \) are the vertices of an equilateral triangle of side length 1, and \( u \) is the center of this triangle (see Figure 2.7). Then, as one can easily verify

\[
\ell(V_4') = \ell(u-y-x-z-u) = 2(1+\sqrt{3}/3) > \pi.
\]

Hence, \( \ell(4) \geq \ell(V_4') > \ell(4,4) \). Therefore, a longest set in \( V_4 \) must be in \( V_{4,3} \). It remains to show that \( V_4' \) is, in fact, a longest set: Let \( V_4^* = \{x',y',z',u'\} \) be a longest set, and let \( u' \) be the internal point of \( V_4^* \). Using

(3) In fact, one can show that \( \ell(4,4) = 2 + 1/\cos 15^\circ = 3.0353(-) \ldots \)
Lemma 2.2, one can show that \{x', y', z'\} are the vertices of an equilateral triangle of side length 1. Also, by Lemma 2.3 and its corollary, each of the arcs \( (x'-y') \), \( (x'-z') \), \( (x'-u') \) is non redundant. Hence, all of the possible 3 TSP circuit have the same length, which implies that \( \delta(u', y') = \delta(u', z') = \delta(u', x') \) and hence that \( u' \) is the center of the triangle. The lemma follows.

An interesting corollary to the last two lemmas is the following:

Let \( n \geq 4 \), and let \( V_n^* \) be a longest set in \( V_n \). Then \( V_n^* \) is not in \( V_{n,n} \).

Deriving \( l(n) \) and \( V_n^* \) for \( n \geq 5 \) seems to be hard. Using Lemmas 2.1-2.5, we have been able to prove some results concerning \( l(5) \) and \( V_5^* \). These results are stated below without proofs.

\textbf{Lemma 2.7:} \( l(5,3) = 3.2155 \ldots \). Moreover, \( l(5,3) = l(V_{5,3}^*) \), where \( V_{5,3}^* \) is defined by (see Figure 2.8):

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2_8.png}
\caption{Figure 2.8}
\end{figure}
(1) \( \{x,y,z\} \) are the vertices of an equilateral triangle of side length 1.

(2) \( u \) is the center of the triangle.

(3) \( v \) lies on the height from \( x \) to \( (y-z) \) and

\[
\delta(v,z) - \delta(v,u) = 1 - \sqrt{3}/3 = 0.4266^{(+)}.\]

Let \( \alpha = \angle vxy \). Then one can check that, by (3):

\[
\delta(v,z) - \delta(v,u) = 1/(2\cos\alpha) - [\sqrt{3}/6 - (\tan\alpha)/2]
\]

\[
= \frac{1}{2} (1/\cos\alpha + \tan\alpha - \sqrt{3}/3) = 1 - \sqrt{3}/3 .
\]

This implies that a longest set in this case is obtained when

\( \alpha \approx 20.1^\circ \). The value for \( \ell(\mathcal{V}_5^*) \) follows by computing the length of one of the optimal routes \( (x-u-v-y-z-x) \), say:

\[
\ell(x-u-v-y-z-x) = \sqrt{3}/2 - (\tan 20.1^\circ)/2 + 1/(2\cos(20.1^\circ)) + 2
\]

\[
\approx 3.21548 .
\]

**Lemma 2.8:** Let \( \mathcal{V}_5^* \) be a longest set in \( \mathcal{V}_5 \). Then \( \mathcal{V}_5^* \in \mathcal{V}_{5,4} \).

**Proof:** In view of Lemma 2.7, we have to show that there is a

\( \mathcal{V}_5' \in \mathcal{V}_{5,4} \) such that \( \ell(\mathcal{V}_5') > \ell(5,3) \approx 3.2155 \). Such a \( \mathcal{V}_5' \) is given in Figure 2.9: \( x,y,z \) form an equilateral triangle of side length 1,
u and v lie on the bisector of \( \angle zxy \), \( \delta(x,u) = 1 \) and \( \delta(u,v) = \frac{\delta(y,u)}{2} = 1/(4\cos15^\circ) = 1/(2(3^{1/2} + 2)^{1/2} = (2 - \sqrt{3})^{1/2}/2 \). In \( V' \), all the 4 circuits which do not intersect themselves, are optimal. (Note that, by Lemma 2.1, these are the only candidates for optimal circuits.)

\[
L(V'_5) = \delta(x,y) + \delta(y,u) + \delta(u,v) + \delta(v,z) + \delta(z,x) = 1 + \frac{1}{2\cos15^\circ} + 1/(4\cos15^\circ) + d(v,z) + 1 \approx 2 + 3/(4\cos15^\circ) + 0.5153505
\]

\approx 3.291807 > L(5,3).

We conjecture that \( V'_5 \) is a longest set, though we do not have yet a formal proof for this.
3. A LOWER BOUND ON $\ell^k(n)$

In this section we derive a lower bound on $\ell^k(n)$. The results are stated and proved for $k=2$, but are easily generalized to arbitrary $k$.

Let $V_n \in V_n$ be such that for all $x, y$ in $V_n$, $\delta(x,y) \geq r$. Then, clearly, $\ell(V_n) \geq n \cdot r$. Taking $r$ to be $\delta(n)$, we have that $\ell(n) \geq n \cdot \delta(n)$. Let

$$c_2 = \limsup_{n \to \infty} \delta(n) \sqrt{n}.$$ 

Then, by the discussion above

$$c_2 \leq \limsup_{n \to \infty} \ell(n) / \sqrt{n}.$$ 

We shall generalize this observation to the following stronger result:

**Theorem 3.1:** For all $n$, $\ell(n) \geq c_2 \sqrt{n}$.

Theorem 3.1 follows easily from

**Theorem 3.2:** For all $n$, $\delta(n) \sqrt{n} \geq c_2$.

Theorem 3.2 seems to be of independent interest, since it implies not only that $c_2 = \lim \delta(n) \sqrt{n}$, but also that $\delta(n) \sqrt{n}$ is a lower bound of $\delta(n) \sqrt{n}$. The $k$ dimensional version of Theorem 3.2 is:

**Theorem 3.2':** Let $c_k = \limsup \delta^k(n) n^{1/k}$. Then for all $n$, $\delta^k(n) n^{1/k} \geq c_k$.

The key lemma for the above theorems is Lemma 3.2, which uses a relation between the "packing constants" and densities of "sparse sets", as described below:

Let $R$ be a planar figure of area $A$, and let $S$ be a finite set of points contained in $R$. Then the density of $S$ in $R$ is $|S|/A$. The set $S$ is "sparse" if for each pair of points $x, y$ in $S$, $\delta(x,y) \geq 1$. The packing constants $\delta(n)$ correspond to the density of sparse sets of points in certain planar figures in the following way:
Suppose that for some $n$ and $c$, $\delta(n) \geq \frac{c}{\sqrt{n}}$. Then, by using appropriate scaling, one can obtain a planar figure $R$ of diameter $\sqrt{n}/c$ which contains a sparse set of $n$ points. Using the fact that, by Barbier Theorem, the area of $R$ cannot exceed $\pi n/(4c^2)$, we have that the density of $S$ in $R$ is at least $4c^2/\pi$. On the other hand, if we can embed a sparse set $S$ in a circle $C$ of diameter $\sqrt{n}/c$ such that the density of $S$ in $C \geq 4c^2/\pi$, then, since the area of $C$ is $\pi n/(4c^2)$, $S$ contains at least $n$ points. This implies, (again by scaling), that $\delta(n) \geq \frac{c}{\sqrt{n}}$. The next lemma summarizes the said above.

**Lemma 3.1:***

a) If $\delta(n) \geq \frac{c}{\sqrt{n}}$, then there is a sparse set $S$ of $n$ points contained in a figure $R$ of diameter $\leq \sqrt{n}/c$, and (hence) the density of $S$ in $R$ is at least $4c^2/\pi$.

b) If a sparse set $S$ is contained in a circle $C$ of diameter $\sqrt{n}/c$ such that the density of $S$ in $C$ is at least $4c^2/\pi$, then $\delta(n) \geq \frac{c}{\sqrt{n}}$.

Due to the fact that for all $k$, the $k$ dimensional set of given diameter and maximal volume is a $k$ dimensional sphere, Lemma 3.1 has a simple generalization to the $k$ dimensional case. In this generalization, $\sqrt{n}/c$ is replaced by $n^{1/k}/c$, and $4c^2/\pi$ is replaced by $c^k/\omega_k$, where $\omega_k$ is the volume of the $k$ dimensional sphere of diameter 1.

In view of Lemma 3.1(b) above, Theorem 3.2 will follow if we show that for each $r$, there is circle $C$ of radius $r$ which contains a sparse set $S$, such that the density of $S$ in $C$ is at least $4c^2/\pi$.

(Recall that $c_2 = \liminf_{t \to \infty} \delta(n)\sqrt{n}$.) This will follow from Lemma 3.2, for which we need the following definitions:

---

(4) The author is indebted to M. Perles for bringing this fact to his attention.
Definition 3.1: For each positive real number \( t \), let \( R_t \) be a set of diameter \( t \) and area \( A_t \), and let \( S_t \) be a finite set of points contained in \( R_t \). For a given \( r \geq 0 \), \( R_{t-r} \) is the set obtained by deleting from \( R_t \) all the points whose distance from the boundary of \( R_t \) is less than \( r \). Let \( A_{t-r} \) be the area of \( R_{t-r} \), and let \( S_{t-r} = S_t \cap R_{t-r} \). We say that the family \( \{(R_t, S_t)\} \) is balanced if for each fixed \( r \), the following hold:

\[
\lim_{t \to \infty} \frac{A_{t-r}}{A_t} = 1 \\
\lim_{t \to \infty} \frac{S_{t-r}}{S_t} = 1
\]

Definition 3.2: A family \( \{(R_t, S_t)\} \) as above has density \( \epsilon \) if it is balanced and

\[
\lim_{t \to \infty} \sup \left| \frac{S_t}{A_t} \right| = \epsilon.
\]

Example 1: Let \( m \) be a positive integer, and let \( G_m \) be the lattice

\[ G_m = \{(p/m, q/m) | p, q \text{ are integers}\} \]

For each \( t \), let \( R_t \) be a set of constant width \( t \), and let \( G_{m,t} = G_m \cap R_t \). Then \( \{(R_t, G_{m,t})\} \) has density \( \frac{m^2}{t^2} \).

Example 2: For each \( t \), let \( R_t \) be as in Example 1, and let \( S_t \) be a sparse set contained in \( R_t \). Then \( \{(R_t, S_t)\} \) has a density \( \leq \frac{4c_2^2}{\pi} \).

Definition 3.3: Let \( G_m \) be as in Example 1 above, and let \( B \) be a bounded region of the plane. Then

\[ N(B, m) = |G_m \cap B| \]

The proofs of the following propositions are easy and omitted.
Proposition 3.1: Let $B$ be a convex figure of area $F > 0$. Then

$$\lim_{m \to \infty} \frac{N(B, m)}{m^2} = F,$$

and the convergence is uniform (that is, it does not depend on the specific location of $B$ in the plane).

Proposition 3.2: Let $(S_t, R_t)$ be a family of density $e$ and let $r$ be a fixed number. Then

$$\lim_{t \to \infty} \frac{S_t - r}{A_t} = e.$$

Lemma 3.2: Let $(S_t, r_t)$ be a family of density $e$. Then for each $F > 0$, there is a circle $C$ of area $F$ and a number $t$ such that $|C \cap S_t| \geq \lceil eF \rceil^5$.

Proof: For simplicity, let $e = 1$. Assume for contradiction that for all $C$ of area $F$, and for all $t$, $|C \cap S_t| < F$. Let $F = I + h$, where $I$ is an integer and $0 < h \leq 1$. Let $\delta = h/F > 0$. Then for each $t$, each circle of area $F$ contains at most $I = F(1-\delta)$ points of $S_t$. Let $\varepsilon > 0$ be such that $(1-\varepsilon)^2 > (1-\delta)(1+\varepsilon)$, and let $r = (R/\pi)^{1/2}$ (i.e., $\pi r^2 = F$). By Propositions 3.1 and 3.2 there exist $t$ and $m$ ($m$ depends on $t$) such that:

(i) $\left| \frac{N(C,m)}{m} - F \right| < \varepsilon F$ for every circle $C$ of area $F$.

(ii) $\left| \frac{|S_t - r|}{A_t} - 1 \right| < \varepsilon$.

(iii) $\left| \frac{N(R_t, m)}{m^2} - A_t \right| < \varepsilon A_t$.

(5) $\lceil x \rceil$ denotes the smallest integer not smaller than $x$. 
Let $G_{m,t} = G_m \cap R_t$ and let $C_{m,t}$ be the set of all circles of radius $r$ (and area $F$) whose centers belong to $G_{m,t}$.

For each $x \in S_t$ and $C \in C_{m,t}$ let:

$$n_x = |\{ C \mid C \in C_{m,t}, C \ni x \} |$$

$$n_C = |\{ x \mid x \in S_t \cap C \} |$$

Note that if $x$ is in $R_{t-r}$, then:

$$n_x = | \{ u \mid u \in G_m, d(u,x) \leq r \} |$$

and that, under the assumption that the lemma is false, $n_C \leq F(1-\delta)$ for all $C \in C_{m,t}$.

Finally, let

$$P = \{ (x,C) \mid C \in C_{m,t}, x \in C \cap S_t \},$$

and let $p = |P|$.

We derive a contradiction by computing $p$ by two different methods:

**Method 1:** $p = \sum_{C \in C_{m,t}} n_C \leq F(1-\delta) |C_{m,t}| < F(1-\delta)m^2(1+\epsilon)A_t$.

The last inequality follows from (iii), since $|C_{m,t}| = N(R_t,m)$.

**Method 2:** $p = \Sigma_{x \in S_t} n_x \geq \Sigma_{x \in S_{t-r}} |S_{t-r}| m^2 F(1-\epsilon) > A_t (1-\epsilon)m^2 F(1-\epsilon)$.

The second inequality follows from (i), and the last inequality from (ii).

By combining the above results and canceling equal terms, we get:

$$(1-\epsilon)^2 < (1-\delta)(1+\epsilon),$$

which contradicts the assumption on $\epsilon$. 

\[ \square \]
Proof of Theorem 3.2: By Lemma 3.1(b), it is enough to show that for every circle $C$ there is a sparse set $S$ whose density in $C \geq 4c^2_2 / \pi$.

For each positive real number $t$, let $S_t$ be a sparse set of width $t$ and of maximum possible cardinality; and let $R_t$ be a set of constant width $t$ containing $S_t$. Then, by Lemma 3.1(a) and the definitions the family $(R_t, S_t)$ has density $4c^2_2 / \pi$. Let $C$ be a given circle of area $F$. Then by Lemma 3.2, there is a replica $C'$ of $C$ and a number $t$ such that $C' \cap S_t \geq \lceil F \cdot 4c^2_2 / \pi \rceil \geq F \cdot 4c^2_2 / \pi$. This implies that the density of $C' \cap S_t$ in $C'$ is at least $4c^2_2 / \pi$. Since $S_t$ is a sparse set, so is $C' \cap S_t$. The theorem follows.

\[ \square \]

Corollary: For each $n$, $\delta(n) \geq (\pi^2/12)^{1/4} n^{-1/2}$, and $\lambda(n) \geq (\pi^2/12)^{1/4} n^{1/2}$.

Proof: By Theorems 3.1 and 3.2, using the result (due to Thue and others, see [2]) that $\lim_{n \to \infty} \delta(n) / n = (\pi^2/12)^{1/4}$.

\[ \square \]
4. AN UPPER BOUND ON \( k(n) \)

In this section first we derive an upper bound on \( k(n) \) expressed in terms of \( \delta n(n) \), and then derive from it an upper bound expressed in terms of \( k \) and \( n \) only. We show that our result improves a similar result of Few [5] for large \( k \), and then use the technique of Few to improve our result for \( k=2 \).

**Lemma 4.1:** For each \( k \) and \( n \) \((k,n \geq 2)\),

\[
k^k(n) \leq k^k(n-1) + (3 - \sqrt{3}) \delta^k(n).
\]

**Proof:** As before, we shall prove the lemma for \( k=2 \), since the generalization to arbitrary \( k \) will be obvious.

Let \( V_n \in V_n \). We shall prove that:

\[
l^k(V_n) \leq l^k(n-1) + (3 - \sqrt{3}) \delta n(n)
\]

Let \( x, y \in V_n \) be such that \( \delta(x,y) \) is minimized. Then \( \delta(x,y) \leq \delta(n) \).

Let \( z \) be the median of the interval \((x,y)\) and let \( V_{n-1} = \{V_n \setminus \{x,y\}\} \cup \{z\} \). Then \( l(V_{n-1}) \leq l(n-1) \). Hence, it is suffices to prove that \( l(V_n) \leq l(V_{n-1}) + (3 - \sqrt{3}) \delta(x,y) \). Let \( H \) be an optimal TSP circuit in \( V_{n-1} \), and let \( u, v \in V_{n-1} \) be such that the path \((u-z-v)\)

is included in \( H \) (see Figure 4.1). A TSP circuit \( H' \) for \( V_n \) is obtained by replacing \((u-z-v)\) in \( H \) by either \((u-x-y-v)\) or \((u-y-x-v)\), whichever is shorter.

![Figure 4.1](image)
Without loss of generality assume that \((u-x-y-v)\) is the shorter one—that is:

\[ (1) \quad \lambda(u-x-y-v) \leq \lambda(u-y-x-v) \]

Then \(\lambda(V_n) \leq \lambda(H') = \lambda(H) + \lambda(u-x-y-v) - \lambda(u-z-v)\).

Hence, it is sufficient to show that:

\[ \lambda(y-x-y-v) - \lambda(y-z-v) \leq (3 - \sqrt{3}) \delta(x,y) \]

Note also that \(x\) and \(y\) were chosen so that:

\[ (2) \quad \delta(x,y) = \min\{\delta(s,t) | s \neq t, \{s,t\} \subseteq \{u,v,x,y\}\} \]

For a given pair of points \((u,v)\), let \(f(u,v) = \lambda(u-x-y-v) - \lambda(u-z-v)\), and let \(M = \max\{f(u,v) | u,v \text{ satisfies (1) and (2) above}\}\). (Note that even if \(u, v, x, y\) are not restricted to be co-planar \(f(u,v)\) is maximized when \(u, v, x, y\) are co-planar.) To prove the lemma, it is sufficient to show that \(M \leq (3 - \sqrt{3})\delta(x,y)\).

Let \(\hat{u}, \hat{v}\) be such that \(\hat{u}, x, \hat{v}, y\) form a parallelogram, in which \(\delta(\hat{u},x) = \delta(\hat{v},x) = \delta(\hat{v},y) = \delta(x,y)\) (see Figure 4.2). Then:

\[ f(\hat{u},\hat{v}) = 3\delta(x,y) - \sqrt{3}\delta(x,y) = (3 - \sqrt{3})\delta(x,y) \]

Hence, the lemma will follow if we can show that \(M = f(\hat{u},\hat{v})\).

To prove this, we prove the following claim:

Claim: \(f(\hat{u},\hat{v}) \geq f(u,v)\) for all \(u, v\) which satisfies (1) and (2) above.
Proof of the Claim: Note that $f(u,v)$ can be written as:

$$f(u,v) = f_1(u) + f_2(v) + \delta(x,y)$$

where

$$f_1(t) = \delta(t,x) - \delta(t,z); \quad f_2(t) = \delta(t,y) - \delta(t,z).$$

The claim now follows by the following observations:

Observation (a): Let $t$ be any point, and let $t'$ be in the arc $(t-z)$, (see Figure 4.3). Then, by the triangle inequality, $f_1(t') \geq f_1(t)$ and $f_2(t') \geq f_2(t)$.

Observation (b): Let $r = \delta(x,y)$, and let $C_x$ ($C_y$) be circles of radius $r$ and centers $x$ ($y$) respectively. Let $D_x$ ($D_y$) denote the discs whose boundaries are $C_x$ ($C_y$). Then, by (2), $u$ and $v$ cannot lie in the interior of $D_x$ ($D_y$), (see Figure 4.4).
Let \( D = D_x \cup D_y \), and let \( C \) be the boundary of \( D \). Observations (a) and (b) above imply that \( f_1(v) \) and \( f_2(u) \) (and hence also \( f(u,v) \)), are maximized for \( u \) and \( v \) which satisfy (2) above when both \( u \) and \( v \) are in \( C \).

Observation (c): Let \( L \) be the line containing \((x-y)\), and let \( a \) be the unique point in \( C_x \cap C \cap L \), and \( b \) be the unique point in \( C_y \cap C \cap L \) (see Figure 4.4). Then, when \( u \) moves from \( a \) to \( b \) along \( C \), \( f_1(u) \) increase monotonic and \( f_2(u) \) decrease monotonic.

Observation (d): If both \( u \) and \( v \) are in \( C \), then \( d(u-x-y-v) \leq d(u-y-x-v) \) is equivalent to \( d(u,x) \leq d(v,x) \). This, together with Observations (b) and (c) above, implies that under constraints (1) and (2), \( f(u,v) \) is maximized when \( d(u,x) = d(v,x) \), that is - when \( u \) is the reflection of \( v \) in \( L \). Note that in this case \( f_1(u) = f_1(v) \) and \( f_2(u) = f_2(v) \). Moreover, by symmetry, \( f(u,v) \) attains its maximum for some \( u, v \) in \( C \cap C_x \). Thus, the problem of computing \( M \) reduces to the following maximization problem:

\[
\text{maximize } f_1(u) + f_2(u) (= f_1(u) + f_2(v)).
\]

Subject to \( u \in C \cap C_x \).
Using polar coordinates by substituting \( u = (r, \alpha) \) (see Figure 4.5), we get:

\[
\maximize \quad r(1 + 1 / \cos(\alpha/2) - (5 + 4 \cos \alpha)^{1/2}) = f(\alpha)
\]

Subject to \( 0 \leq \alpha \leq 2\pi/3 \quad (r = \delta(x, y)) \).

(In polar coordinates, \( \delta(u, x) = r, \delta(u, y) = r \sin \alpha / \sin(\alpha/2), \delta(u, z) = \frac{r}{2} (5 + 4 \cos \alpha)^{1/2} \).

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.5}
\caption{Figure 4.5}
\end{figure}
\]

Taking derivative with respect to \( \alpha \), we get:

\[
f'(\alpha) = \frac{r}{2} \left( \frac{\tan(\alpha/2)}{\cos(\alpha/2)} + \frac{\sin \alpha}{(5 + 4 \cos \alpha)^{1/2}} \right)
\]

Since \( f'(\alpha) > 0 \) for \( \alpha \in (0, 2\pi/3) \), \( f \) obtains its maximum when \( \alpha = 2\pi/3 \), i.e., when \( u = \hat{u}, v = \hat{v} \). This completes the proof of the claim, and hence of the lemma.

\[\]

\textbf{Theorem 4.1:} For all \( k \) and \( n \)

\[
\delta(k(n)) / (n \delta(k(n))) \leq (3 - \sqrt{3})k / (k - 1) + o(1).
\]

\textbf{Proof:} Let \( C_k = \lim_{n \to \infty} \delta_k(n) 1/k \) (the existence of \( C_k \) follows from Theorem 3.2'). Then

\[
\delta_k(n) = C_k n^{-1/k} + o(n^{-1/k}).
\]

The theorem will follow if we can show that:

\[
\delta_k(n) \leq \{(3 - \sqrt{3})k / (k - 1)\} C_k n^{-1/k} + o(n^{-1/k}).
\]
By Lemma 4.1, \( x^k(n) \leq x^k(n-1) + (3 - \sqrt{3}) \delta^k(n) \). Hence:

\[
x^k(n) \leq (3 - \sqrt{3})[\delta^k(2) + \ldots + \delta^k(n)] = (*)
\]

Let \( \delta^k(x) \) be a continuous, non-increasing real extension of \( \delta^k(n) \).

Then \( \delta^k(x) = c_k x^{-1/k} + o(x^{-1/k}) \). We get that:

\[
x^k(n) \leq (*) < (3 - \sqrt{3}) \int_2^n \delta^k(x) \, dx = (3 - \sqrt{3}) \int_2^n [c_k x^{-1/k} + o(x^{-1/k})] \, dx
\]

\[
= \{(3 - \sqrt{3})k/(k-1)\} c_k n^{1-1/k} + o(n^{1-1/k}).
\]

\[\square\]

**Theorem 4.2**: For all \( k \), \( x^k(n) \leq \{(3 - \sqrt{3})k/(k-1)\} n^{1-1/k} + o(n^{1-1/k}) \).

**Proof**: In view of Theorem 4.1, it is enough to show that for large enough \( n \), \( \delta^k(n) < n^{-1/k} \).

Let \( \delta^k(n) = d \). Then it is possible to pack \( n \) disjoint \( k \) dimensional spheres of diameter \( d \) in a \( k \) dimensional sphere of diameter \( 1 + d/2 \).

It is known that the ratio between the sum of the volumes of the small spheres and the volume of the large sphere cannot exceed some constant \( E_k < 1 \). The ratio between the volumes of one small sphere and the large sphere is \( [d/(1+d/2)]^k \).

Hence:

\[
n[d/(1 + d/2)]^k \leq E_k \]

or, equivalently:

\[
d^k \leq (1 + d/2)^k E_k / n.
\]

For large enough \( n \), \( (1 + d/2)^k E_k < 1 \) (recall that \( d = \delta^k(n) \)) which implies that, for large enough \( n \), \( \delta^k(n) < n^{-1/k} \).

\[\square\]

A problem similar to the one discussed in this section was discussed in [5, 8], where a bound on the length of the shortest road through \( n \) points in the \( k \) dimensional unit cube was investigated. In [5] it...
was shown that this bound cannot exceed

\[(4.1) \quad \{k(2(k-1))(1-k)/2k + o(1)\}n^{1-1/k}\]

Using argument similar to the one in Theorem 4.2, but replacing the k-dimensional sphere of diameter \(1 + d/2\) by a k-dimensional cube of side length \(1 + d/2\), and using the fact that the volume of the k-dimensional sphere of radius 1 is \(\pi^{k/2}/\Gamma(k/2 + 1)\), one can show that this bound cannot exceed

\[(4.2) \quad \{2(3 - \sqrt{3})k/(k-1)\}(F(k/2 + 1)\}^{1/2}/(r-1/2) + o(1)\}n^{1-1/k}\]

For large \(k\) we have:

\[(4.1) \Rightarrow (k/2)^{1/2} + o(1)n^{1-1/k} \approx 0.7071\sqrt{k} n^{1-1/k}.\]

\[(4.2) \Rightarrow (3 - \sqrt{3})(2k/\pi e)^{1/2} + o(1)n^{1-1/k} \approx 0.6136/k n^{1-1/k}.\]

However, for small \(k\), the technique used in [5] provides a better bound on \(k^k(n)\). In particular, one can use that technique to prove:

**Theorem 4.3:** \(k(n) = (\pi n/2)^{1/2} + o(1)\).

**Proof:** (sketch): Let \(V_n \in \mathcal{V}_n\) be given. Then \(V_n\) can be embedded in a figure \(C\) of diameter 1 which, by Barbier's theorem, is of area \(\leq \pi/4\).

Let \(t = (\pi/2n)^{1/2}\), and let \(L_0, L_1\) be the sets of lines defined by:

\[L_0 = \{(x,y) | y = nt \quad \text{for some integer} \ n\}\]

\[L_1 = \{(x,y) | y = (n + 1/2)t, \quad \text{for some integer} \ n\}\].

For a point \(v\) in \(\mathbb{R}^2\), let \(\delta(v, L_1)\) be the shortest distance from \(v\) to a line in \(L_1\). Then for each \(v\), \(\delta(v, L_1) + \delta(v, L_2) = t/2\).
Hence
\[ \sum_{v \in V_n} \delta(v, L_0) + \sum_{v \in V_n} \delta(v, L_1) = n' t/2. \]

Hence, for \( i = 0 \) or \( i = 1 \), it holds that:

(4.3.1) \[ \sum_{v \in V_n} \delta(v, L_i) \leq n' t/4. \]

Without loss of generality, assume that 4.3.1 holds for \( i = 0 \).

Consider the TSP circuit composed of:

(a) the line segments in \( L_0 \cap C \).

(b) portions of the boundary of \( C \) connecting these line segments to a path.

(c) for each point \( v \) in \( V_n \), the shortest line segment connecting \( v \) to the path described above, each such segment counted twice.

(d) a segment connecting the first and last points of \( V_n \) traversed along the described path.

The sum of the length of the line segments described in (a) is equal approximately to the area of \( C \) divided by \( t \) — that is \( \pi/4t + O(1) \).

The sum of the lengths of the segments in (b) is \( O(1) \). The sum of the lengths of the segments in (c), (each taken twice), is \( nt/2 \), and the segment (d) is of length \( \leq 1 \). Altogether, the total length of the described circuit is \( \pi/4t + nt/2 + o(1) \). The theorem follows. \( \square \)
5. TWO RELATED RESULTS

Two problems which are related to the problem discussed in this paper are:

1) Minimal tree: Given a set \( V \) of \( n \) points in \( \mathbb{R}^k \), find a tree (that is: a connected graph without circuits) on \( V \) such that the length of the tree, defined as the sum of the lengths of its arcs, is minimal. Denote this length by \( l_T^k(V) \).

2) Steiner tree: Given a set \( V \) as above, find a set \( V' \supseteq V \) such that \( l_T^k(V') \) is minimal. Formally, for a given \( V \), the length of the Steiner tree of \( V \) is defined by:

\[
l_S^k(V) = \min_{V' \supseteq V} \{ l_T^k(V') \}.
\]

Note: The existence of a set \( V' \supseteq V \) such that \( l_T^k(V') \) is minimal follows from the observation that for every set \( V' \) which contains \( V \) there exists a set \( V'' \) which contains \( V \) such that \( l_T^k(V'') \leq l_T^k(V') \) and \( |V''| \leq 2|V| - 2 \). (Thus, in the definition of Steiner tree we can add the restriction: \( |V'| \leq 2|V| - 2 \), which implies that the minimum is attained). \( V'' \) is constructed from \( V' \) in the following manner:

Let \( T \) be a tree of minimal length on \( V' \). Delete from \( V' \) all the points which are not in \( V \) and have degree at most 2 in \( T \). In the resulted tree every point not in \( V \) has a degree at least 3.

The observation follows.

The corresponding problems for graph of bounded diameter are:

For each \( n \), find:

1) \( l_T^k(n) = \max_{V \in \mathcal{V}_{n}^k} \{ l_T^k(V) \} \)

2) \( l_S^k(n) = \max_{V \in \mathcal{V}_{n}^k} \{ l_S^k(V) \} = \max_{V \in \mathcal{V}_{n}^k} \{ \min_{V' \supseteq V} \{ l_T^k(V') \} \} \).
Theorem 5.1: For each $k$ and $n$

$$1 - 0(n^{-1/k}) \leq \lambda^k_T(n)/n^\delta^k(n) \leq (3 \sqrt{3})k/(k-1) + o(1).$$

Proof: The upper bound follows immediately from the upper bound on $\lambda^k(n)$ (Theorem 4.1). The lower bound follows from the proof of the lower bound for $\lambda^k(n)$ (Theorem 3.1), noting that a tree on $n$ points has $n-1$ edges. 

Theorem 5.2: For each $k$ and $n$

$$1/2 \leq \lambda^k_S(n)/n^\delta^k(n) \leq k/(k-1) + o(1).$$

Proof: The upper bound follows from the observation that $\lambda^k_S(n) \leq \lambda^k(n-1) + \delta^k(n)$. (In the proof of Lemma 4.1, simply add $z-x$ and $z-y$ to the Steiner tree for $V_{n-1}$, see Figure 5.1).

![Figure 5.1](image)

The lower bound follows from the lower bound on $\lambda^k(n)$, by the fact that the existence of Steiner tree of length $\lambda$ implies the existence of Hamiltonian circuit of length $\leq 2\lambda$. 

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