AN OPTIMIZATION PROBLEM IN GEOMETRY WITH APPLICATION TO COMMUNICATION IN NETWORKS WITH MOBILE NODES

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1. INTRODUCTION

In this paper we consider an optimization problem which occurs in the study of designing access protocols for systems of nodes (or users) with access to a common communication channel. In such a system, a set of nodes use a common communication channel, where they can transmit and receive messages to and from any other node in the system. In the process of communication, a "collision" occurs when 2 or more nodes attempt to transmit a message through the channel simultaneously - a case in which all of the messages transmitted may get lost. One way to avoid collisions is to assign a certain order to the nodes in the system, and each node then may use the channel to transmit a message only in its turn, according to that order. A node may choose not to transmit any message in its turn, and then the next node may use the channel.

Let A, B and C be 3 nodes in the system, and assume that B is scheduled to use the channel before A and A is scheduled to use the channel before C. Let A(t), B(t) and C(t) denote the locations of nodes A, B and C at time t, and let a(t), b(t) and c(t) denote the side lengths of the triangle (A(t), B(t), C(t)) (see Fig. 1.1). Let \( \Delta(t) \) be a non-negative number defined by:

\[
\Delta(t) = b(t) + c(t) - a(t).
\]

When transformed to the right units, \( \Delta(t) \) expresses approximately the time difference between the moment C received a message X transmitted by B at time t and the moment C received a message Y transmitted by A immediately after A received the message X transmitted by B. Thus, if \( \Delta(t) \) or more time after
Figure 1.1
C received message $X$, C had not received yet any message from A, then C can conclude that A did not transmit any message after he received message $X$, and hence that A chose not to use the channel this time, so C can use it (see [2]). It is important for node C, therefore, to keep the value of $\Delta(t)$. However, if the nodes in the system are mobile, the value of $\Delta(t)$ may vary in time, and it is hard to keep and update it. An alternative approach, which was suggested in [2], is to keep an upper bound on $\Delta(t)$, using the following method:

At time $t = 0$ the exact value of $\Delta(t)$ is computed. This value is updated once in a (rather long) while, and between 2 successive updates there are several adjustments. In each adjustment the value of the bound on $\Delta(t)$ is incremented. The amount by which this bound is incremented at each adjustment must be large enough to guarantee that no collisions occur (that is: to guarantee that C transmits its message only when it is clear that A had either transmitted his message or decided not to use the channel this time), but not too large, to provide an efficient use of the channel. The value of the increment must be decided by the following information:

(i) The time difference between 2 successive adjustments, to be denoted as $t_1$.
(ii) The bounds on the velocities of nodes A, B and C, denoted by $v_A$, $v_B$ and $v_C$.
(iii) The maximal possible distance at any time between any 2 nodes in the system, denoted by $R$.

Note that this information does not vary in time, in general, and hence it does not require additional memory or processing units to keep and update it.
Given information (i) and (ii) above, we can find the values $r_A$, $r_B$, and $r_C$ of the maximum possible distances between the locations of nodes A, B and C respectively in 2 successive adjustments. The problem of finding the value of the increment in each adjustment reduces, therefore, to the following geometric problem (to be defined rigorously in the next section):

Find the smallest number $M$ such that for all pairs of triangles $(A_0, B_0, C_0), (A_1, B_1, C_1)$ whose sides-lengths are at most $R$ and for which the distances between $A_0$ and $A_1$, $B_0$ and $B_1$ and $C_0$ and $C_1$ are bounded by $r_A$, $r_B$ and $r_C$ respectively, the difference $(b_1 + c_1 - a_1) - (b_0 + c_0 - a_0)$ is at most $M$.

A solution to the problem above in the special case where $R = \infty$ is given in [2], where it is shown that $M = 2(r_A + \sqrt{r_B^2 + r_C^2})$. This solution provides a fairly reasonable bound on the increment when $R$ is relatively large with respect to $r_A$, $r_B$ and $r_C$ but may be much too large otherwise. In this paper we solve the problem for the case where $R$ is finite. This case seems to be much more complicated than the special case with $R = \infty$, and has a less simple solution.

The paper has 5 sections including this introductory first section. The next section contains preliminary definitions and notations, and gives a formal representation of the problem. In Section 3 we give the solution of the problem in some extremal cases. Section 4, which is the core of the paper, gives the geometric characterization of the solution, that is the geometric properties of the displacements of nodes A, B and C which yield a maximal increment in the value of $\Delta(t)$. The results of Section 4 are used in Section 5 to provide a numerical method to obtain the least upper bound on that increment.
2. PRELIMINARIES

Let \( R \) be a positive number which denotes the maximum possible distance between any 2 nodes in the system. Then a "Legal Configuration", denoted by \( \text{CONF} \), is a triangle \((A,B,C)\) all of whose side-lengths are at most \( R \). Let \( t_1 \) denote the time difference between 2 successive adjustments, which are assumed to take place at times \( t = 0 \) and \( t = t_1 \). Then \( \text{CONF}_0 = (A_0,B_0,C_0) \) and \( \text{CONF}_1 = (A_1,B_1,C_1) \) represent possible configurations of nodes \( A, B \) and \( C \) at times \( t = 0 \) and \( t = t_1 \) respectively.

The corresponding sides of the triangles are denoted either by lower case letters \( a_0, b_0, c_0 \), etc., or by the pairs of capital letters which denote their endpoints. Thus, \( a_0 \) can be represented also as \( B_0C_0 \). \( \alpha, \beta \) and \( \gamma \) denote the angles near \( A_0, B_0 \) and \( C_0 \) respectively.

We shall not distinguish, in general, between line segments and their lengths. Thus, \( A_1B_1 = a_0 \) means that the length of the segment \( A_1B_1 \) (which can be denoted also as \( c_1 \)) is equal to the length of the segment \( a_0 \) (which can be denoted also as \( B_0C_0 \)).

Let \( r_A, r_B \) and \( r_C \) denote the maximum possible distances traversed by \( A, B \) and \( C \) respectively in \( t_1 \) time. (That is: \( r_A = V_A t_1, r_B = V_B t_1, \) and \( r_C = V_C t_1 \), where \( V_A, V_B \) and \( V_C \) are the bounds on the velocities of nodes \( A, B \) and \( C \).) Then a "Legal Displacement" \( D \) is a pair: \( D = (\text{CONF}_0, \text{CONF}_1) \) which satisfies the following:

(i) \( \text{CONF}_0 \) and \( \text{CONF}_1 \) are legal configurations;
(ii) \( A_0A_1 \leq r_A, \ B_0B_1 \leq r_B, \ C_0C_1 \leq r_C \).

For a given configuration \( \text{CONF} = (A,B,C) \), \( \Delta \text{LTA}(\text{CONF}) \) is defined by:

\[
\Delta \text{LTA}(\text{CONF}) = b + c - a.
\]
For a given displacement \( D = (\text{CONF}_0, \text{CONF}_1) \), INC(D) is defined by:

\[
\text{INC}(D) = \text{DELTA}(\text{CONF}_1) - \text{DELTA}(\text{CONF}_0).
\]

A legal displacement, \( D \) is superior to a displacement \( D' \) if \( \text{INC}(D) > \text{INC}(D') \). \( D^* \) is an optimal displacement if it is a legal displacement, and there is no legal displacement \( D' \) which is superior to \( D^* \).

A displacement \( D \) is "Planar" if all of the 6 nodes of the displacement lie in the same plane.

For a given plane \( P \) and a node \( X \), \( d(X,P) \) denotes the distance from \( X \) to \( P \). Similarly, for a line \( \ell \), \( d(X,\ell) \) denotes the distance from \( X \) to \( \ell \).

In Section 4 we investigate the geometric properties of an optimal displacement. For this sake, we shall make use of the following geometric rule, to be denoted as "The Principle of Reflection" [1].

The Principle of Reflection: Let \( \ell \) be a line which separates points \( A \) and \( B \) in a given plane. Let \( B' \) be the reflection of \( B \) in \( \ell \). (That is: the segment \( BB' \) is bisected by and is perpendicular to \( \ell \).) Then:

(i) For each point \( X \) on \( \ell \), \( XB = XB' \)

(ii) \( AB' < AB \).

(See Figure 2.1.)

We shall also use, without mentioning it, the following fact:

Let \( c \) be the hypotenuse in a right angle triangle with sides \( a \) and \( h \) (Figure 2.2). Then \( c - a < \frac{h^2}{2a} \). In particular, if \( a \) is bounded from below by some positive value, then \( c - a = O(h^2) \).
3. SOME EXTREMAL CASES

In this section we consider the problem in the cases where \( r_A + r_B \) are relatively large with respect to \( R \). Both the geometrical and numerical solution in this case are given in:

**Proposition 3.1:** Let \( M \) be the least upper bound on \( \text{INC}(D) \), where \( D \) is a legal displacement. If \( r_A + r_B \geq R \) or \( r_A + r_C \geq R \), then \( M = 2R \).

**Proof:** Clearly, for each legal configuration \((A, B, C)\), \( 0 \leq \Delta((A, B, C)) \leq 2R \). Hence, \( M \leq 2R \), no matter what \( r_A, r_B \) and \( r_C \) are.

Assume that \( r_A + r_B \geq R \). (The case where \( r_A + r_C \geq R \) is similar.) Let \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \) be the configuration defined as follows. (We represent each node by its coordinates in the x-y plane (see Fig. 3.1).)

\[
A_0 = B_0 = (0, r_A); \quad (O = (0, R)); \quad A_1 = (0,0), B_1 = C_1 = (0, R).
\]

It is easily verified that:

(i) \( \Delta(A_0, B_0, C_0) = 0; \)
(ii) \( \Delta(A_1, B_1, C_1) = 2R; \)
(iii) \( D^* \) is a legal displacement

which implies the proposition. \( \square \)

It should be noted that, unless \( r_A + r_B = R \) and \( r_C = 0 \), the geometry of the optimal displacement in the case discussed in the proposition above is not unique. As we shall see, the geometry of the optimal displacement when \( r_A + r_B < R \) and \( r_A + r_C < R \) is unique (up to translation and rotation).

The optimal displacement which was given in Proposition 3.1 had the nice property that all its nodes were colinear. We conclude
this section by showing that, in general, optimal displacements do not have this property.

Lemma 3.1: Let \( D = (((A_0, B_0, C_0), (A_1, B_1, C_1)) \). If \( A_0, B_0 \) and \( C_0 \) are colinear, then \( \text{INC}(D) \leq \text{Max}\{2(r_A + r_B), 2(r_A + r_C)\} \).

Proof: We note first that, by the proof of Proposition 3.1, a displacement \( D^* \) whose nodes are colinear and which satisfy \( \text{INC}(D^*) = \text{Max}\{2(r_A + r_B), 2(r_A + r_C)\} \) exists. To prove that this is the maximum possible increment for such a displacement, we first prove the following claim.

Claim: If in \( D \) \( A_0, B_0 \) and \( C_0 \) are colinear, then there is a legal displacement \( D' = (((A_0', B_0', C_0'), (A_1', B_1', C_1')) \) such that \( \text{INC}(D') \geq \text{INC}(D) \), and either \( A_0' = B_0' \) or \( A_0' = C_0' \).

Proof of the Claim: We consider two cases:

Case 1: In \( D \) \( A_0 \) is not between \( B_0 \) and \( C_0 \) (Fig. 3.2a). WLG let 
\[ A_0 = (0,0), B_0 = (0,d), C_0 = (0,e) \quad (0 < d \leq e), \]
\[ A_1 = (x_1,y_1), \]
\[ B_1 = (x_2,y_2), C_1 = (x_3,y_3). \]
Let \( A_0' = B_0' = B_0 = (0,d), C_0' = C_0, \)
\[ A_1' = (x_1,y_1+d), \]
\[ B_1' = B_1, C_1' = C_1. \]
Then \( \text{DELTA}(A_0,B_0,C_0) - \text{DELTA}(A_0',B_0',C_0') = 2d \).
On the other hand, by the triangle inequality, \( \text{DELTA}(A_1,B_1,C_1) - \text{DELTA}(A_1',B_1',C_1') \leq 2d. \) Hence, \( \text{INC}(D') \leq \text{INC}(D) \). It is also easy to check that if \( D \) is a legal displacement, so is \( D' \).

Case 2: In \( D \) \( A_0 \) is between \( B_0 \) and \( C_0 \) (Fig. 3.2b). Let \( B_0' \) be the closest point to \( A_0 \) which satisfies the following:
(i) \( B_0' \) is colinear with \( A_0, B_0 \) and \( C_0 \);
(ii) \( B_0'B_1 \leq r_B \).

and let \( C_0' \) be defined similarly (see Fig. 3.2b). Let \( D' = (((A_0,B_0',C_0'), (A_1,B_1',C_1')) \). Then \( D' \) is a legal displacement, and \( \text{INC}(D) = \text{INC}(D') \).
since \( \text{DELTA}(A_0,B_0,C_0) = \text{DELTA}(A_0,B_1,C'_1) = 0 \). If \( A_0 = B_0 \) or \( A_0 = C'_0 \), then \( D' \) is the desired displacement, so assume that \( A_0 \neq B_0 \) and \( A_0 \neq C'_0 \). In this case, \( \angle A_0B_0B_1 \geq 90^\circ \) and \( \angle A_0C'_0C_1 \geq 90^\circ \). This implies that \( a_1 \geq B_0C'_0 = a'_1 \). Hence, \( \text{INC}(D') \leq (b_1-b'_0) + (c_1-c'_0) \leq (r_A+r_B) \leq \max(2(r_A+r_B), 2(r_A+r_C)). \) \( b'_0 \) denotes \( A_0B'_0 \), \( c'_0 \) denotes \( A_0C'_0 \). The claim follows from the fact that, by the proof of Proposition 3.1, there is a legal displacement \( D' \) in which either \( A'_0 = B'_0 \) or \( A'_0 = C'_0 \), and \( \text{INC}(D') = \max(2(r_A+r_B), 2(r_A+r_C)). \) □

In view of the claim above, it is sufficient to prove the lemma for a displacement \( D \) in which \( A_0 = B_0 \) or \( A_0 = C_0 \). Assume first that \( A_0 = B_0 \) (see Fig. 3.3). In this case the following inequalities hold:

\[
\begin{align*}
    c_1 &= A_1B_1 \leq c_0 + r_A + r_B = r_A + r_B \\
    b_1 &= A_1C_1 \leq c_1A_0 + r_A \\
    a_1 &= B_1C_1 \geq c_1B_0 - r_B = c_1A_0 - r_B.
\end{align*}
\]

Hence, \( \text{INC}(D) = \text{DELTA}(A_1,B_1,C_1) - \text{DELTA}(A_0,B_0,C_0) = \text{DELTA}(A_1,B_1,C_1) = r_c + (r_b-r_a) \leq (r_A+r_B) + (r_A+r_B) = 2(r_A+r_B). \)

Similarly, if \( A_0 = C_0 \) then \( \text{INC}(D) \leq 2(r_A+r_C) \). This implies the lemma. □

**Lemma 3.2:** If \( r_A + r_B < R, \ r_A + r_C < R, \ r_B > 0 \) and \( r_C > 0 \), then there exist a legal displacement \( D \) for which \( \text{INC}(D) > \max\{2(r_A+r_B), 2(r_A+r_C)\} \).

**Proof:** Assume that the hypothesis of the lemma holds. We shall construct a legal displacement \( D' \) for which \( \text{INC}(D') > 2(r_A+r_B) \).

A similar construction provides a legal displacement \( D'' \) with \( \text{INC}(D'') > 2(r_A+r_C) \). This implies the lemma.
Figure 3.3

Figure 3.4
Let $d = R - (r_A + r_B)$. $D' = ((A_0', B_0', C_0'), (A_1', B_1', C_1'))$ is defined by:

- $A_0' = (0, r_B + d)$; $B_0' = (0, r_B)$; $C_0' = (x, y)$
- $A_1' = (0, R)$; $B_1' = C_1' = (0, 0)$. (See Fig. 3.4.)

$x$ and $y$ in the definition above are any 2 numbers which satisfy the following:

(i) $x \neq 0$ (and hence $A_0$, $B_0$, and $C_0$ are not colinear);
(ii) $C_o A_0 < R$ and $C_o B_0 < R$;
(iii) $C_o C_1 \leq r_C$.

The existence of such $x$ and $y$ follows from the fact that $r_C > 0$.

In $D'$ above we have that $\Delta((A_0', B_0', C_0')) = 2R$ and $\Delta((A_1', B_1', C_1')) = c_o' + (b_o' - a_o') < 2c_o' = 2d$. Hence:

$$\text{INC}(D') = \Delta((A_1', B_1', C_1')) - \Delta((A_0', B_0', C_0')) > 2R - 2d = 2(r_A + r_B).$$

From Lemmas 3.1 and 3.2 we have the following:

Lemma 3.3: Let $D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1))$ be an optimal displacement. If $r_A + r_B < R$, $r_A + r_C < R$, $r_B > 0$ and $r_C > 0$, then $A_0$, $B_0$ and $C_0$ are not colinear.

Proof: By Lemma 3.2, $\text{INC}(D^*) > \max\{2(r_A + r_B), 2(r_A + r_C)\}$. On the other hand, by Lemma 3.1, if $A_0$, $B_0$ and $C_0$ are colinear, then $\text{INC}(D^*) \leq \max\{2(r_A + r_B), 2(r_A + r_C)\}$. □

In the sequel we shall assume that (i) $r_A + r_C < R$, $r_A + r_C < R$ and that (ii) $r_A$, $r_B$ and $r_C$ are positive. Assumption (ii) is not necessary, but it simplifies the discussion. It also does not affect the generality of the geometric and numerical solutions of the problem.
since the solutions are easily verified to be continuous in $r_A$, $r_B$, and $r_C$. Hence, the solutions in the cases where at least one out of $r_A$, $r_B$, and $r_C$ is 0 can be obtained by substituting 0 in the appropriate place in the general solution (which will be obtained under the assumption that $r_A$, $r_B$, and $r_C$ are positive).
4. THE GEOMETRY OF AN OPTIMAL DISPLACEMENT

In this section we investigate the geometric properties of an optimal displacement in the case where $r_A + r_B < R$, $r_A + r_C < R$ and $r_A$, $r_B$ and $r_C$ are positive. First we prove that an optimal displacement exists, and then we prove its geometric properties by showing that any displacement with different properties cannot be optimal. This proof also implies the uniqueness of the optimal displacement, up to translation and rotation. The proofs are based on arguments from elementary geometry and calculus.

Throughout this section the values of $R$, $r_A$, $r_B$ and $r_C$ are fixed, and $r_A + r_B$, $r_A + r_C < R$, $(r_A, r_B, r_C > 0)$.

Lemma 4.1: There exists an optimal displacement $D^*$.

Proof: The value of $\text{INC}((A_0, B_0, C_0), (A_1, B_1, C_1))$ is completely determined by the side lengths of the triangles involved, and hence $\text{INC}$ can be considered as a function of the side-lengths (rather than the vertices) of those triangles. Moreover, $\text{INC}((a_0, b_0, c_0), (a_1, b_1, c_1)) = (a_0 - a_1) + (b_1 - b_0) + (c_1 - c_0)$ is a continuous function of its arguments.

The set of all tuples $((a_0, b_0, c_0), (a_1, b_1, c_1))$ which represent legal displacements is easily verified to be a compact set. Hence, $\text{INC}$ attains its maximum on some element $D^*$ of this set [4].

Lemma 4.2: An optimal displacement is planar.

Proof: Let $D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1))$ be an optimal displacement. We shall prove first that $A_1$ lies in the plane determined by $A_0 B_0 C_0$ (see Fig. 4.1).
Suppose that $A_1$ does not lie on that plane. Then $A_0$ does not lie on the plane $P_0$ of $A_1B_0C_0$. Let $A'_0$ be the projection of $A_0$ on $P_0$, and let $D' = ((A'_0, B'_0, C'_0), (A_1, B_1, C_1))$. The by the "larger side against larger angle" rule, $A'_0A_1 < A_0A_1$. Similarly, $c'_0 = A'_0B'_0 < A_0B_0 = c_0$, and $b'_0 < b_0$. The last 3 inequalities imply that if $D^*$ is a legal displacement so is $D'$, and that $\text{DELT}(A'_0, B'_0, C'_0) < \text{DELT}(A_0, B_0, C_0)$. This implies that $\text{INC}(D') > \text{INC}(D^*)$, and hence that $D'$ is superior to $D^*$, in contradiction to the optimality of $D^*$.

To complete the proof of the lemma it suffices to show that both $B_0$ and $C_0$ lie in the plane of $A_1B_1C_1$. We shall prove it only for $B_0$, since the other case is similar.

Assume that $B_0$ does not belong to that plane. Then $B_1$ does not belong to the plane $P_1$ of $A_1B_0C_1$ (see Fig. 4.2). Let $B_1E$ be the height from $B_1$ in the triangle $A_1B_1B_0$. Let $B'_1$ be the unique point in $P_1$ satisfying:

(i) $B'_1B_0 = B_1B_0$
(ii) $B'_1A_1 = B_1A_1$
(iii) $\angle B_1EC_1 < 90^\circ$.

It is easy to see that $B'_1E$ is perpendicular to $B_0A_1$. Hence, the projection of $B_1E$ or $P_1$ lies on $B'_1E$. This implies that $\angle B'_1EC_1 < \angle B_1EC_1$. Combining this inequality with the fact that $B_1E = B'_1E$, we obtain:

(iv) $B'_1C_1 < B_1C_1$.

Let $D'$ be the displacement obtained by replacing $B_1$ by $B'_1$ in $D^*$. Then by (i), (ii) and (iv) above, $D'$ is a legal displacement which, by (iv), is superior to $D^*$. A contradiction.

We shall need the following definition.
Definition 4.1: Let \((A,B,C)\) be a legal configuration. Then \(A\) is an "internal" node if there is an open neighbourhood \(B\) of \(A\) such that for each \(A'\) in \(B\), the configuration \((A',B,C)\) is a legal configuration. \(B(C)\) are internal if a similar condition apply to them.

A characterization of internal nodes is given by:

**Lemma 4.3:** \(A\) is internal iff \(b < R\) and \(c < R\). \((B(C)\) are internal iff \(a\) and \(b\) \((a\) and \(c\)) are less than \(R\).

**Proof:** Omitted.

The importance of the concept of "internal node" is displayed in the following:

**Lemma 4.4:** Let \(D^* = ((A_0,B_0,C_0),(A_1,B_1,C_1))\) be an optimal displacement. Then:

(a) \(A_1A_0 = r_A\), \(B_0B_1 = r_B\) and \(C_0C_1 = r_C\).

(b) If \(A_0\) is internal, then \(A_0A_1\) bisect the angle opposite to \(a\).

If \(B_0\) \((C_0)\) is internal, then \(B_0B_1\) \((C_0C_1)\) bisects the external angle near \(\beta(\gamma)\), (see Fig. 4.3).

**Proof:** (a) Assume that \(A_0A_1 < r_A\). Then we can replace \(A_0\) by \(A'_0\) such that

(i) \(A'_0\) is located inside triangle \((A_0,B_0,C_0)\), and \(A'_0B_0 < A_0B_0\), \(A'_0C_0 < A_0C_0\). (Here we used the fact that, by Lemma 3.3, \(A_0\), \(B_0\), and \(C_0\) are not collinear.)

(ii) \(A'_0A_1 < r_A\). 

(i) implies that \(\Delta(A'_0,B_0,C_0) < \Delta(A_0,B_0,C_0)\), and hence \(\text{INC}((A'_0,B_0,C_0),(A_1,B_1,C_1)) < \text{INC}(D^*)\). (i) and (ii) implies that \(D' = ((A'_0,B_0,C_0),(A_1,B_1,C_1))\) is a legal displacement (provided \(D^*\)
Figure 4.3
is a legal displacement). Hence, \( D' \) is superior to \( D^* \). A contradiction. The proofs that \( B_0 B_1 = r_B \) and \( C_0 C_1 = r_C \) are similar and are omitted.

(b) Assume that \( A_0 A_1 \) does not intersect with the angle opposite to \( \alpha \), that is: that \( \alpha_1 \neq \alpha_2 \) in Fig. 4.3. WLG we may assume that \( \alpha_1 > \alpha_2 \).

Let \( h \) be a small positive value, and let \( A_0' \) be a node satisfying:

\[
A_o A_o' = h \quad \text{and} \quad A_o' A_1 = A_o A_1.
\]

Let \( b_0' = A_0'C_0 \) and \( c_0' = A_0'B_0 \). Then one can verify that:

(i) \( b_0' = b_0 - h \sin \alpha_1 + O(h^2) \);  
(ii) \( c_0' = c_0 + h \sin \alpha_2 + O(h^2) \).

Let \( D' = ((A_o'B_o'C_0), (A_1B_1C_1)) \). By the fact that \( A_o \) is internal \( D' \) is a legal displacement for small enough \( h \). Also, \( \text{INC}(D') - \text{INC}(D^*) = h(\sin \alpha_1 - \sin \alpha_2) + O(h^2) \), which is a positive value for small enough \( h \) — in contradiction with the optimality of \( D^* \).

The proof of part (b) of the lemma for \( B_o \) and \( C_o \) is similar, and (hence) is omitted.

**Lemma 4.5:** Let \( D^* = ((A_o'B_o'C_0), (A_1B_1C_1)) \) be an optimal displacement. Then \( A_o \) is internal.

**Proof:** By Lemma 4.3, it is enough to prove that \( b_0 < R \) and \( c_0 < R \). We shall prove only that \( b_0 < R \), since the other case is similar.

So assume that \( b_0 = R \).

**Claim:** \( \angle A_0 C_0 C_1 < 90^\circ \) (see Fig. 4.4).

**Proof of the Claim:** Let \( \ell_1 \) and \( \ell_2 \) be 2 lines perpendicular to \( b_0 \) through \( A_o \) and \( C_0 \) respectively (see Fig. 4.4). Then \( A_1 \) must be above \( \ell_1 \), otherwise \( A_o \) can be replaced by \( A_o' \) to obtain a legal
We consider two cases:

Case a: \( A_0 \) and \( C_1 \) do not lie on the same side of the line containing \( a_0 \) (see Fig. 4.5). Then, by the claim above, \( \angle A_0 C_0 C_1 < 90^\circ \).

This implies that \( 0^\circ < \angle B_0 C_0 C_1 < \angle A_0 C_0 C_1 < 90^\circ \). Now, \( \angle B_0 C_0 C_1 = \angle A_0 C_0 C_1 \) only if \( A_0, B_0 \) and \( C_0 \) are collinear, which, by Lemma 3.3, is impossible and hence \( \angle B_0 C_0 C_1 < \angle A_0 C_0 C_1 \). Since both angles are less than \( 90^\circ \),

\[
\sin \angle B_0 C_0 C_1 < \sin \angle A_0 C_0 C_1.
\]

Let \( h \) be a small positive number, and let \( C_0' \) be the unique point (in the plane) satisfying:

(i) \( C_0' C_0 = C_1 C_0' \)
(ii) \( C_0' C_0 = h \)
(iii) \( C_0' A_0 < C_0 A_0 \) (see Fig. 4.5).

Then:
\[
\begin{align*}
A_0 C_1 &= A_0 C_0 - h \sin \angle A_0 C_0 C_1 + O(h^2) \\
B_0 C_1 &= B_0 C_0 - h \sin \angle B_0 C_0 C_1 + O(h^2).
\end{align*}
\]

Let \( D' = ((A_0, B_0, C_0'), (A_1, B_1, C_1)) \). Then by the equalities above, for small enough \( h \) \( D' \) is a legal displacement, and:

\[
\text{INC}(D') - \text{INC}(D^*) = h(\sin \angle A_0 C_0 C_1 - \sin \angle B_0 C_0 C_1) + O(h^2).
\]
Figure 4.5
which is a positive number for small enough \( h \), in contradiction to the optimality of \( D^* \).

Case b: \( A_0 \) and \( C_1 \) lie on the same side of the line containing \( a_0 \) (see Fig. 4.6). Let \( C'_0 \) be the reflection of \( C_0 \) in the line containing \( C_1B_0 \), and let \( D' = ((A_0,B_0,C'_0),(A_1,B_1,C_1)) \). By the principle of reflection one can show that \( D' \) is a legal displacement which is superior to \( D^* \). A contradiction. This completes the proof of the lemma.

Corollary 1: Let \( D^* \) be an optimal displacement. Then \( A_0A_1 = r_A \) and \( A_0A_1 \) bisect the angle opposite to \( \alpha \).

Proof: \( A_0A_1 = r_A \) by Lemma 4.4(a). \( A_0A_1 \) bisects the angle opposite to \( \alpha \) since \( A_0 \) is internal node, by Lemma 4.4(b).

The proofs of Cases a and b in Lemma 4.5 imply also the following corollary, which will be used later:

Corollary 2: Let \( D^* = ((A_0,B_0,C_0),(A_1,B_1,C_1)) \) be an optimal displacement. Then:

(i) \( \angle A_0B_0B_1 > 90^\circ \) and \( \angle A_0C_0C_1 > 90^\circ \).
(ii) The line through \( a_0 \) separates \( A_0 \) from \( B_1 \) and \( C_1 \).

The locations of \( B_1 \) and \( C_1 \) in an optimal displacement are determined in the following 3 lemmas.

Lemma 4.6: Let \( D^* = ((A_0,B_0,C_0),(A_1,B_1,C_1)) \) be an optimal displacement. Then:

a) \( A_1 \) and \( B_1 \) do not lie on the same side of the line containing \( B_0C_1 \).
b) $A_1$ and $C_1$ do not lie on the same side of the line containing $C_0B_1$.

Proof: It is enough to prove a). We assume that a) is false and derive a contradiction. We consider two cases:

Case (i): $B_0B_1 \geq B_0C_1$ (Fig. 4.7a). Let $B_1' = C_1$. Then we have the following inequalities:

1. $B_0B_1' \leq B_0B_1 \leq r_B$ ;
2. $B_1'C_1 = 0$ ;
3. $b_1 = A_1B_1' > c_1-a_1$.

(The last inequality follows from the fact that in the triangle $A_1C_1B_1$ the side lengths are $A_1B_1'$, $c_1$ and $a_1$.)

Let $D' = ((A_0,B_0,C_0), (A_1,B_1',C_1))$. Then by the inequalities above $D'$ is a legal displacement, and $\text{INC}(D') - \text{INC}(D^*) = a_1-B_1'C_1 + A_1B_1' - c_1 = a_1 + A_1B_1' - c_1 > 0$, a contradiction.

Case (ii): $B_0B_1 < B_0C_1$ (Fig. 4.7b). Let $E$ be the reflection of $B_1$ in the line containing $B_0C_1$. By Corollary 2 to Lemma 4.5 we have that $\not\exists A_1B_0B_1 \geq 90^\circ$, which can be shown to imply that the line containing $B_1E$ separates $A_1$ from $C_1$. This implies that there are points $X$ on $B_1E$ which satisfy: $X \neq B_1$, and $XA_1 < C_1A_1 \leq R$. We also have, by the principle of reflection, that $A_1E > A_1B_1'$. Hence, there exists a point $B_1'' \neq B_1$ on $B_1E$ which satisfies: $B_1''A_1 \leq R$, and for each point $X$ on $B_1E$ with $XA_1 \leq R$, $XA_1 \leq B_1''A_1$. If $B_1'' = E$ then $A_1B_1'' > A_1B_1$. If $B_1'' \neq E$ then $A_1B_1'' = R \geq A_1B_1$, and $B_1''C_1 < B_1C_1$. In both cases, replacing $B_1$ by $B_1''$ yields a displacement $D''$ which is legal (since $B_1''B_0 \leq B_1B_0$), and is superior to $D^*$. A contradiction. □
Corollary: Let \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \) be an optimal displacement, and assume that \( B_1 \neq C_1 \). Then if the line through \( a_1 \) intersects the line through \( b_0(c_0) \) at point \( E(F) \), then \( C_0(B_0) \) lies between \( E(F) \) and \( A_0 \) (see Fig. 4.8).

Proof: If \( E(F) \) lies in any other portion of the line containing \( b_0(c_0) \), then \( C_1A_1(B_1A_1) \) lie on the same side of the line containing \( C_0B_1(B_0C_1) \), in contradiction to Lemma 4.6. \( \square \)

Lemma 4.7: Let \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \) be an optimal displacement. Then \( C_1 \) and \( B_1 \) lie in angle \( C_0A_1B_0 \).

Proof: (An outline): Using the principle of reflection, it can be shown that any displacement which does not satisfy the lemma is not optimal. e.g., if \( C_1 \) lies in \( \angle C_0A_1B_0 \) and \( B_1 \) lies out of it (see Fig. 4.9), then replacing \( B_1 \) by its reflection \( B'_1 \) in the line containing \( B_0A_1 \) yields a displacement which is superior to \( D^* \). \( \square \)

Lemma 4.8: Let \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \) be an optimal displacement. Then \( B_1 = C_1 \).

Proof: Assume that \( B_1 \neq C_1 \) in \( D^* \). We shall derive a contradiction by showing that \( D^* \) is not optimal.

Let \( E \) be the point where the line containing \( a_1 \) intersects the line containing \( c_0 \), and let \( \delta = \angle B_0EB_1 \). WLG assume that \( \delta < \beta \) (see Fig. 4.10). (If \( a_1 \parallel c_0 \), the \( E \) is a point in infinity and \( \delta = 0^\circ \).)

By the corollary to Lemma 4.6 above, \( B_0 \) lies between \( A_0 \) and \( E \), and hence \( \beta' = \angle A_1B_1C_1 \) is external to a triangle containing \( \delta \) (the shaded triangle in Fig. 4.10). Hence, \( \beta' > \delta \). Let \( h \) be a small positive number, and let \( B'_0 \) and \( B'_1 \) be defined by:

\[
B'_1B'_1 = B_0B'_1 = h
\]

\[
\angle A_0B'_0B'_0 = \angle A_1B'_1B'_1 = \delta/2
\]
Figure 4.10
Note that $B_1B'_1 || B_0B'_0$, since both are parallel to the bisector of $\delta$. Hence, $B'_0B'_1 = B_0B_1$, which implies that the displacement $D' = ((A_0, B'_0, C'_0), (A'_1, B'_1, C'_1))$ is a legal displacement.

Let $a'_1 = B'_1C'_1$, $c'_1 = A'_1 B'_1$, $a'_0 = B'_0 C'_0$, $c'_0 = A'_0 B'_0$. Then:

$$a'_1 = a_1 - h \cos \delta/2 + O(h^2) \quad ; \quad c'_1 = c_1 - h \cos(\beta'-\delta/2) + O(h^2)$$

$$a'_0 = a_0 - h \cos(\beta-\delta/2) + O(h^2) \quad ; \quad c'_0 = c_0 - h \cos \delta/2 + O(h^2),$$

which implies that:

$$\text{INC}(D') - \text{INC}(D^*) = h(2 \cos \delta/2 - \cos(\beta-\delta/2) - \cos(\beta'-\delta/2)) + O(h^2).$$

Since $\beta \geq \delta$ and $\beta' > \delta$, $\text{INC}(D') > \text{INC}(D^*)$ for small enough $h$. This is the desired contradiction. □

**Lemma 4.9:** In an optimal displacement $a_0 < R$.

**Proof:** By Lemma 4.8, $B_1 = C_1$ in an optimal displacement (see Fig. 4.11). Let $B_0B_1$ intersect the external angle near $B_0$ to angles $x$ and $x_1$, and $C_0C_1$ intersect the external angle near $C_0$ to angles $y$ and $y_1$ (see Fig. 4.11). We first prove that $x \geq x_1$ and $y \geq y_1$. Assume that $x < x_1$. Then, by replacing $B_0$ by $B'_0$ such that $B_0B'_0 = h$ and $B'_0B'_1 = B_0B_1$ (Fig. 4.11), we obtain a displacement $D' = ((A_0, B'_0, C'_0), (A'_1, B'_1, C'_1))$ such that:

$$\text{INC}(D') - \text{INC}(D^*) = \text{DELTA}((A_0, B'_0, C'_0)) - \text{DELTA}((A_0, B_0, C_0)) = h(\cos x - \cos x_1) + O(h^2).$$

For small enough $h$, $\text{INC}(D') > \text{INC}(D^*)$ — in contradiction to the optimality of $D^*$. A similar argument show that $y \geq y_1$.

Assume now that $a_0 = R$. Then $\alpha \geq \beta$ and $\alpha \geq \gamma$, since otherwise either $b_0$ or $c_0$ are greater than $R$. Hence $\alpha \geq 60^\circ$. This implies that $x + x_1 + y + y_1 \geq 240^\circ$. Since $x \geq x_1$ and $y \geq y_1$,
Theorem 4.1: There is an optimal displacement \( \theta^* = (A_0, B_0, C_0), (A_1, B_1, C_1) \).

Lemma 4.10: In an optimal displacement \( \theta^* = (A_0, B_0, C_0), (A_1, B_1, C_1) \),

\[ b_1 = c_1 = R. \] (Note that by Lemma 4.8, \( B_1 = C_1 \), and hence \( b_1 = c_1 \).)

Proof: By Lemmas 4.5 and 4.9, \( B_0 \) and \( C_0 \) are internal nodes in an optimal displacement. The corollary follows by Lemma 4.4.

We need one more lemma before we can fully characterize the geometric properties of an optimal displacement.

Lemma 4.10: In an optimal displacement \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \),

\[ b_1 = c_1 = R. \] (Note that by Lemma 4.8, \( B_1 = C_1 \), and hence \( b_1 = c_1 \).)

Proof: Assume that \( b_1 < R \). By Lemma 5, \( b_0 < R \) and \( c_0 < R \). Hence, for small enough \( h \), the displacement \( D' = ((A_0, B_0', C_0'), (A_1, B_1', C_1')) \) shown in Figure 4.12, where \( B_0B_0' = C_0C_0' = B_1B_1' = h, b_1B_0'O'C_0'C_1'B_1' \) is a legal displacement. Also, \( b_1' = c_1' = b_1 + h, h_0' < b + h \), \( c_0' < c_0 + h \). This implies that: \( a_0 = a_0', b_1 - b_0 < b_1' - b_0', \) and \( c_1 - c_0 < c_1' - c_0' \). It follows that \( D' \) is superior to \( D^* \) — a contradiction.

Combining the above 10 lemmas and their corollaries, we get the following main result of this section.

Theorem 4.1: There is an optimal displacement \( D^* = ((A_0, B_0, C_0), (A_1, B_1, C_1)) \).

\( a_0' \) denotes \( B_0'C_0' \), \( b_0' \) denotes \( A_0'C_0' \), etc.
which satisfies the following:

(i) \( \mathbf{D}^* \) is planar;

(ii) \( A_1 \) lies on the bisector of the angle opposite to \( \alpha \);

(iii) \( B_1 = C_1 \), and \( b_1 = c_1 = R \);

(iv) \( B_1 \) lies on the point where the bisector of \( \alpha \) meet with the
               bisectors of the external angles near \( \beta \) and \( \gamma \).

(v) \( A_0 A_1 = r_A, B_0 B_1 = r_B, C_0 C_1 = r_C \).

Moreover, the optimal displacement is unique (up to rotations and translations). The optimal displacement is shown in Figure 4.13.

Proof: By the lemma above.
5. THE NUMERICAL SOLUTION

In this section we consider the following problem: Given $R$, $r_A$, $r_B$ and $r_C$, what is the least upper bound on $\text{INC}(D)$ for all legal displacements $D$? We assume, as before, that $r_A + r_B < R$ and that $r_A + r_C < R$ (otherwise this upper bound is $2R$, as shown in Proposition 3.1).

Let $D^*$ be an optimal displacement. Then it follows directly from the definitions that the value we are looking for is $\text{INC}(D^*)$. In the special case where $R = \infty$ the solution was found in [2], and is $2(r_A + \sqrt{r_B^2 + r_C^2})$. (In this case $D^*$ is an ideal configuration, where $b_0 = c_0 = b_1 = c_1 = \infty$.) Unfortunately, we were not able to derive such a simple formula for the case where $R < \infty$. However, a solution in this general case can be obtained by numerical methods, along the outline suggested in this section.

Lemma 5.1: Let $D^*$ be an optimal displacement as shown in Fig. 4.13. Let $\beta = \angle C_0B_0B_1$, $\gamma = \angle B_0C_0B_1$. Then:

$$\text{INC}(D^*) = 2(r_A + r_B\cos\beta + r_C\cos\gamma + (R-r_A)\cos\alpha/2).$$

Proof: $\text{INC}(D^*) = (a_o-a_1) + (b_1-b_0) + (c_1-c_0)$. One can verify by inspection that:

$$a_0 = r_C\cos\gamma + r_B\cos\beta \quad a_1 = 0$$
$$b_0 = (R-r_A)\cos\alpha/2 - r_C\cos\gamma \quad b_1 = R$$
$$c_0 = (R-r_A)\cos\alpha/2 - r_B\cos\beta \quad c_1 = R.$$

The lemma follows. \(\square\)

By the lemma above, one can find $\text{INC}(D^*)$ if one can solve for $\alpha$, $\beta$ and $\gamma$. By the Sine formula and the fact that $\sin(180^\circ-x) = \sin x$, \(\dot{\omega} = \dot{x} + \dot{\omega} \int \sin x\)
the following equalities are derived:

(i) \[ \frac{r_B}{\sin \frac{\alpha}{2}} = \frac{R - r_A}{\sin \beta} \]

(ii) \[ \frac{r_C}{\sin \frac{\alpha}{2}} = \frac{R - r_A}{\sin \gamma} \]

We also get by inspection that \( \alpha = \beta + \gamma - 180^\circ \), or:

(iii) \( \frac{\alpha}{2} = \beta + \gamma - 90^\circ \).

The above equations can be solved by first finding \( \beta \), and then substituting \( \sin \alpha/2 = \frac{r_B \sin \beta}{R - r_A} \) and \( \sin \gamma = \frac{r_C}{r_B} \sin \beta \).

\( \sin \beta \) can be found by the following:

**Lemma 5.2:** Let \( q = \frac{r_C}{r_B} \) and \( M = \frac{r_B}{r - r_A} \). Then \( \sin \beta \) is the positive solution \( y \) of the equation:

(S) \[ (1 - q^2)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}} - qy^2 - My = 0. \]

**Proof:** From (iii) above we have that \( \sin \alpha/2 = \cos(\gamma + \beta) \). By substituting in (i) we obtain:

\[ \frac{r_B}{\cos(\gamma + \beta)} = \frac{R - r_A}{\sin \beta}. \]

This can be rewritten as:

(iv) \[ \frac{\cos \gamma \cos \beta - \sin \gamma \sin \beta}{\sin \beta} = \frac{r_B}{R - r_A} = M. \]
By the Sine formula, \( \frac{\sin \gamma}{\sin \beta} = \frac{r_C}{r_B} = q \). Hence, (iv) can be written as:

\[
\frac{(1-q^2 \sin^2 \beta)^{1/2}(1-\sin^2 \beta)^{1/2} - q \sin^2 \beta}{\sin \beta} = M
\]

which implies the lemma.

An analytic solution to (S) requires to find a root of a polynomial of degree 4, and hence is hard (though possible - see e.g. [3]). It seems, therefore, that solutions by numerical methods are more appropriate for this task. In the special case where \( r_B = r_C \) (that is \( q = 1 \)), (S) reduces to the much simpler equation:

(S') \[
2y^2 + My - 1 = 0,
\]

and

\[
\sin \beta = \frac{-M + \sqrt{M^2 + 8}}{4}.
\]

Note that when \( R \to \infty \) \( M \) tends to 0, and the solution to (S') tends to \( \sqrt{2}/2 \) - which fits with the solution to the special case where \( R = \infty \) given in [2].
REFERENCES


