THE POWER OF LAMBDA-LABELS IN PETRI NETS

by

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ABSTRACT

In this paper we prove that labelled generalized Petri nets, including λ-labels, are more powerful than λ-free labelled generalized Petri nets.

We thus solve an open problem, formulated by M. Hack and J.L. Peterson.
1. INTRODUCTION

In this paper we solve an open problem in the theory of Petri nets. Namely, we show that labelled generalized Petri nets (including λ-labels) are more powerful than λ-free labelled generalized Petri nets. This problem, formulated in [HACK] and [PET], was open till now.

In Section 2, we introduce the necessary concepts and definitions. In Section 3 we state and prove our results.
2. GENERALIZED PETRI NETS AND THEIR LANGUAGES

Definition 2.1: A Generalized Petri Net (GPN) is a 4-tuple \( N = (P,T,V,M_0) \), where
- \( P \) and \( T \) are finite sets of places and transitions, respectively.
- \( P \cap T = \emptyset \), \( P \cup T \neq \emptyset \).
- \( V \) is a function,
  \[ V: (P \times T) \cup (T \times P) \to \omega \] (\( \omega \) denotes the set of non-negative integers).
- \( M_0 \), the initial marking, is a function \( M_0: P \to \omega \).

A GPN is represented graphically as follows:
1. Places are represented by circles.
2. Transitions are represented by bars.
3. The place \( p \in P \) is connected by a directed arc to the transition \( t \in T \) iff \( V(p,t) > 0 \); The arc is labelled by \( V(p,t) \), if \( V(p,t) > 1 \).
4. The transition \( t \in T \) is connected by a directed arc to the place \( p \in P \) iff \( V(t,p) > 0 \); The arc is labelled by \( V(t,p) \), if \( V(t,p) > 1 \).
5. The integer \( m = M_0(p) \) is written inside the circle representing \( p \).
   Usually, one does not write 0 inside the circle.

Definition 2.2: Let \( M \) be a marking of \( N \), i.e. a function \( M: P \to \omega \).
A transition \( t \in T \) is enabled in \( M \) iff
\[ (\forall p \in P)[M(p) \geq V(p,t)]. \]

Definition 2.3: Let \( M, M' \) be two markings of \( N \). We say that \( M' \) is obtained from \( M \) by firing \( t \) (notation: \( M[t \to M'] \)) iff
1. \( t \) is enabled in \( M \)
2. \( (\forall p \in P)[M'(p) = M(p) - V(p,t) + V(t,p)]. \)
Definition 2.4: Let \( w \in T^* \), i.e. \( w \) is a finite string of transitions \( w = t_1 t_2 \cdots t_r \). \( w \) is called a firing sequence of the GPN \( N \) iff there exist markings \( M_1, M_2, \ldots, M_r \) such that

\[
M_0[t_1 > M_1, M_1[t_2 > M_2, \ldots, M_{r-1}[t_r > M_r]
\]

In this case we write \( M_0[w > M_r] \).

We also write \( M[\lambda > M^\lambda] \) for every marking \( M \), where \( \lambda \) denotes the empty sequence.

Definition 2.5: Let \( N = (P, T, V, M_0) \) be a GPN. We define its language \( L(N) \) as follows:

\[
L(N) = \{ x \in T^* | (\exists M)[M_0[x > M]] \}
\]

Definition 2.6: A labelled GPN is a triple \( \Gamma = (N, \Sigma, \eta) \), where \( N = (P, T, V, M_0) \) is a GPN; \( \Sigma \) is a finite alphabet and \( \eta \) is a mapping \( \eta: T \rightarrow \Sigma \cup \{ \lambda \} \). \( \Gamma \) is called \( \lambda \)-free iff \( \eta(T) \subseteq \Sigma \). If \( M_0[w > M'] \), where \( w \in T^* \), we also write \( M[\eta(w) > M'] \).

Definition 2.7: The language of \( \Gamma \) is defined by

\[
L(\Gamma) \triangleq \eta(L(N)) = \{ \eta(x) | x \in L(N) \}
\]

Definition 2.8: Let \( F \) be a finite set of markings of \( N \). We define the terminal language \( L_F(\Gamma) \) as follows:

\[
L_F(\Gamma) \triangleq \{ \eta(x) | (x \in T^*) \land (\exists M \in F)[M_0[x > M]] \}
\]

Definition 2.9: Let \( L \) be a language over some finite alphabet \( \Sigma \). We say that \( L \) is \( \lambda \)-GPN-realizable iff \( \lambda = L(\Gamma') \) for some labelled \( \lambda \)-free GPN \( \Gamma = (N, \Sigma, \eta) \). \( L \) is \( \lambda \)-GPN-realizable iff there exists an arbitrary labelled GPN \( \Gamma \) with \( L = L(\Gamma) \).

We denote by \( L^\lambda_{\text{GPL}} \) and \( L^\lambda_{\text{GPN}} \) the sets of all \( \lambda \)-GPN-realizable and \( \lambda \)-GPN-realizable languages, respectively. In a similar way we define the sets \( L^\lambda_{\text{GPL}_F} \) and \( L^\lambda_{\text{GPN}_F} \).
3. MAIN THEOREM

**Definition 3.1:** Let \( \Gamma = (P, T, V, M_0, \Sigma, \eta) \) be a labelled \( \Gamma PN \). Assume \( w = t_1 t_2 \ldots t_r \in T^+ \) and \( \eta(t_i) = \sigma \in \Sigma \). We say that \( i \) is a \( \sigma \)-index of \( w \). \( I(\sigma, w) \) will denote the set of all \( \sigma \)-indices of \( w \).

**Lemma 3.1:** Let \( N = (P, T, V, M_0) \) be a \( \Gamma PN \).

Let \( x = t_1 t_2 \ldots t_r \) be a firing sequence of \( N \).

Let \( y \) be a permutation of \( x \). If \( y \) is also a firing sequence of \( N \), then

\[ M_0[x] M \iff M_0[y] M. \]

**Proof:** This is an immediate consequence of Definitions 2.3 and 2.4.

**Theorem 3.1:** \( L^{\lambda} GPL \) properly includes \( LGPL \), i.e.

\[ L^{\lambda} GPL \supset \subseteq LGPL. \]

**Proof:** Clearly, \( L^{\lambda} GPL \supset \subseteq LGPL \).

Now, consider the labelled \( \Gamma PN \) \( \Gamma_1 \) of Figure 3.1. It will be shown that \( L(\Gamma_1) \subseteq \subseteq LGPL \).

Assume that \( L(\Gamma) = L(\Gamma_1) \), where \( \Gamma = (P, T, V, M_0, \Sigma, \eta) \) is a labelled \( \lambda \)-free \( \Gamma PN \), and \( \Sigma = \{a, b, c\} \).

We define the following sets.

\[ A = \{ x \in (a + b)^* | i = \sum_{j \in I(a, x)} 2^j \} \]

\[ B = \{ x \in (a + b)^* | i > \sum_{j \in I(a, x)} 2^j \} \]

One easily verifies that \( A \subseteq L(\Gamma_1) \) and \( B \cap L(\Gamma_1) = \emptyset \).
The condition \( L \cap \ (1.1') \neq \emptyset \).
For any positive integer $n$, let

$$C_n = \{x \in (a + b)^* \mid \text{length of } x \text{ is } n\}$$

$$D_n = \{M \mid (M_0[x > M] \land (x \in C_n) \land (\exists i \in \omega)((xc^i \in A) \land (\exists M')(M[c^i > M']))\}$$

**Claim 1:** $|D_n| \geq |C_n| = 2^n$.

**Proof:** Assume $i < j$, $x_i \cdot c^i \in A, x_j \cdot c^j \in A$, where $x_i, x_j \in C_n$. Let $M_0[x_i > M_i, M_0[x_j > M_j].$ If $M_i = M_j$, we have $M_i[c^j > M_j$ for some $M$. Hence, $x_i \cdot c^j \in L(I)$. But this contradicts the condition $B \cap L(I) = \emptyset$. Therefore, $M_i \neq M_j$. Hence, the number of markings in $D_n$ equals to or is greater than the number of words in $C_n$, which is $2^n$. \[\square\]

On the other hand, the number of combinations to choose $n$ t's such that $t \in T$ is $(\frac{|T| + n - 1}{n})$. Therefore, by Lemma 3.1, $|D_n| \leq (\frac{|T| + n - 1}{n})$. But if we choose $n$ big enough we will have $(\frac{|T| + n - 1}{n}) < 2^n$, contradicting Claim 1. Hence $l(I) \notin L_{GPL}$. Thus $L_{GPL} \subset L_{GPL}$.

**Theorem 3.2:** $L^\lambda_{GPL_F} \supset L_{GPL_F}$.

**Proof:** Clearly, $L^\lambda_{GPL_F} \supset L_{GPL_F}$.

Now, consider the labelled GPN $I_1$ of Figure 3.1, with the set

$$F = \{M \mid (\forall p \in P)M(p) = 0\}.$$

Similarly, to the proof of Theorem 3.1, it is easily shown that

$L_F(I_1) \notin L_{GPL_F}$. Hence

$L^\lambda_{GPL_F} \supset L_{GPL_F}$. \[\square\]

It would be interesting to prove our results by means of nets which are simpler than the one of Figure 3.1. In particular, we have the following conjecture, but were not yet able to prove it.
Consider the labelled GPN $\Gamma_2$ of $\Gamma_3$.

Conjecture: Consider the labelled GPN $\Gamma_2$ of $\Gamma_3$.

that

$L(\Gamma_2) \in \text{LGPL}.$

**Figure 3.2**: Example of a labelled GPN($\Gamma_2$).

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