SUPER-NETS AND THEIR HIERARCHY

by

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ABSTRACT

The paper considers various modifications and extensions of the concept of Petri nets, however without labelling; and establishes a hierarchy of the corresponding language families.

The last part of the paper deals with closure properties of net languages.
1. INTRODUCTION

In this paper we consider various extensions of the concept of Petri nets, however without labelling, and the languages associated with such nets.

The main part of the paper is concerned with establishing a hierarchy of the corresponding language families.

The concept of "Super-Net" introduced in this paper combines various modifications and generalizations of Petri nets, which have appeared in the literature so far (see [BRA], [PET]).

The last part of the paper deals with closure properties of net languages.

2. SUPER-NETS

We denote by \( \omega \) the set of non-negative integers.

**Definition 2.1:** A SUPER-Net (SUPER-Net) is a 4-tuple \( N = (P,T,V,K) \), where

1. \( P \) and \( T \) are finite sets of places and transitions, respectively.
2. \( P \cap T = \emptyset, \ P \cup T \neq \emptyset \).
3. \( V \) is a function,
   \[ V: (P \times T) \cup (T \times P) \to \omega \cup \{I,E,L\} \]
   Note I,E,L are symbols indicating "Inhibiting", "Emptying", and "Logical" arcs, respectively.
4. \( V(T \times P) \subseteq \omega \).
5. \( K \) is a function,
   \[ K: P \to \{\omega\} \cup \omega \times \{A,R\} \]
   Note A,R are symbols indicating "Absorbing", and "Restricting" places.
If \( K(p) = \infty \), we say that the place \( p \) has infinite capacity. If \( K(p) = (k,A) \) or \( K(p) = (k,R) \) we say that the place \( p \) has finite capacity \( k \in \omega \). We denote by \( k(p) \) the capacity \((\infty \text{ or } k \in \omega)\) of the place \( p \).

Definition 2.2: A marked SUP-Net is a pair \( S = (N,M) \), where \( N \) is a SUP-Net and \( M \) is a marking of \( N \), i.e. a function \( M: P \to \omega \), satisfying the condition

\[
(\forall p \in P) [k(p) \in \omega \Rightarrow M(p) \leq k(p)].
\]

A marked SUP-Net \( S = (P,T,V,K,M) \) is represented graphically as follows:

1. Places are represented by circles \( (\bigcirc) \).
2. Each place \( p_i \) is labelled by \( p_i/K(p_i) \).
3. A transition \( t_j \) is represented by a bar, labelled by \( t_j \).
4. The place \( p \in P \) is connected by a directed arc to the transition \( t \in T \), iff \( V(p,t) \neq 0 \). The arc is labelled by \( V(p,t) \).
5. The transition \( t \in T \) is connected by a directed arc to the place \( p \in P \), iff \( V(t,p) > 0 \). The arc is labelled by \( V(t,p) \).
6. The integer \( m = M(p) \) is written inside the circle representing \( p \). Usually, one does not write 0 inside the circle.

An example of a marked SUP-Net is shown in Figure 2.1.

In the sequel we need the following:

Definition 2.3: Let \( S = (P,T,V,K,M) \) be a marked SUP-Net. We define a function \( W: P \times T \to \omega \) as follows:

\[
W(p,t) = \begin{cases} 
V(p,t) & \text{if } V(p,t) \in \omega \\
0 & \text{if } V(p,t) = 1 \\
M(p) & \text{if } V(p,t) = E \\
1 & \text{if } V(p,t) = L \land M(p) > 0 \\
0 & \text{if } V(p,t) = L \land M(p) = 0.
\end{cases}
\]
Figure 2.1 Example of marked SUP-Net.
Definition 2.4: Let \( S = (P, T, V, K, M) \) be a marked SUP-Net. A transition \( t \in T \) is enabled if the following conditions are satisfied:

1. \( (\forall p \in P) [V(p, t) \in w + M(p) \geq V(p, t)] \).
2. \( (\forall p \in P) [V(p, t) = I + M(p) = 0] \).
3. \( (\forall p \in P) [V(p, t) = E + M(p) > 0] \).
4. \( (\exists p \in P) V(p, t) = L + (\exists p \in P) [V(p, t) = L \land M(p) > 0] \).
5. \( (\forall p \in P) [K(p) \in (\omega, \infty) + M(p) + V(t, p) - W(p, t) \leq k(p)] \).

Definition 2.5: Let \( S = (P, T, V, K, M) \) be a marked SUP-Net and \( t \in T \) an enabled transition of \( S \). We define the marking \( M' \) of \( N = (P, T, V, K) \) as follows:

\[
(\forall p \in P) M'(p) = \min[M(p) + V(t, p) - W(p, t), k(p)].
\]

We say that \( M' \) is obtained from \( M \) by firing \( t \) (notation: \( M(t) = M' \)). Frequently, it is convenient to represent a marking \( M \) by the vector \( [M(p_1), M(p_2), \ldots, M(p_n)] \), where \( P = \{p_1, \ldots, p_n\} \).

For the example of Figure 2.1 we obtain the following "firing sequence".

\[
[0, 3, 1] \ [t_2] > [0, 5, 1] \\
[0, 5, 1] \ [t_2] > [0, 6, 1] \\
[0, 6, 1] \ [t_3] > [0, 6, 0] \\
[0, 6, 0] \ [t_4] > [0, 2, 1] \\
[0, 2, 1] \ [t_1] > [1, 2, 1] \\
[1, 2, 1] \ [t_1] > [2, 2, 1] \\
[2, 2, 1] \ [t_1] > [3, 2, 1] \\
[3, 2, 1] \ [t_3] > [2, 2, 0] \\
[2, 2, 0] \ [t_3] > [0, 2, 0]
\]

Definition 2.6: Let \( N(P, T, V, K) \) be a SUP-Net. For every \( t \in T \) we set

\[ t' \triangleq \{p \in P | V(t, p) \neq 0\} \]
\[ t^- \triangleq \{p \in P | V(p, t) \neq 0\} \]

We call every \( p \in t^- \) an output place of \( t \), and every \( p \in t^- \) an input place of \( t \).
3. SUPER-NET LANGUAGES

With a given marked SUP-Net $S = (P, T, V, K, M)$ we associate a language $L(S)$ over the alphabet $T$ in the usual way.

**Definition 3.1:** Let $N = (P, T, V, K)$ be a SUP-Net, and let $w \in T^+$, i.e. $w$ is a finite string of transitions $w = t_1 t_2 \ldots t_r$. $w$ is called a firing sequence of the marked SUP-Net $S = (N, M)$ iff there exist markings $M_1, \ldots, M_r$ such that

$$M[t_1 > M_1, M_1[t_2 > M_2, \ldots, M_{r-1}[t_r > M_r].$$

In this case we write $M[w > M_r]$, and say that $M_r$ is reachable from $M$. We also write $M[\lambda > M]$ for every marking $M$, where $\lambda$ denotes the empty sequence.

**Definition 3.2:** Let $S = (P, T, V, K, M)$ be a marked SUP-Net. We define its language $L(S)$ as follows:

$$L(S) \triangleq \{ x \in T^* | \exists M' \} M[x > M'].$$
4. CLASSIFICATION AND HIERARCHY OF SUPER-NETS

In this section we represent various types of nets as special cases of SUP-Nets, and study their hierarchy.

Definition 4.1: The following table defines various types of nets as special cases of SUP-Nets, by restricting range(V) and range(K).

<table>
<thead>
<tr>
<th>TYPE OF NET</th>
<th>RANGE(V)</th>
<th>RANGE(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GP-Net</td>
<td>( \omega )</td>
<td>( {=} )</td>
</tr>
<tr>
<td>P-Net</td>
<td>( {0,1} )</td>
<td>( {=} )</td>
</tr>
<tr>
<td>I-Net</td>
<td>( {0,1} \cup {I} )</td>
<td>( {=} )</td>
</tr>
<tr>
<td>E-Net</td>
<td>( {0,1} \cup {E} )</td>
<td>( {=} )</td>
</tr>
<tr>
<td>L-Net</td>
<td>( {0,1} \cup {L} )</td>
<td>( {=} )</td>
</tr>
<tr>
<td>A-Net</td>
<td>( {0,1} )</td>
<td>( \omega \times {A} )</td>
</tr>
<tr>
<td>R-Net</td>
<td>( {0,1} )</td>
<td>( \omega \times {R} )</td>
</tr>
</tbody>
</table>

In the sequel we study the hierarchy of the above net types.

Clearly, the above net types can be combined into more complex classes of SUP-Nets. For example, the Place-Transition (P/T) Net of [BRA] is the combination of the GP-type and the R-type.

In the graphical representations of the first five types of SUP-Nets we label places by \( p \) instead of \( p/\omega \). In the diagrams representing any type of Definition 4.1 we omit the label \( 1 \) of any arc.

Definition 4.2: Let \( L \) be a language over some finite alphabet \( \Sigma \).

We say that \( L \) is \textit{GP-realizable} iff \( L = L(S) \) for some marked GP-Net \( S = (P,T,V,K,M) \), with \( T = \Sigma \). We denote by \( \text{GPL} \) the set of all GP-realizable languages.

In a similar way we associate sets of languages with the other types of SUP-Nets defined above (Definition 4.1).
Proposition 4.1: The languages in RL and AL are regular.

Proof: This result follows immediately, since the corresponding SUP-Nets all have finite sets of reachable markings. Hence each such SUP-Net may be viewed as finite automaton with the set of reachable markings as its state set.

The following proposition is an immediate consequence of a result derived in [POR-YOE].

Proposition 4.2: Let \( L \) be a language over \( \Sigma \) in either RL or PL or IL or GPL. Assume \( x\sigma \in L \), where \( x \in \Sigma^+ \) and \( \sigma \in \Sigma \).

Furthermore, let \( y \) be a permutation of \( x \), with \( y \in L \). Then \( y\sigma \in L \).

For two sets \( A \) and \( B \), we write \( A \subseteq B \) to state that \( A \) is a subset of \( B \), and \( A \subset B \) to state that \( A \subseteq B \) and \( A \neq B \).

The following result is proven in [POR-YOE].

Theorem 4.1:
(a) PL \( \subseteq \) IL
(b) PL \( \subseteq \) LL
(c) IL and LL are not comparable, i.e. neither IL \( \subseteq \) LL nor LL \( \subseteq \) IL holds.

Theorem 4.2:
(a) RL \( \subseteq \) PL
(b) RL \( \subseteq \) AL
(c) AL and PL are not comparable.

Proof: (a) Let \( S \) be a marked R-Net. One easily verifies that there exists a marked P-Net \( S' \), which is equivalent to \( S \), i.e. \( L(S) = L(S') \). This is illustrated in Figure 4.1. Thus, RL \( \subseteq \) PL.
Figure 4.1  
(a) Example of a marked R-Net $S_1$. 
(b) An equivalent marked P-Net $S_2$, i.e. $L(S_1) = L(S_2)$.
On the other hand, there exist languages in PL which are not regular, e.g. the language of the marked p-Net $S_3$ of Figure 4.2.

Figure 4.2: Example of a marked p-Net ($S_3$).

Thus, by Proposition 4.1, we have $RL \subseteq PL$.

(b) With every marked R-Net one easily associates an equivalent marked A-Net (see Figure 4.3). Hence $RL \subseteq AL$.

Figure 4.3 - A marked A-Net $S_4$ equivalent to the marked R-Net $S_1$ of Figure 4.1 (a).
To show that $RL \subseteq AL$, we consider the marked A-Net $S_5$ of Figure 4.4.

![Figure 4.4 - Example of a marked A-Net ($S_5$)](image)

Clearly $abab \in L(S_5)$ and $aab \in L(S_5)$, but $aabb \notin L(S_5)$. Hence, by Proposition 4.2, $L(S_5) \notin RL$. Consequently, $RL \subseteq AL$.

(c) Consider the marked P-Net $S_3$ of Figure 4.2. Since $L(S_3)$ is not regular, $PL \subseteq AL$ does not hold, in view of Proposition 4.1. The above argument in the proof of (b) about $S_5$, also shows that $L(S_5) \notin PL$. Hence $AL \subseteq PL$ does not hold. It follows that $AL$ and $PL$ are not comparable.

Theorem 4.3: $PL \subseteq EL$.

Proof: Clearly, $PL \subseteq EL$, by Definition 4.1. Consider now the marked E-Net $S_6$ of Figure 4.5.

![Figure 4.5 - Example of a marked E-Net ($S_6$)](image)

We have $L(S_6) = L(S_5)$. Hence $L(S_6) \notin PL$. Thus, $PL \subseteq EL$. □
Proposition 4.3: Let $L$ be a language over $\Sigma$ in either $AL$ or $PL$ or $EL$ or $LL$. Assume $\sigma_1 \sigma_2 \in L$, where $\sigma_1 \in \Sigma$ and $\sigma_2 \in \Sigma$. Then $\sigma_1 \sigma_2 \in L$.

Proof: Let $L = L(S)$, where $S'$ is the relevant $SUP$-Net. Assume $\sigma_1 \sigma_2 \notin L$, but $\sigma_1 \sigma_1 \sigma_2 \in L$. After firing $\nu_1$ in $S$ for the first time, there exists a set of input places of $\sigma_2$ which are not marked, preventing $\sigma_2$ from firing. But this set of input places of $\sigma_2$ remains not marked, after $\sigma_1$ is fired a second time, in contradiction with our assumption that $\sigma_1 \sigma_1 \sigma_2 \in L$. Hence $\sigma_1 \sigma_2 \in L$. □

Theorem 4.4: $PL \subseteq GPL$.

Proof: Evidently, $PL \subseteq GPL$, by Definition 4.1. We now consider the marked GP-Net $S_7$ of Figure 4.6.

![Figure 4.6 - Example of a marked GP-Net $S_7$.](image)

We have $ab \in L(S_7)$, but $aab \notin L(S_7)$. Hence, by Proposition 4.3, $L(S_7) \notin PL$. It follows that $PL \subseteq GPL$. □

Let $S = (P,T,V,K,M)$ be a marked L-Net. For any $t \in T$ we define an equivalence relation $E_t$ on $^*t$ as follows:

$$E_t = \{(p,p') | p \in ^*t \land p' \in ^*t \land [p = p' \lor V(p,t) = V(p',t) = L]\}.$$  

We denote by $Q_t$ the partition $^*t/E_t$ of $^*t$. For any $q \in Q_t$, we set

$$M(q) = \max\{M(p) | p \in q\}.$$
We say that \( q \in Q_t \) is an output of \( t \), iff \( (\exists p \in q) \ V(t,p) = 1 \). Let \( S = (P,T,V,K,M) \) be a marked E-Net. In the following two theorems we make use of the fact that \( \ldots \) implies \( b^i \in L(S) \) for every \( i > 1 \), and \( b - b^i = 0 \) in \( S \) implies the following: if \( x b \in L(S) \), where \( x \in T^* \), then also \( x b^i \in L(S) \) for every \( i > 1 \).

**Theorem 4.5:** The language types EL and LL are not comparable.

**Proof:** (a) Consider the marked E-Net \( S_8 \) of Figure 4.7.

![Figure 4.7 - Example of a marked E-Net (S_8)](image)

We wish to show that \( L(S_8) \notin LL \).

Assume that \( L(S_8) = L(S) \), where \( S = (N,M) \) is a marked L-Net. We have \( a^4b^4 \in L(S) \). Let \( M[a^4>M' \) in \( S \). Then, for every \( q \) in \( Q_b \) which is not an output of \( b \), we must have \( M'(q) \geq 4 \). Now, \( a^4cab \in L(S) \). Let \( M'[cab>M'' \) in \( S \). Then, for every \( q \) in \( Q_b \) which is not an output of \( b \), \( M''(q) \geq 1 \). Also, since \( M'' \) was obtained by firing \( b \), we must have \( M''(q) \geq 1 \) for every \( q \) in \( Q_b \) which is an output of \( b \). Hence \( b \) is enabled by \( M'' \). It follows that \( a^4cab^2 \in L(S) \), but \( a^4cab^2 \notin L(S_8) \), contradicting our assumption that \( L(S_8) = L(S) \).

Consequently, \( L(S_8) \notin LL \), i.e. EL $\neq$ LL.
(b) To show that $LL \not\subseteq EL$, we consider the marked $L$-Net $S_9$ of Figure 4.8.

![Figure 4.8 - Example of a marked L-Net $(S_9)$](image)

One easily verifies that $L(S_9) \not\subseteq EL$. Indeed, assume $L(S_9) = L(S)$, where $S = (N, M)$ is a marked E-Net. Let 

$$k = \max \{M(p) \mid p \in 'b - b'\}.$$ 

Since $b \in L(S)$, we have $k \geq 1$ as well as $(\forall p \in 'b) M(p) = 1$.

Now, $a^{k+1}b^{k+1} \in L(S)$. Therefore,

$$(\forall p \in 'b - b')(p \in 'a' - 'a').$$

Let $M[a^{k+1}>M']$. It follows that $(\forall p \in 'b - b'[M'(p) = k + 2]$.

Since $a^{k+1}b^{k+2} \in L(S)$, we must have

$$(\forall p \in 'b - b') V(p, b) \neq E.$$ 

Let $M'[b^{k+1}>M'']$. Then $(\forall p \in 'b - b')[M''(p) = 1]$. Since $M''$ was obtained by firing $b$, we must also have

$$(\forall p \in 'b \cap b') M''(p) = 1.$$ 

It follows that $b$ is enabled in $M''$, hence $a^{k+1}b^{k+2} \in L(S)$.

But $a^{k+1}b^{k+2} \in L(S_9)$. Consequently, $L(S_9) \not\subseteq EL$. Thus, $IL \not\subseteq EL$.

Since $LL \not\subseteq EL$ and $EL \not\subseteq LL$, Theorem 4.5 is proven. □
Theorem 4.6: \( A_1 \) is not comparable with either \( GPL \) or \( IL \) or \( LL \) or \( EL \).

**Proof:** (a) Consider the marked \( P \)-Net \( S_3 \) of Figure 4.2. \( S_3 \) is also a marked \( GP \)-Net, \( I \)-Net, \( L \)-Net and \( E \)-Net. Since \( L(S_3) \) is not regular, neither \( GPL \subseteq A_1 \) nor \( IL \subseteq A_1 \) nor \( LL \subseteq A_1 \) nor \( EL \subseteq A_1 \) can hold, in view of Proposition 4.1.

(b) Consider the marked \( A \)-Net \( S_5 \) of Figure 4.4. The argument in the proof of Theorem 4.2(b) about \( S_5 \), also shows that \( L(S_5) \notin GPL \) and \( L(S_5) \notin IL \). Consequently neither \( A_1 \subseteq GPL \) nor \( A_1 \subseteq IL \) can hold.

(c) Consider the marked \( A \)-Net \( S_{10} \) of Figure 4.9.

![Figure 4.9 - Example of a marked A-Net (S_{10})](image)

Assume that \( L(S_{10}) = L(S) \), where \( S = (N,M) \) is a marked \( L \)-Net. We have \( a^3 b^2 \in L(S) \). Let \( M'[a^3\gamma] \) and \( M'[b^2\gamma] \) in \( S \). Since \( a^3 b^2 \in L(S) \), there exists an element \( q \) of \( Q_b \), which is not an output of \( b \), such that \( M'(q) = 2 \) and \( M''(q) = 0 \). Since \( M'(q) = 2 \) we have \((\forall p \in q)[V(a,p) = 1 \rightarrow V(p,a) \neq 0] \). We also have \( a^3 b^2 a^2 \in L(S) \). Let \( M'[a^2\gamma\gamma] \). Thus, we have \((\forall p \in q)[M''(p) \leq 1] \). It follows that \( a^3 b^2 a^2 b^2 \in L(S) \), but \( a^3 b^2 a^2 b^2 \in L(S_{10}) \), contradicting our assumption that \( L(S_{10}) = L(S) \). Consequently, \( A_1 \subseteq LL \) does not hold.

(d) Consider the marked \( A \)-Net \( S_{10} \) of Figure 4.9. Assume that \( L(S_{10}) = L(S) \), where \( S = (N,M) \) is a marked \( E \)-Net. We have \( a^3 b^2 \in L(S) \). Let \( M[a^3\gamma] \) and \( M'[b^2\gamma] \) in \( S \). Since \( a^3 b^3 \in L(S) \), we have \((\exists p \in (b-b))[V(p,b) \notin E \land M'(p) = 2 \land M''(p) = 0] \). Let \( p \)

satisfy this condition. Since \( M'(p) = 2 \) we have \( V(a,p) = 1 \rightarrow V(p,a) = 1 \).
We distinguish between two cases:

**Case I:** \( V(a,p) = 1 \land V(p,a) = 1 \).

Since \( M''(p) = 0 \) it follows that \( a^3 b^2 a \notin L(S) \).

**Case II:** \( V(a,p) = 0 \).

We have \( a^3 b^2 a \in L(S) \). Let \( M'[a>M'' \) in \( S \). Hence, \( M''(p) = 0 \).

It follows that \( a^3 b^2 ab \notin L(S) \).

But, since \( a^3 b^2 ab \in L(S_{10}) \), both cases I and II contradict our assumption that \( L(S_{10}) = L(S) \). Consequently, \( AL \subseteq EL \) does not hold.

Parts (a), (b), (c) and (d) complete the proof of Theorem 4.6.

The following proposition is an immediate consequence of a result derived in [POR-YOE].

**Proposition 4.4:** Let \( L \) be a language over \( \Sigma \) in either EL or GPL. Let \( S = (N,M) \) be a marked E-Net or a marked GP-Net.

Let \( y_1, y_2 \in L(S) \) and \( M[y_1>M_1, M[y_2>M_2 \) in \( S \). If \( M_2 \geq M_1 \) and \( y_1 y_3 \in L(S) \) then \( y_2 y_3 \in L(S) \).

**Theorem 4.7:** IL is not comparable with either EL or GPL.

**Proof:** (a) Consider the marked I-Net \( S_{11} \) of Figure 4.10.

![Figure 4.10 - Example of a marked I-Net (S₁₁)](image)

It can easily be shown that \( L(S_{11}) \notin EL \) and \( L(S_{11}) \notin GPL \), similarly to the proof in [POR-YOE] that \( L(S_{11}) \notin LL \) and \( L(S_{11}) \notin PL \).

(b) Consider the marked E-Net, \( S_6 \) of Figure 4.5. Clearly \( abab \in L(S_6) \) and \( aab \in L(S_6) \), but \( aabb \notin L(S_6) \). Hence, by Propos. 4.2, \( L(S_6) \notin IL \).
(c) Consider the marked GP-Net $S_7$ of Figure 4.6. Assume that $L(S_7) = L(S)$, where $S = (N, M)$ is a marked I-Net. We have $a \in L(S)$ but $ab \notin L(S)$. Let $M[a \rightarrow M]$, so we have one of the following two cases:

**Case I:** $(\exists p \in 'b)[V(p, b) = 1 \land M'(p) = 0]$. Let $p$ denote a place satisfying this condition. Now, $a^2 \in L(S)$, and let $M'[a \rightarrow M']$. We still have $M''(p) = 0$, hence $a^2 b \notin L(S)$, but $a^2b \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

**Case II:** $(\exists p \in 'b)[V(p, b) = 1 \land M'(p) > 0]$. Let again $p$ denote a place satisfying this condition and $M'[a \rightarrow M']$. Since $a^2 b \in L(S)$ we have $p \in 'a$, $V(p, a) = 1$ and $M''(p) = 0$. Hence $a$ is not enabled in $M''$, but $a^3 \in L(S_7)$, contradicting our assumption that $L(S_7) = L(S)$.

We have thus proved that GPL $\subseteq$ IL does not hold.

Parts (a), (b) and (c) complete the proof of Theorem 4.7.

\[\Box\]

**Theorem 4.8:** GPL is not comparable with either LL or EL.

**Proof:** (a) Consider the marked GP-Net $S_7$ of Figure 4.6. We have $ab \notin L(S_7)$, but $aab \in L(S_7)$. Hence, by Proposition 4.3 $L(S_7) \notin LL$ and $L(S_7) \notin EL$.

(b) Consider the marked L-Net $S_9$ of Figure 4.8. Clearly $bab \notin L(S_9)$ and $ab \in L(S_9)$, but $abb \notin L(S_9)$. Hence, by Proposition 4.2 $L(S_9) \notin GPL$.

(c) Consider the marked E-Net $S_6$ of Figure 4.5. Clearly, $abab \in L(S_6)$ and $aab \in L(S_6)$, but $aabb \notin L(S_6)$. Hence, by Proposition 4.2 $L(S_6) \notin GPL$.

Parts (a), (b) and (c) complete the proof of Theorem 4.8.

\[\Box\]

The theorems of Section 4 yield the hierarchy of SUP-Nets illustrated in Figure 4.11, where $\longrightarrow$ denotes proper inclusion, and $XL \dashv \vdash YL$ indicates that $XL$ and $YL$ are not comparable.
Figure 4.11 - Hierarchy of SUB-Nets
5. SOME CLOSURE PROPERTIES OF GPL

The following results provide solutions to open problems mentioned in [PET].

Theorem 5.1: GPL is not closed under union.

Proof: Consider the marked GP-Nets $S_{12}$ and $S_{13}$ of Figure 5.1.

![Diagram of GP-Nets $S_{12}$ and $S_{13}$](image)

Figure 5.1 - (a) Example of a marked GP-Net $(S_{12})$

(b) Example of a marked GP-Net $(S_{13})$

We have $abb \in L(S_{12}) \cup L(S_{13})$ and $ba \in L(S_{12}) \cup L(S_{13})$ but $bab \notin L(S_{12}) \cup L(S_{13})$. Hence, by Proposition 4.2 $L(S_{12}) \cup L(S_{13}) \notin GPL$. Consequently, GPL is not closed under union. □

Theorem 5.2: GPL is not closed under concatenation.
Clearly $abc \in L(S_{15}) \cup L(S_{16})$ and $ba \in L(S_{15}) \cup L(S_{16})$, but $bac \in L(S_{15}) \cup L(S_{16})$. Hence, by Proposition 4.2 $L(S_{15}) \cup L(S_{16}) \notin \text{GPL}$. Consequently, GPL is not closed under the concurrency operator. \hfill \Box
Theorem 5.4: GPL is not closed under prefix regular substitution (see [HACK]).

Proof: Consider the marked GP-Net $S_{17}$ of Figure 5.4.

![Diagram of GP-Net $S_{17}$]

Figure 5.4 - Example of a marked GP-Net ($S_{17}$).

Consider the prefix regular substitution $f(a) = \{\lambda, a, ab, abb, b, ba\}$. We have $abb \in f(L(S_{17}))$ and $ba \in f(L(S_{17}))$ but $bab \notin f(L(S_{17}))$. Hence, by Proposition 4.2, $f(L(S_{17})) \notin$ GPL. Consequently, GPL is not closed under prefix regular substitution. 

Theorem 5.5: GPL is not closed under Kleene star (iteration).

Proof: Consider the marked GP-Net $S_{14}$ of Figure 5.2. We have $abab \in (L(S_{14}))^*$ and $aab \in (L(S_{14}))^*$ but $aabb \notin (L(S_{14}))^*$. Hence, by Proposition 4.2 $(L(S_{14}))^* \notin$ GPL. Consequently, GPL is not closed under Kleene star.

In [PET] Petri nets with final markings and the corresponding languages are discussed. It is noteworthy that the results and proofs of Theorems 5.2, 5.3 and 5.5 also apply to the family of languages denoted by $L^f$ in [PET, p. 186].
REFERENCES


