BIN PACKING:
AN ANALYSIS OF THE NEXT-FIT ALGORITHM

by

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ABSTRACT

An infinite supply of pieces, with i.i.d. sizes, is packed under the Next-Fit packing procedure. The first two moments of $A_n$, the number of bins to pack $n$ pieces are computed when the piece sizes are uniformly distributed. A simple approximate pgf for $A_n$ is also given, valid for general piece size distribution.
1. INTRODUCTION

1.1 The problem of bin packing is probably one of the "celebrated problems of Computer Science" - reference [3] furnishes a sweeping overview. It can be defined as follows: Given a set of \( n \) pieces \( \{R_i\} \), each of length \( X_i \), and a set of identical bins \( \{B_i\} \), each of size \( b \), determine which subsets of \( \{R_i\} \) should be packed in each bin such that

a) The capacity of each bin is not exceeded;
b) The minimum number of bins is used.

This problem has been shown to be "NP-hard", i.e. it essentially requires the (possibly implicit) enumeration of all packing schemes [2].

An example with \( n = 8 \), \( X_i = \{3,1,4,2,4,4,4,4\} \), \( b = 9 \) is in Fig. 1a.

Recently there is increasing interest in estimating the performance of simple packing (and scheduling) algorithms, especially where the simplicity of those is traded-off consciously against the higher complexity of optimal procedures. See e.g. [1] and further references there.

The problem is normally attacked on one of two fronts:
a) estimating worst case behavior, which implied the consideration of some specific set of pieces (tasks) contrived to show the algorithm's poorest performance;
b) postulating a probability measure over the possible sets of pieces and computing distributions (or at least moments) of performance criteria with respect to this measure.

The latter is the approach taken here.

* This is a one-dimensional problem; the width or any other descriptors of the pieces are immaterial.
1.2 This note is in a sense a continuation of [1]; there the main effort was to prove that subject to the assumptions detailed below, the processes that describe the Next-Fit packing procedure are well behaved, from the analytic point of view.

None the less, only average values could be obtained, and these gave no information on probable deviations from the mean. Such is the case for practically all the evaluations of heuristic algorithms we have seen. In the sequel we go beyond mean value results. Both worst-case bounds and mean value results can be said to provide limited guidance in the process of selecting an algorithm. We wish to show how this process can be better aided.

Figure 1 - Example of bin packing

a. Optimal Packing  
b. Next Fit Packing
1.3 Below we define precisely the environment and the Next-Fit (NF) algorithm. From these we derive equations for the number of bins used under NF to pack \( n \) pieces, \( A_n \). It seems to be rather easy to evaluate \( E(A_n) \), but the evaluation of higher moments and certainly the distribution is less transparent. For the former we outline a procedure and use it to calculate \( V(A_n) \); for the distribution we suggest an approximate generating function which turns out to be easy to compute and fairly accurate (except that it over-estimates the tails of the distribution). For the special case of uniformly distributed piece sizes we derive exact results, albeit in a rather hard-to-use format.

1.4 Specifically we consider the following situation: An infinite sequence of pieces \( \{R_i, i \geq 1\} \) of sizes \( \{X_i, i \geq 1\}, X_i \sim F \) independently is to be packed into an infinite set of identical bins \( B_i \), each of size 1, using the Next-Fit algorithm, nearly the most primitive one possible:

Next-Fit Packing

1) Set \( k = 1, i = 1 \)

2) Pack pieces \( R_k, R_{k+1}, \ldots, R_j \) into \( B_i \), as long as \( \sum_{x=k}^{j} X_x \leq 1 \);

3) Set \( k = j+1, i = i+1 \), repeat step 2.

Thus a 'discarded' bin is never reexamined. An example is in Fig. 1b. Note that this algorithm is a 'real-time' one - i.e. for each piece it requires approximately the same (little amount here, of) processing. Its low efficiency in terms of usage of bins is directly related to this.
1.5 In [1] the process \( \{L_i\} \)-occupancy of discarded bins, \( B_i \) and \( \{N_i\} \)-number of pieces in such bins were considered. Here we focus on:

\[ A_n \] - the number of bins used to pack \( n \) pieces.

It will become apparent that the following variable is of interest as well:

\[ T_n \] - occupancy of the last bin used, when \( n \) pieces are packed.

The basic relations will be written for a general piece size distribution, but useful results are mostly obtained here for the case \( X_i \sim U(0,1) \), independently. More of this in Section 4.

2. THE BASIC EQUATION

2.1 The process \( A_n \) evolves according to

\[ A_n = A_{n-1} + \delta_n \]  \hspace{1cm} (1)

where

\[ \delta_n = \begin{cases} 1 & \text{when } R_n \text{ overflows to a new bin}, \\ 0 & \text{otherwise}. \end{cases} \]

Here the auxiliary variable \( T_n \) defined above is useful in relating \( \delta_n \) to \( A_{n-1} \):

Define

\[ P_n(\varepsilon,x) = \text{Prob}(A_n = \varepsilon, T_n \leq x), \ 1 \leq \varepsilon \leq n, \ 0 \leq x \leq 1. \]  \hspace{1cm} (2)

Clearly, \( P_1(\varepsilon,x) = F(x) \) for \( \varepsilon = 1 \) and zero otherwise.

In the sequel we shall always assume \( X_i \) to have a probability density function (pdf), and then, so does \( T_n \). We shall use the notation:
\[ f(x) = \frac{dF(x)}{dx} \]

\[ p_n(x,x) = \frac{dP_n(x,x)}{dx} \]

\[ p_n(x) = p_n(x,1) \]

\[ F_{T_n}(x) = \sum_{\ell=1}^{n} p_n(\ell,x) \]

\[ f_{T_n}(x) = \frac{dF_{T_n}(x)}{dx} \]

The complete dynamics of the packing process are expressed in the following relation, obtained by conditioning on the manners by which the packing of \( n \) pieces can give rise to \( \ell \) occupied bins with the last one filled to level \( x \):

\[ p_n(\ell,x) = \int_{s=0}^{x} p_{n-1}(\ell,s)f(x-s)ds + f(x) \int_{1-x}^{1} p_{n-1}(\ell-1,s)ds \]

\[ = f^* p_{n-1}(\ell,x) + f(x)[p_{n-1}(\ell-1) - p_{n-1}(\ell-1,1-x)], \quad (4) \]

\[ p_1(1,x) = f(x), \quad 1 \leq \ell \leq n, \quad 0 \leq x \leq 1. \]

The value for \( n = 1 \) is evident from (2). The first part in (4) corresponds to the case where \( R_1 \) fits into \( B_{A_{n-1}} \) and the second one takes care of an overflowing piece.

For uniformly distributed piece size (4) reduces to

\[ p_n(\ell,x) = p_{n-1}(\ell,x) + p_{n-1}(\ell-1) - p_{n-1}(\ell-1,1-x). \quad (5) \]

The marginal density of \( T_n \) is, from (4) using (3)

\[ f_{T_n}(x) = f^* f_{T_{n-1}}(x) + f(x)[1-F_{T_{n-1}}(1-x)], \quad f_{T_n}(x) = f(x). \quad (6) \]

\[ 0 \leq x \leq 1. \]
For $X_i \sim U(0,1)$ (6) has the somewhat surprisingly stable solution

$$f_n(x) = \begin{cases} 
1 & n = 1 \\
2x & n > 1 \quad 0 \leq x \leq 1.
\end{cases} \quad (7)$$

No closed form was found here for general $f(*)$.

2.2 Define the generating functions:

$$\beta_n(z,x) = \sum_{\ell=1}^{\ell=n} p_{\ell}(x,x) z^\ell \quad (8)$$

Equation (4) written in terms of $\beta$ is

$$\beta_{n,x}(z,x) = f(0)\beta_{n-1}(z,x) + \int_0^x f'(x-s)\beta_{n-1}(z,s)ds + z f(x)[\beta_{n-1}(z,1) - \beta_{n-1}(z,1-x)] \quad n \geq 1, (9)$$

and equation (5) in terms of $\beta$ becomes

$$\beta_{n,x}(z,x) = \beta_{n-1}(z,x) + z[\beta_{n-1}(z,1) - \beta_{n-1}(z,1-x)], \quad n \geq 1, \quad (10)$$

where subscript $x$ denotes differentiation,

$$\beta_0(z,x) = z; \quad \beta_n(z,0) = 0, \quad n \geq 1.$$

3. SOLUTION FOR THE CASE OF UNIFORM DISTRIBUTION

3.1 We present a solution for equation (10), i.e. for uniformly distributed pieces.

Define $\psi(u,z,x) = \sum \beta_n(z,x) u^n.$
Multiplying (10) by $u^n$ and summing for $n \geq 1$ yields

$$
\varphi_x(u,z,x) = z + u\varphi(u,z,x) + u\varphi(x,1) + u\varphi(u,1-x) \quad \text{ (11)}
$$

To simplify notation write $\varphi(x) = \varphi(u,z,x)$, since (11) is homogeneous in the suppressed variables in $\varphi$. Differentiating (11) (wrt $x$) to obtain a local equation:

$$
\varphi''(x) = u\varphi'(x) + uz\varphi'(1-x)
$$

which on substitution from (11) becomes

$$
\varphi''(x) = u^2(1-z^2)\varphi(x) + uz(1+z) + u^2z(1+z)\varphi(1). \quad \text{ (12)}
$$

Equation (12) together with the initial conditions $\varphi(0) = z$, $\varphi'(0) = z(1+u)$ yield a second order ODE of the form $f'' = af + b$

which has the solution

$$
f = c_1 e^{x\sqrt{a}} + c_2 e^{-x\sqrt{a}} - b/a. \quad \text{ (13)}
$$

Using the initial conditions we obtain

$$
c_1 = \frac{z}{2u} \left( \frac{1+u\varphi(1)}{1-z} + \frac{1+u}{\sqrt{1-z^2}} \right), \quad \text{ (14)}
$$

$$
c_2 = \frac{z}{2u} \left( \frac{1+u\varphi(1)}{1-z} - \frac{1+u}{\sqrt{1-z^2}} \right).
$$

It is $\varphi(1)$, or rather $\varphi(u,z,1)$ that encapsulates the distribution of $A_n$. From (13) and (14) one gets

$$
\varphi(u,z,1) = \frac{z}{u} \frac{e^{u\sqrt{1-z^2}[1+(1+u)\sqrt{1-z^2}]} + e^{-u\sqrt{1-z^2}[1-(1+u)\sqrt{1-z^2}]} \cdot z e^{u\sqrt{1-z^2}} - ze^{-u\sqrt{1-z^2}}}{2ze^{u\sqrt{1-z^2}} - ze^{-u\sqrt{1-z^2}}} \quad \text{ (15)}
$$
3.2 While the probabilities \( P(A_n = k) \) can, in principle, be extracted from (15), no convenient form was obtained, neither for the moments. To obtain the latter we return to eq. (5).

If (5) is integrated over \( 0 \leq x \leq 1 \) one obtains

\[
p_n(x) = \int_0^1 p_{n-1}(x,x) \, dx + n - 1 - \int_0^1 p_{n-1}(x-1,x) \, dx.
\]

(16)

Multiplying by \( x \) and summing for \( x \geq 1 \) yield

\[
\mathbb{E}(A_n) = \sum_{x \geq 1} x \int_0^1 p_{n-1}(x,x) \, dx + \sum_{x \geq 1} (x-1)p_{n-1}(x-1) + \sum_{x \geq 1} p_{n-1}(x-1) - \sum_{x \geq 1}
\]

\[
\int_0^1 p_{n-1}(x-1,x) \, dx - \sum_{x \geq 1} \int_0^1 p_{n-1}(x-1,x) \, dx
\]

\[
= \mathbb{E}(A_{n-1}) + 1 \int_0^1 F_T(x) \, dx
\]

\[
= \mathbb{E}(A_{n-1}) + \mathbb{E}(T_{n-1}).
\]

(17)

Since \( \mathbb{E}(T_{n-1}) \) is available from (7) we have

\[
\mathbb{E}(A_n) = \begin{cases} 1 & n = 1 \\ \frac{2}{3} n + \frac{1}{6} & n > 1. \end{cases}
\]

(18)

3.3 An alternative way to obtain (17) is to note that (and this is true for general \( f(\cdot) \) as well) taking the expectation of (1),

\[
\mathbb{E}(A_n) = \mathbb{E}(A_{n-1}) + \text{Prob}(R_n \text{ starts a new bin})
\]

\[
= \mathbb{E}(A_{n-1}) + \int_0^1 f_{T_{n-1}}(u)P(X>1-u) \, du
\]

\[
= \mathbb{E}(A_{n-1}) + 1 - F_{T_{n-1}}(1).
\]

(19)
For \( f(\cdot) = 1 \) (19) reduces to (17). This is also the only nontrivial case where \( E(A_n) - E(A_{n-1}) \) is constant (for \( n > 2 \)).

3.4 The interest in the second moment is nearly as great as in \( E(A_n) \), since it will give a probabilistic statement on the extent to which we may assume \( E(A_n) \) to characterize the evolution of \( A_n \), especially for large \( n \). From (1), on squaring and taking expectation we get:

\[
V(A_n) = V(A_{n-1}) + E(\delta_n)[1-E(\delta_n) - 2E(A_{n-1})] + 2E(A_{n-1}\delta_n). \quad (20)
\]

For this recursion the initial value is \( V(A_1) = 0 \). The last term in (20) is the only one not known so far:

\[
\gamma_{n-1} = E(A_{n-1}\delta_n) = \sum_{\ell=1}^{n-1} \ell \int_0^1 x\rho_{n-1}(\ell,x)dx.
\]

Multiplying equation (5) by \( \ell x \), summing for all \( \ell \leq 1 \) and integrating \( x \) over \( (0,1) \) one obtains after some routine manipulations

\[
\gamma_n = \gamma_{n-1} + \left\{ \frac{1}{2} [E(A_{n-1}) - E(T_{n-1}^2) + 2E(T_{n-1})] \right\} - \sum_{\ell=1}^{n-1} \ell \int_0^1 x^2\rho_{n-1}(\ell,x)dx \quad (21)
\]

so we only netted a more complicated term! In working out (21) a hint is suggested: call this last term in (21) \( \theta_{n-1} \), and use (5) multiplied by \( \ell x^2 \) to write a recursion for it. This time, because

* This follows from the applicability of the Central Limit Theorem to \( A_n = \sum_{i=1}^{n} \delta_i \).
of the even power of $x$ in the definition of $\theta_n$ no higher degree terms are preserved, and we obtain the following relation

$$\theta_n = \gamma_{n-1} - \theta_{n-1} + \{E(T_{n-1}) - E(T_{n-1}^2) + \frac{1}{3} \{E(A_{n-1}) + E(T_{n-1}^3)\}\}. \quad (22)$$

Denoting the terms within braces in (21) and (22) by $f_1(n)$ respectively, we immediately obtain

$$\gamma_n = f_1(n) - f_2(n-1) + f_1(n). \quad (23)$$

Collecting all these and evaluating (23) produces

$$\gamma_n = \begin{cases} 1/2 & n = 1 \\ 1 & n = 2 \\ 17/12 & n = 3 \\ (20n+4)/45 & n \geq 4 \end{cases}$$

and (20) duly results in

$$V(A_n) = \begin{cases} 0 & n = 1 \\ 1/4 & n = 2 \\ 17/36 & n = 3 \\ (32n-13)/180 & n \geq 4 \end{cases} \quad (24)$$

3.5 Since both $E(A_n)$ and $V(A_n)$ are linear in $n$ it means that for not too small $n$ likely deviations of $A_n$ from $E(A_n)$ are $O(n^{1/2})$, and their relative magnitude is therefore $O(n^{-1/2})$. This allows us to say that $A_n$ is well represented by $E(A_n)$ of (18), for sufficiently large $n$. Moreover, it also allows us to go beyond mean values and evaluate probabilities such as $P(A_{100} > 75)$, where equation (18) tells us $E(A_{100}) = 66.83$. Note that we can use here confidently the central
limit theorem, $[A_n - E(A_n)]/\sqrt{\text{Var}(A_n)} \sim N(0,1)$.

Hence, $\text{Prob}(A_{100} > 75) \approx 1 - \phi(\frac{75.5 - 40/6}{\sqrt{3187/180}}) \approx 0.0197$.

4. APPROXIMATE GENERATING FUNCTIONS

4.1 The method of 3.4 seems to be applicable to higher moments as well, but with rapidly increasing complexity, and could only be effected for uniformly distributed piece sizes.

Thus, while the possibility exists to calculate $\beta_n(\cdot)$ of paragraph 2.2 from eq. (15), or term by term, the prospect is not attractive. Examining equation (1) we note that the obstacle is the dependence between $A_{n-1}$ and $\epsilon_n$, or alternatively $A_{n-1}$ and $T_{n-1}$.

Intuitively, except for the first few values of $n$, unless $A_n$ is at - or near - its unlikely extreme values (1 and $n$), $T_n$ should not be too sensitive to its exact value. Thus the following approximation suggests promising. We demonstrate it first for $\chi \sim U(0,1)$, taking up the general case in 4.4. Consider again equation (5). Note that

\begin{equation}
\int_0^1 p_{n-1}(\xi, x)dx = p_{n-1}(\xi)[1 - \alpha_{n-1}(\xi)]
\end{equation}

where $\alpha_n(\xi) = E(T_n|A_n = \xi)$.

Integrating $x$ out in (5) we have

\begin{equation}
p_n(\xi) = p_{n-1}(\xi)[1 - \alpha_{n-1}(\xi)] + p_{n-1}(\xi-1)\alpha_{n-1}(\xi-1).
\end{equation}

Using the 'near insensitivity' of $T$ to $A$ we write

\begin{equation}
\hat{p}_n(\xi) = \hat{p}_{n-1}(\xi)[1 - \alpha_{n-1}] + \hat{p}_{n-1}(\xi-1)\alpha_{n-1}
\end{equation}
where the circumflex denotes the approximate values. The unconditional
\( \alpha_n \) are simply \( E(T_n) \), available from (7), and thus
\[
\hat{p}_n(k) = \frac{1}{3} \hat{p}_{n-1}(k) + \frac{2}{3} \hat{p}_{n-1}(k-1) \quad n > 2.
\] (28)

Defining
\[
\hat{\beta}_n(z) = \sum_{k=1}^{n} \hat{p}_n(z) z^k
\]
one gets
\[
\hat{\beta}_n(z) = \hat{\beta}_{n-1}(z) \frac{1+2z}{3} = \beta_a(z) \left( \frac{1+2z}{3} \right)^{n-a} \quad n \geq a,
\] (29)
where: \( \beta_a(z) \) is the exact generating function, found from eq. (8)
by recursion with \( x = 1 \); \( a \) is chosen low enough for this recursion
to be painless, under the consideration that higher values for \( a \)
would yield a somewhat more accurate approximation, since then the
contribution of extreme values of \( A_n \) decreases.

4.2 It turns out that quite useful values may be obtained with rather
low \( a \). The derivation suggests that the \( \hat{p}_n(k) \) will be better
approximated for \( k \) which is not very close to 1 or \( n \). Actually
one may expect on some reflection those tail probabilities to be
overestimated.

Thus, for \( n = 8 \) we find the typical pattern taking \( a = 2,4 \)
Note the quite small improvement by increasing $a$ from 2 to 4. One might expect the approximation to improve, as the extreme values become less likely, with increasing $n$.

If (29) is differentiated at $z = 1$, we obtain the correct result, (3) for $E(A_n)$. This can more simply be gleaned from (28).

4.3 As the table above suggests we should expect the variance to be overestimated by $\hat{p}_n(z)$. From (29) we obtain for $n > 2$, $V(A_n) \approx \frac{2}{5} n + \text{terms constant in } n \text{ but dependent on } a$. 

<table>
<thead>
<tr>
<th>$z$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_8(z)$</td>
<td>$0.248 \cdot 10^{-4}$</td>
<td>$0.31498 \cdot 10^{-2}$</td>
<td>$0.37426 \cdot 10^{-1}$</td>
<td>$0.15305$</td>
</tr>
<tr>
<td>$\hat{p}_8(z)$</td>
<td>$0.6858 \cdot 10^{-3}$</td>
<td>$0.89163 \cdot 10^{-2}$</td>
<td>$0.4938 \cdot 10^{-1}$</td>
<td>$0.150891$</td>
</tr>
<tr>
<td>$\hat{p}_8(z)$</td>
<td>$0.5144 \cdot 10^{-3}$</td>
<td>$0.7716 \cdot 10^{-2}$</td>
<td>$0.4681 \cdot 10^{-1}$</td>
<td>$0.15072$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$z$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_8(z)$</td>
<td>$0.3007$</td>
<td>$0.30945$</td>
<td>$0.16247$</td>
<td>$0.3435 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$\hat{p}_8(z)$</td>
<td>$0.274348$</td>
<td>$0.296296$</td>
<td>$0.17558$</td>
<td>$0.4389 \cdot 10^{-1}$</td>
</tr>
<tr>
<td>$\hat{p}_8(z)$</td>
<td>$0.2798$</td>
<td>$0.30041$</td>
<td>$0.17283$</td>
<td>$0.4115 \cdot 10^{-1}$</td>
</tr>
</tbody>
</table>
Thus, $V(A_n)$ would be overestimated by 25% even for large $n$. Higher moments will be probably even further off, but the probabilities near the mode ($\approx \frac{2}{3} n$) are quite well approximated.

4.4 Let us consider the procedure of paragraph (4.1) when piece sizes have the density $f(x)$. Integration over $x, 0 \leq x \leq 1$, in eq. (4) yields

$$p_n(z) = p_{n-1}(z-1) + \int_0^1 f(x) \int_0^{1-x} [p_{n-1}(x, u) - p_{n-1}(x-1, u)] du dx. \quad (30)$$

Again, the "near independence" of $T_n$ and $A_n$, for $n$ not too small, allows us to approximate $p_{n-1}(z, u)$ by $p_{n-1}(z) f_{T_{n-1}}(u)$, with (30) rewritten now as

$$\hat{p}_n(z) = \hat{p}_{n-1}(z-1) + (\hat{p}_{n-1}(z) - \hat{p}_{n-1}(z-1)) \int_0^1 f(x) \int_0^{1-x} f_{T_{n-1}}(u) du dx. \quad (31)$$

Denote the integral by $C_{n-1}$, and rewrite (31):

$$\hat{p}_n(z) = \hat{p}_{n-1}(z) + \hat{p}_{n-1}(z-1) \left[ 1 - C_{n-1} \right]$$

or, in terms of $\hat{\beta}_n(z)$:

$$\hat{\beta}_n(z) = \beta_a(z) \prod_{j=a}^{n-1} \left[ C_j + z(1-C_j) \right]. \quad (32)$$

A further assumption, that $C_n \to C$ rapidly will provide a result just like eq. (29)

$$\hat{\beta}_n(z) = \beta_a(z)[C + z(1-C)]^{n-a}. \quad (33)$$
The last assumption is reasonable when 1) the support of $f(x)$ is not small compared with 1, 2) the variance of $X$ is also not "too small" - say $\geq 0.1$. Recursive calculation of $g_a(z)$ from (4) and (30), and of a few values to $C_n$ from (31) will complete all that is required.

4.5. Example: Let $f(x) = 3x^2$, a choice which assures the last assumption to be satisfied for low value of $n$. Here we evaluated $P_{10}(z)$ directly (via (30)) and $\hat{P}_{10}(z)$ with $a = 2$. The relative difference in $\hat{P}_{10}(z)$ when using $C_3 = 0.0400298$ or $C_{10} = 0.039839$ was rather less than one percent - the probabilities obtained with $C_3$ are presented below:
<table>
<thead>
<tr>
<th>( \ell )</th>
<th>1</th>
<th>2</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{10}(\ell) )</td>
<td>(0.2280 \cdot 10^{-24} )</td>
<td>(0.7193 \cdot 10^{-16} )</td>
<td>(0.3993 \cdot 10^{-11} )</td>
<td>(0.6516 \cdot 10^{-8} )</td>
<td>(0.7531 \cdot 10^{-5} )</td>
<td>(0.0001061 )</td>
<td>(0.003063 )</td>
<td>(0.0421 )</td>
<td>(0.7740 )</td>
<td>(0.5808 )</td>
</tr>
<tr>
<td>( P_{10}(\ell) )</td>
<td>(0.32 \cdot 10^{1-2} )</td>
<td>(0.67 \cdot 10^{-10} )</td>
<td>(0.63 \cdot 10^{-8} )</td>
<td>(0.3458 \cdot 10^{-6} )</td>
<td>(0.222 \cdot 10^{-4} )</td>
<td>(0.0002868 )</td>
<td>(0.004486 )</td>
<td>(0.0451 )</td>
<td>(0.7539 )</td>
<td>(0.5862 )</td>
</tr>
</tbody>
</table>
Again, the same phenomenon - \( \hat{p}_n(x) \) are useless at (the very thin) tail of the distribution but fairly accurate near its mode.

5. COMMENTS

5.1 The unique role of the \( U(0,1) \) distribution in the foregoing may seem a bit surprising. Note that we insisted not only on the uniformity, but that the support of the piece size distribution equals the bin size. Any deviation seems to land one in a quagmire of complex calculations. While these can be pushed recursively (in (4)-(6)), no pattern seems to emerge - we iterated (5) up to \( n = 32 \). In particular, eq. (23) has no apparent equivalent. We tend to ascribe this specificity to the simple form of the distribution of \( S_n \) - the sums of \( n \) piece sizes:

\[
F_{S_n}(x) = x^n/n!, \quad \text{on} \quad x \in [0,1],
\]

but it is probably a matter of taste.

5.2 The form of eqs. (4) and (24) suggests a simple diffusion approximation for \( A_n \), but it does not seem to serve any use for our purposes.

5.3 While Next-Fit seems to be the only reasonable "real time" algorithm, i.e., one that packs pieces as they come, and requires approximately the same effort for each, with a slightly higher expenditure of processing the packing can be materially improved [3]. Examples are - sorting the pieces in increasing (or decreasing) piece size - and then effecting the Next-Fit procedure; or when not all the pieces are immediately available for sorting, scanning with each piece for the first
bin (the one with the lowest index) that will accommodate it. This is the First-Fit packing algorithm. Both algorithms and numerous variations have been shown to improve materially over Next-Fit in terms of worst case behavior, and can be expected to do so on the average as well. Their stochastic analysis however is quite intricate.
REFERENCES

