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CONVERGENCE OF PRODUCT INTEGRATION RULES FOR
FUNCTIONS WITH INTERIOR AND ENDPOINT SINGULAR-
ITIES OVER BOUNDED AND UNBOUNDED INTERVALS

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ABSTRACT

We investigate convergence of product integration rules (based on Gaussian quadratures) for functions with interior and endpoint singularities, over bounded and unbounded intervals. A new result is also obtained for convergence of general Gaussian quadratures on singular integrands.

Running Title: Product Integration Rules

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1. INTRODUCTION

The standard product integration rule has the form
\[ \int_a^b f(x)k(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \]

where the \( \{x_i\} \) are usually zeroes of orthogonal polynomials and the \( \{w_i\} \) are chosen to make the rule exact for polynomials of degree less than \( n \). A thorough investigation of convergence appears in Sloan and Smith [7] for the case where (i) \((a,b)\) is bounded; (ii) \( f(x) \) is bounded and Riemann integrable; (iii) the orthogonal polynomials correspond to a weight \( w(x) \) positive almost everywhere in \((a,b)\); (iv) \( k(x) \) satisfies a weak integrability condition.

In this note, we allow (i) \((a,b)\) to be unbounded; (ii) \( f(x) \) to have finitely many interior and endpoint singularities; (iii) we consider calculation of the more general integral \( \int_a^b f(x)\theta(x)dx \). The methods used are those of Sloan and Smith [7] together with an old lemma of Shohat, and a generalized Markov-Stieltjes inequality first proved by Posse (see Freud [3, pp. 33, p. 92]). We use also some convergence theorems for Gaussian quadrature and Lagrange interpolation due to Shohat and Freud (see Freud [3, pp. 92-98]). In proving a new convergence theorem for Gaussian quadrature of singular integrands, we correct an error in a result of Freud [3, pp. 132-133, Problems 12, 13].
2. NOTATION

(i) Throughout \((a,b)\) will be a fixed real interval \((-\infty < a < b < \infty)\) and \(\alpha: (a,b) \rightarrow \mathbb{R}\) will be right continuous, monotone increasing with infinitely many points of increase such that
\[
\alpha_j = \int_a^b x^j \, d\alpha(x) < \infty, \quad j = 0, 1, 2 \ldots
\] (2.1)

We assume that, apart from normalization, \(\alpha\) is the unique solution of the (Hamburger) moment problem (2.1) - see for example Freud [3, Chapter 2].

(ii) \(\varphi_0, \varphi_1, \varphi_2 \ldots\) will be the sequence of orthonormal polynomials for \(\alpha\), and \(a < x_{n 1} < x_{n 2} < \ldots < x_{nn} < b\) will denote the zeroes of \(\varphi_n\), while we write \(x_{n 0} = a, x_{n n+1} = b, n = 1, 2, 3, \ldots\). The Gauss-Jacobi quadrature rule of order \(n\) is
\[
\int_a^b g(x) \, d\alpha(x) = K_n(g) = \sum_{j=1}^{n} \lambda_{nj} g(x_{nj})
\] (2.2)

and is exact for polynomials of degree less than \(2n\). When \(g(x)\) has (finitely many) singularities in \((a,b)\), we must modify (2.2) as follows: Suppose for some positive integer \(\ell\)
\[
a < \gamma_1 < \gamma_2 < \ldots < \gamma_{\ell} < b \quad \text{are those points in} \ (a,b)
\]

for which \(\limsup_{x \to \gamma_i} |g(x)| = \infty, i = 1, 2, \ldots \ell\) \quad (2.3)

Define \(\tau(n,g)\) to be the subset of \(\{1, 2, \ldots n\}\) such that \(j \in \tau(n,g)\) if either
\[
\{\gamma_1, \gamma_2, \ldots, \gamma_{\ell}\} \cap (x_{n j-1}, x_{n j+1}) = \emptyset
\] (2.4A)
or if for some \(1 \leq i \leq \ell, \gamma_i \in (x_{n j-1}, x_{n j+1})\)
\[
\text{but } |\alpha(x_{nj}) - \alpha(y_i)| \geq \lambda_{nj}
\] (2.4B)
Define \( K_n^*(g) = \sum_{j \in \tau(n,g)} \lambda_{nj} g(x_{nj}) \). 

(2.5)

The condition (2.4A) ensures that, in forming \( K_n^*(g) \), we include those abscissas \( x_{nj} \) which are not the closest among the \( \{x_{ni}\} \) from the left or right, to any singularity of \( g \). The condition (2.4B) ensures that we include also those abscissas \( x_{nj} \) which are closest from the left or right to some singularity of \( g \), provided they are not "too close" to the singularity.

One can modify (2.4B); \( \lambda_{nj} \) can be replaced by \( c\lambda_{nj} \) (\( c \) a positive constant). The main results still hold then, though one has to modify Lemma 3.2.

Note that \( \tau(n,g) \) omits at most \( 2k \) integers, that is at most two integers, per singularity of \( g \) interior to \((a,b)\). When \( g \) has no singularities interior to \((a,b)\), then \( \tau(n,g) = \{1,2,...,n\} \) and \( K_n^*(g) = K_n(g) \).

We shall see that when more is assumed about \( a(x) \), then (2.4B) can be modified so that \( \tau(n,g) \) omits at most one integer per singularity of \( g \) interior to \((a,b)\).

(iii) Whenever the (Lebesque-Stieltjes) integrals are defined, we set

\[
S_n[g](x) = \sum_{j=0}^{n} \int_{a}^{b} g(t)\varphi_j(t)\,d\alpha(t)\varphi_j(x) 
\]

(2.6)

that is, \( S_n[g] \) is the \((n+1)\)th partial sum of \( g \)'s orthogonal series expansion in the \( \{\varphi_j\} \).

(iv) Set \( \xi_{n1}(x) = \prod_{\substack{j=1 \atop j \neq i}}^{n} \left( \frac{x-x_{nj}}{x_{ni}-x_{nj}} \right), \quad 1 \leq i \leq n, \)

so that \( L_n(g) = \sum_{i=1}^{n} \xi_{n1}(x)g(x_{ni}) \) is the Lagrangian interpolation polynomial to \( g \) at the \( \{x_{ni}\} \). When \( g \) has singularities interior to \((a,b)\), we define
\[
L_n^*(g) = \sum_{j \in \mathcal{E}(n, g)} L_{n_j}^*(x_j) g(x_j)
\]  
(2.7)

with the notation of (2.3), (2.4A,B). Note that

\[
\lambda_j = \int_a^b L_{n_j}^*(x) d\alpha(x) = \int_a^b L_{n_j}^2(x) d\alpha(x), \quad 1 \leq j \leq n
\]  
(2.8)

(see for example Freud [3, Theorem 1.3.2, p. 21]) and hence

\[
\begin{align*}
K_n(g) &= \int_a^b L_n(g) d\alpha(x) \\
K_n^*(g) &= \int_a^b L_n^*(g) d\alpha(x)
\end{align*}
\]  
(2.9)

(v) As usual \( \|g\|_{\alpha, p} = \left( \int_a^b |g(x)|^p d\alpha(x) \right)^{1/p} \) whenever \( p > 0 \) and the Lebesque-Stieltjes integral is defined and finite. For \( p = \infty \),

\[ \|g\|_{\alpha, \infty} = \sup\{|g(x)| : x \in (a, b)\} . \]

(vi) We shall seek to approximate integrals of the form

\[
\begin{align*}
I[\beta; f] &= \int_a^b f(x) d\beta(x) \\
\end{align*}
\]  
(2.10)

where \( \beta: (a, b) \to \mathbb{R} \) is right continuous and of bounded variation in \( (a, b) \) and where, in the Lebesque-Stieltjes sense,

\[
\int_a^b t^j d\beta(t) \text{ exists and is finite,} \quad j = 0, 1, 2, \ldots
\]  
(2.11)

Sloan and Smith [7] considered the case \( d\beta(x) = k(x) dx \). The more general form (2.10) allows \( d\beta(x) \) to be discrete, so that \( I[\beta; f] \) is a series. Approximating infinite series by rewriting them as Stieltjes integrals, and then using Gaussian quadrature on the Stieltjes integral, is of practical and theoretical interest.
(vii) We shall require that $\beta(x)$ is absolutely continuous with respect to $\alpha(x)$, that is:

$$\text{Given } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that given any number of disjoint subintervals } [x_1, z_1] \ldots [x_n, z_n] \text{ of } (a, b) \text{ such that } \sum_{i=1}^{n} |\alpha(z_i) - \alpha(x_i)| < \delta, \text{ then we have }$$

$$\sum_{i=1}^{n} |\beta(z_i) - \beta(x_i)| < \varepsilon \quad (2.12)$$

In particular this implies that the Radon-Nikodym derivative $\frac{d\beta}{d\alpha}(x)$ is defined almost everywhere with respect to $d\alpha(x)$ and, whenever $f(x)$ is Lebesgue-Stieltjes integrable in $(a, b)$ with respect to $d\beta(x)$,

$$\int_{a}^{b} f(x) d\beta(x) = \int_{a}^{b} f(x) \frac{d\beta}{d\alpha}(x) d\alpha(x). \quad (2.13)$$

See Riesz-Sz.Nagy [6, p.127] for (2.13), though they do not use the name "Radon-Nikodym derivative" - for the latter, see de Barra [2, Chapter 8]. Note also that if $d|\beta|(x)$ is the total variation of $d\beta(x)$, then

$$\frac{d|\beta|}{d\alpha}(x) = \left|\frac{d\beta}{d\alpha}(x)\right| \text{ almost everywhere with respect to } d\alpha(x) \quad (2.14)$$

(see de Barra [2, p.166]) and hence

$$\int_{a}^{b} f(x) d|\beta|(x) = \int_{a}^{b} f(x) \left|\frac{d\beta}{d\alpha}(x)\right| d\alpha(x) \quad (2.15)$$

whenever $f(x)$ is Lebesgue-Stieltjes integrable in $(a, b)$ with respect to $d\beta(x)$.

(2.12) forces $\beta$ to be continuous at every point of continuity of $\alpha$ (among other things). So given $\beta(x)$, we have to choose the $\alpha(x)$ on
which we base the product integration rule in such a way that (2.12) holds. In the case where \( d\beta(x)^i = k(x) \, dx \), we see \( \frac{d\beta}{dx} (x) = k(x)/a'(x) \). Thus (2.12) is satisfied if \( k(x) \) is integrable and vanishes almost everywhere in \( \{ x : \alpha'(x) = 0 \} \). This weakens (in a trivial way) Sloan and Smith's requirement that \( \alpha'(x) \) be positive almost everywhere in \( (a,b) \).

(viii) The product rules considered will have the form

\[
I_n[\beta; f] = \sum_{i=1}^{n} w_{ni} f(x_n)
\]

(2.16)

where the \( \{w_{ni}\}_i \) are chosen to make the rule exact for polynomials of degree less than \( n \). It is easy to see that

\[
I_n[\beta; f] = \int_a^b L_n(f) \, d\beta(x)
\]

(2.17)

and

\[
w_{ni} = \lambda_n S_{n-1} \left[ \frac{d\beta}{dx} \right](x_{ni}), \quad 1 \leq i \leq n
\]

(2.18)

- Sloan and Smith [7] proved these for \( d\beta(x) = k(x) \, dx \). Note that in view of (2.11), (2.13), \( \int_a^b t^j d\beta(t) = \int_a^b t^j \frac{d\beta}{dx} (t) \, dx(t) \) exists for \( j = 0,1,2,\ldots \). Hence the right member of (2.18) is defined.

Associated with (2.16), there is the "companion rule", first introduced by Sloan and Smith [7]:

\[
J_n[\beta; f] = \sum_{i=1}^{n} |w_{ni}| f(x_n)
\]

(2.19)

When \( f \) has singularities interior to \( (a,b) \), we define, with the notation of (2.3), (2.4A,B):

\[
I_n^{\#}[\beta; f] = \sum_{j \in \mathcal{E}(n,f)} w_{nj} f(x_{nj})
\]

(2.20)

\[
J_n^{\#}[\beta; f] = \sum_{j \in \mathcal{E}(n,f)} |w_{nj}| f(x_{nj})
\]

(2.21)
3. CONVERGENCE OF GAUSSIAN QUADRATURES

In this section, we use a lemma due to Shohat and a generalized Markov-Stieltjes inequality first proved by Posse (see Freud [3, p.33, p.92]) to prove a convergence theorem for Gaussian quadrature and Lagrange interpolation of singular integrands - Theorem 3.5. This theorem is the corrected version of a result stated by Freud [3, pp. 132-133, Problems 12, 13] which we show, by counterexample, is incorrect. For functions with endpoint singularities only, detailed results appear in Freud [3, pp. 92-98]. These results are superior to the "dominated integrability" results established in recent years, in the sense that they do not require certain inequalities between all weights and abscissas near the singularity, but the methods work only for Gaussian quadrature.

Lemma 3.1 (Shohat) Suppose \( G(x) \) is infinitely differentiable in \((a,b)\) with
\[
G^{(2k)}(x) \geq 0 \text{ in } (a,b), \quad k = 0,1,2,...
\]
Then
\[
K_n(G) \leq \int_a^b G(x)dx, \quad n = 0,1,2,...
\]
provided the integral is convergent.

Proof: See for example Freud [3, p.92].

Q.E.D.

The following lemma is based on a generalized Markov-Stieltjes inequality first proved by Posse (Freud [3, p.33, p.53]).

Lemma 3.2 Suppose \( G(x) \) is infinitely differentiable in \((-\infty,\infty)\) with \( y \in (a,b) \) and
\[
G^{(k)}(x) \geq 0 \text{ in } (-\infty,y); \quad (-1)^k G^{(k)}(x) \geq 0 \text{ in } (y,\infty)
\]

\[
k = 0,1,2...
\]
Then
$$K_n^*(G) \leq \int_a^b G(x) \, d\alpha(x), \quad n = 1, 2, 3, \ldots$$  \hfill (3.2)

More precisely if \( y \in (x_{nt}, x_{n \cdot t+1}) \) some \( 1 \leq t < n \), then
$$K_n(G) \leq \int_a^b G(x) \, d\alpha(x) + \lambda_{nt} G(x_{nt}) + \lambda_{n \cdot t+1} G(x_{n \cdot t+1})$$  \hfill (3.3)

while if \( y = x_{n \cdot t+1} \) some \( 0 \leq t < n \) and we set \( G(y) = 0 \) then
$$K_n(G) \leq \int_a^b G(x) \, d\alpha(x).$$

**Proof:** Suppose first \( y \in (x_{nt}, x_{n \cdot t+1}) \), some \( 0 \leq t \leq n \) (where \( a = x_{nt}, b = x_{n \cdot t+1} \)). We now apply Posse's inequality, which is stated as inequality (5.10) in Freud [3, p.33]. For Freud's notation, which is a little different from that here, the reader is referred to Freud [3, pp. 17-22]. Taking \( \zeta = x_{nt} \) and \( f = G \) in Freud's (5.10) gives
$$\sum_{j=1}^{t-1} \lambda_{nj} G(x_{nj}) = \sum_{x_{nj} < \zeta} \lambda_{nj} G(x_{nj}) \leq \int_{-\infty}^{x_{nt}} G(x) \, d\alpha(x)$$  \hfill (3.4)

(the sum is taken as \( 0 \) if \( t = 0,1 \)). Freud proves (3.4) under the stronger assumption that strict inequality holds in (3.1), but as he remarks [3, p.50, Problem 16] consideration of \( G_\varepsilon(x) = G(x) + \varepsilon e^x \) with \( \varepsilon \to 0^+ \) gives (3.4) under the weaker assumption \( G(k)(x) \geq 0 \) in \((-\infty, y)\), \( k = 0,1,2,\ldots\). To deal with \( \sum_{j=t+2}^n \lambda_{nj} G(x_{nj}) \), we consider the transformation \( x \rightarrow -x \). It is well known, and easy to see, that for the distribution function
\[\gamma(x) = \alpha(\infty) - \alpha(-x)\],
the orthonormal polynomials are the \( \{\varphi_j(-x)\} \)
and the abscissas \( z_{nj} \) and weights \( v_{nj} \) in \( \gamma \)'s quadrature formula satisfy \( z_{nj} = -x_{nk} \) \( n \) \( k = n+1-j \) and \( 1 \leq j \leq n \).

Further \( H(x) = G(-x) \) satisfies
If both (3.6), (3.8) hold, then (3.4), (3.5), (3.7), (3.9) give (3.2), (3.3) as $G$ is non-negative. If (3.6) (respectively (3.8)) fails then (2.48) shows that $x_{nt}$ (respectively $x_{nt+1}$) is omitted in forming the sum $K^x_n(G)$ and (3.2), (3.3) still follow from (3.4), (3.5).

Finally, if $y = x_{nt+1}$, taking $\zeta = x_{nt+1}$ and $f = G$ in (5.10) in Freud [3, p.33] gives

$$
\sum_{j=1}^{t} \lambda_n j G(x_{nj}) \leq \int_{-\infty}^{x_{nt+1}} G(x) \alpha(x) 
$$

(3.10)
and the inequality
\[ \sum_{j=t+2}^{n} \lambda_{nj} G(x_{nj}) \leq \int_{x_{n,t+1}}^{x_{n,t}} G(x) \, d\alpha(x) \]

follows from (3.10), as before, by considering the transformation \( x \to -x \).

\[ \text{Q.E.D.} \]

The function \( G(x) \) in Lemma 3.2 is "absolutely monotone" in \((-\infty, y)\) and "completely monotone" in \((y, \infty)\). This, together with the above lemmas, motivates the following definition:

**Definition 3.3:** We shall say \( g(x) \) is monotone integrable if

(a) there exists a non-negative integer \( \ell \) and points \( a = y_0 < y_1 < \ldots < y_{\ell} < y_{\ell+1} = b \) such that \( \alpha \) is continuous at, and strictly increasing from the left and right at, \( y_1 \ldots y_{\ell} \). Further we assume \( g \) is properly Riemann-Stieltjes integrable with respect to \( d\alpha(x) \) in each compact subinterval of \((y_i, y_{i+1}), i = 0,1, \ldots \ell \) and such that

\[ \lim_{A \to y_i^+} \int_{A}^{B} |g(x)| \, d\alpha(x) < \infty, \quad i = 0,1, \ldots \ell \quad (3.11) \]

(\( \text{where } (-\infty) = -\infty; \infty = \infty \)).

(b) there exist functions \( G_i(x), i = 0,1, \ldots \ell+1 \) such that

\[ \lim_{x \to y_i} \sup_{x \in (a,b)} \frac{|g(x)|}{G_i(x)} < \infty, \quad i = 0,1, \ldots \ell+1 \quad (3.12) \]

and such that for \( i = 0, \ell+1 \)
Lemma 3.4:

\( G_i(x) \) is infinitely differentiable in \((a,b)\) with \( G_i^{(2k)}(x) \geq 0 \) in \((a,b)\), \( k = 0,1,2,... \)

but for \( i = 1,2,...,\ell \)

\( G_i(x) \) is infinitely differentiable in \((-\infty,\infty)\setminus\{y_i\}\) with

\[
G_i^{(k)}(x) \geq 0 \text{ in } (-\infty,y_i); \quad (-1)^k G_i^{(k)}(x) \geq 0 \text{ in } (y_i,\infty)
\]

\( k = 0,1,2,... \)

We assume also

\[
\int_a^b G_i(x) d\alpha(x) < \infty, \quad i = 0,1,...,\ell+1.
\] (3.15)

Instead of (3.12), Freud assumes more-namely that the lim sup in (3.12) is 0. Our weaker condition simplifies some theoretical aspects, but offers little real advantage in practice. By saying \( \alpha \) is strictly increasing from the left and right at \( y_1,...,y_{\ell} \), we mean

\( \alpha(x_1) < \alpha(y_i) < \alpha(x_2) \) for all \( x_1 < y_i < x_2, \quad i = 1,2,...,\ell \). This is a natural requirement — if for example \( \alpha \) was constant near \( y_i \), we could alter \( g \)'s values near \( y_i \) without affecting \( \int_a^b g(x) d\alpha(x) \).

Typically for \((a,b) = (0,\infty)\) one would choose \( G_0(x) = x^{-\delta} \);

\( G_i(x) = |x-y_i|^{-\delta}, \quad i = 1,2,...,\ell \) (some \( 0 < \delta < 1 \)) and \( G_{\ell+1}(x) = e^{\epsilon x} \) (some \( \epsilon > 0 \)) or \( x^s \) (\( s \) an even positive integer). See Freud [3, p. 96] for a table of typical \( G_0(x), G_{\ell+1}(x) \) for the classical weights.

The following lemma lists some closure properties of the class of monotone integrable functions.

Lemma 3.4:

(a) Let \( f(x) \) be properly Riemann-Stieltjes integrable with respect to \( d\alpha(x) \) in each compact subinterval of \((a,b)\). Further suppose there
exist $C, D > 0$ and a positive even integer $s$ such that $|f(x)| \leq C + Dx^s$ for all $x \in (a,b)$. Then $f(x)$ is monotone integrable.

(b) If $f(x), g(x)$ are monotone integrable and $c, d \in \mathbb{R}$, then so are $cf(x) + dg(x); |f(x)|$.

(c) If $f^2(x)$ is monotone integrable, and $R(x)$ is a polynomial, then $(f|P|)(x)$ and $(f-P)^2(x)$ are monotone integrable.

**Proof:**

(a) Since $f(x)$ has no singularities in $(a,b)$, we have $z = 0$ in Definition 3.3 and can clearly choose $G_0(x) = G_{\frac{x}{2} + 1}(x) = C + Dx^s$.

(b) Is easy.

(c) As $g(x) = f^2(x)$ is monotone integrable, there exist $\{y_\alpha\}, \{G_{\alpha}\}$ as in Definition 3.3. Using Hölder's inequality, and (2.1), we see $f|P|$ satisfies (3.11). Further the $G_1, G_2, \ldots, G_k$ that satisfy (3.12), (3.14), (3.15) for $g = f^2$ also satisfy (3.12), (3.14), (3.15) for $g = f|P|$, as

$$\limsup_{x \to y_\alpha} \frac{|f|P|)(x)/G_1(x)|}{|f^2(x)/G_1(x)|||P|/(x)/f(x)|} < \infty$$

We need modify only $G_0(x)$ (respectively $G_{k+1}(x)$) and only in the case that $a$ (respectively $b)$ are infinite. Suppose for example $a = -\infty$. Choose an even integer $s$ larger than the degree of $P^2$, and consider $G(x) = G_0(x) + x^s$, which obviously satisfies (3.13) and (3.15). Now

$$|G(x)/(f|P|)(x)| = |G_0/f^2(x)| + [x^s/P^2(x)]|P/f|/(x)$$

$$\geq \min\{G_0(x)/f^2(x), x^s/P^2(x)\}$$

$$> c > 0 \text{ for large negative } x.$$
Hence, we can use $G(x)$ as $G_0(x)$ for $g = f|P|$. Finally $(f-P)^2 = f^2 - 2fP + P^2$ is a linear combination of monotone integrable functions.

**Theorem 3.5** (a) Let $g(x)$ be monotone integrable. Then
\[ \lim_{n \to \infty} K_n^+(g) = \int_a^b g(x) \, d\alpha(x). \]

(b) Let $g^2(x)$ be monotone integrable. Then
\[ \lim_{n \to \infty} \|L_n^+(g) - g\|_{a,2} = 0. \]

**Proof:** For functions with no interior singularities, these results are Theorem 111.1.6(a), (b) and Theorem 111.2.2 in Freud [3, pp. 93-94, 97].

(a) We assume the $l_i, \{y_i\}, \{G_i\}$ are as in Definition 3.3. Fix $y = y_i$, some $1 \leq i \leq l$. For each $n \geq 1$, let $j$ denote the positive integer such that $y \in (x_nj-1, x_nj+1)$. We claim that for any positive integer $r$,
\[ \lim_{n \to \infty} x_{nj-r} = y = \lim_{n \to \infty} x_{nj+r} \quad \text{(3.16)} \]

(where $x_{nk} = a$ for $k \leq 0$, $x_{nk} = b$ for $k > n$). If, say, the first limit failed to hold for some $r$, we can find a subsequence of integers $S$ and $\delta > 0$ such that $\lim_{n \to \infty} x_{nj-k} = y$, $k = 1, 2, \ldots, r-1$, but $x_{nj-r} < y - \delta$ all $n \in S$.

Then for any $0 < \eta < \delta$, we have
\[ \lim_{n \to \infty} \sum_{n \in S} \lambda_{nj} = 0. \quad \text{(3.17)} \]

But assuming (as we can) that $y-\delta, y-\eta$ are points of continuity of $\alpha$, the limit in (3.17) should be $\alpha(y-\eta) - \alpha(y-\delta)$ (by Theorem 111.1.1 in Freud [3, p.89]). As $\alpha$ is continuous and strictly increasing from the left at $y$, we have $\alpha(y-\eta) - \alpha(y-\delta) > 0$ for small $\eta$, and so
we have a contradiction to (3.17). Hence (3.16) holds. Then by
right continuity of $\alpha$ and the Markov-Stieltjes inequality (see,
for example, Freud [3, p. 29]) we have

$$
\lambda_{n+1} \leq \alpha(x_{n+1}) - \alpha(x_{n}) \to 0 \quad \text{as} \quad n \to \infty,
$$

by (3.16) and continuity of $\alpha$ at $y$.

Since $\tau(n,g)$ omits at most $2\ell$ integers, and $j \notin \tau(n,g)$
$\Rightarrow y_i \in (x_{j-1},x_{j+1})$ some $1 \leq i \leq \ell$, we have

$$
\lim_{n \to \infty} \sum_{j \in \tau(n,g)} \lambda_{n,j} = 0.
$$

(3.18)

Theorem III.1.1 in Freud [3, p. 89] and (3.18) imply

$$
\lim_{n \to \infty} \sum_{j \in \tau(n,g)} \lambda_{n,j} h(x_{n,j}) = \int_{a}^{b} h(x) d\alpha(x)
$$

(3.19)

for any function $h$ bounded in $(a,b)$ and properly Riemann-Stieltjes
integrable with respect to $d\alpha(x)$ in each compact subinterval of
$(a,b)$ and such that $h$ is (improperly) Riemann-Stieltjes integrable
with respect to $d\alpha(x)$ in $(a,b)$.

Next, by (3.12), we can choose $K > 0$ and open intervals, $I_1, I_2, \ldots, I_{\ell}$
containing $y_1, \ldots, y_{\ell}$ and (possibly unbounded) open intervals $I_0$
with left endpoint $a$, $I_{\ell+1}$ with right endpoint $b$, such that

$$
|g(x)| \leq K g_i(x), \quad x \in I_i, \quad i = 0, 1, \ldots, \ell + 1.
$$

(3.20)

By reducing the sizes of the $I_i$, if necessary, we may assume that
their endpoints are points of continuity of $\alpha$ and that $I = \bigcup_{i=0}^{\ell+1} I_i$
has, for some given $\epsilon > 0$,

$$\left| \int_{I} g(x) d\alpha(x) \right| < \epsilon
$$

(3.21A)
and
\[ K \sum_{i=0}^{l+1} \int_I G_i(x) \, d\alpha(x) < \varepsilon \quad \text{(3.21B)} \]

Let \( \chi(x) = \begin{cases} 1 & x \in (a,b) \setminus I \\ 0 & \text{otherwise} \end{cases} \)

Then (2.5) gives
\[
|K^*_n(g) - \int_a^b g(x) \, d\alpha(x)| \\
\leq \left| \sum_{j \in \tau(n,g)} \lambda_{n_j} g(x) \chi(x) - \int_a^b (g \chi)(x) \, d\alpha(x) \right| + \sum_{j \in \tau(n,g)} \lambda_{n_j} g(x) \chi(x) \\
\cdot \left( \sum_{j \in \tau(n,g)} \lambda_{n_j} g(x) \chi(x) \right) \\
+ \left| \int_I g(x) \, d\alpha(x) \right| . \quad \text{(3.22)}
\]

Since \( h(x) = (g \chi)(x) \) is bounded in \((a,b) \setminus I\) and vanishes outside this bounded set, (3.19) shows that the first term in the right member of (3.22) \( \to 0 \) as \( n \to \infty \). Next, the third term in the right member of (3.22) is bounded by \( \varepsilon \), by (3.21A). Finally, we deal with the second term. Now by (3.20) and definition of \( \chi \),
\[
| \sum_{j \in \tau(n,g)} \lambda_{n_j} g(x) \chi(x) | \leq K \sum_{j=0}^{l+1} \sum_{i \in \tau(n, G_i)} \lambda_{n_j} G_i(1-\chi)(x) \\
\chi_{n_j} \\
\leq K \sum_{i=0}^{l+1} \left\{ \sum_{j \in \tau(n, G_i)} \lambda_{n_j} G_i(x) \chi(x) - \sum_{j \in \tau(n, G_i)} \lambda_{n_j} G_i(1-\chi)(x) \right\} \\
\leq K \sum_{i=0}^{l+1} \int_a^b \left\{ G_i(x) - \int_a^b G_i(1-\chi)(x) \, d\alpha(x) + \varepsilon/(K(l+2)) \right\} \\
\leq 2\varepsilon \quad \text{for large } n, \text{ by Lemma 3.2, (2.5) and (3.19)}
\]

This completes the proof of (a).
(b) Let ε > 0. By a theorem of M. Riesz (Freud [3, p. 75]), there exists a polynomial P(x) such that \( \|g-P\|_{\alpha,2} < \varepsilon \). Now, if \( n \) exceeds degree (P), we have \( P = L_n(P) \) and so (2.7) and Minkowski's inequality give

\[
\|L_n^*(g) - g\|_{\alpha,2} 
\leq \| \sum_{j \in \tau(n,g)} \xi_n^j(x)(g-P)(x_n^j)\|_{\alpha,2} + \| \sum_{j \in \tau(n,g)} \xi_n^j(x)P(x_n^j)\|_{\alpha,2} + \|P-g\|_{\alpha,2}.
\]

(3.23)

Writing

\[
h_n(x) = \begin{cases} 0 & \text{if } x = x_n^j \text{ some } j \in \tau(n,g) \\ (g-P)(x) & \text{all other } x, \end{cases}
\]

(3.24)

we see that the first term in the right member of (3.23) is \( \|L_n(h_n)\|_{\alpha,2} \). Now, by equation (2.2) in Freud [3, p. 93] or more simply by exactness of the rule (2.2), we have

\[
\|L_n(h_n)\|_{\alpha,2}^2 = \int_a^b \langle h_n(x) \rangle^2 d\alpha(x)
= K_n(h_n^2)
= K_n^*(g-P^2).
\]

(by (3.24) and as \( \tau(n,(g-P)^2) = \tau(n,g) \))

\[
+ \int_a^b (g-P)^2(x) d\alpha(x) = \|g-P\|_{\alpha,2}^2 \quad \text{as } n \to \infty.
\]

(3.25)

Here we have used Theorem 3.5(a) and Lemma 3.4(c). Next, applying Hölder's inequality to the second term in the right member of (3.23) and using \( \lambda_n^j = \|\xi_n^j\|_{\alpha,2}^2 \) (by 2.8), we see

\[
\| \sum_{j \in \tau(n,g)} \xi_n^j(x)P(x_n^j)\|_{\alpha,2}^2 \leq \left\{ \sum_{j \in \tau(n,g)} \lambda_n^j \right\} \left\{ \sum_{j \in \tau(n,g)} P^2(x_n^j) \right\}
\to 0 \quad \text{as } n \to \infty.
\]
(by (3.18) and boundedness of \( P \) in \([y_1-\delta, y_2+\delta]\) any \( \delta > 0 \), and as
t(n,g) omits at most \( 2\ell \) integers). Together with (3.23), (3.25),
this completes the proof of the theorem.

Q.E.D.

Remarks

(a) Theorem 3.5(a) is not contained in the results of Rabinowitz [5] as
he considers compound rules based on a fixed Gauss rule, and for general
rules places explicit conditions on weights and abscissas near the
singularities.

(b) Along the lines of the example in Davis and Rabinowitz [1], we now
establish a counterexample to the result stated in Freud [3, pp.132-133,
Problems 12-13]. Freud evidently intended to "ignore" the singularity,
by setting \( f(x_n) = 0 \) when \( x_n \) coincided with a singularity of \( f \).
Theorem 3.5 nevertheless shows that Freud's result can be repaired by
omitting at most two points per interior singularity of \( f \).

Example 3.6  Let \( I \) be an open subinterval of \((a,b)\) and assume that
each point of \( I \) is a limit point of \( \{x_{ni}\}_{n,i} \). For example, this
would be true if \( a(x) \) is strictly increasing throughout \( I \). We shall
also assume \( \int_a^b \log|x-y| \, d\alpha(x) < \infty \) for all \( \gamma \in I \). This weak restric-
tion is satisfied if for example \( \alpha(x) \) has a bounded derivative in \( I \).

Now choose \( 1 \leq j(1) \leq n(1) \) such that \( x_{n(1)j(1)} \in I \). Next, having
chosen \( x_{n(k)j(k)} \in I, k = 1,2,\ldots,r \), choose \( x_{n(r+1)j(r+1)} \in I \) s.t.

\[
0 < x_{n(r+1)j(r+1)} - x_{n(r)j(r)} < \min\{x_{n(r)j(r)} - x_{n(r-1)j(r-1)}/2; \exp(-n(r)n(r)/\lambda_n(r)j(r))\} \tag{3.26}
\]
In this way we generate a sequence \( \{x_n(k)j(k)\}_k \) satisfying (3.26) for \( r \geq 1 \). Define

\[
    y = x_n(1)j(1) + \sum_{k=1}^{\infty} (x_n(k+1)j(k+1) - x_n(k)j(k)) = \lim_{k \to \infty} x_n(k)j(k)
\]

which exists by (3.26). Further if \( r = 1,2,\ldots \), then

\[
    0 < y - x_n(r)j(r) = \sum_{k=r}^{\infty} (x_n(k+1)j(k+1) - x_n(k)j(k))
\]

\[
< 2 (x_n(r+1)j(r+1) - x_n(r)j(r))
\]

(appealing (3.26) repeatedly)

\[
< 2 \exp(-n(r)^n(r)/\lambda_n(r)j(r)) .
\]  \( (3.27) \)

Then if \( g(x) = |\log|x-y|| \) and \( n = n(r) \), \( r = 1,2,3,\ldots \) we have

\[
    K_n(g) > \lambda_n(r)j(r)|\log|x_n(r)j(r) - y|
\]

\[
> n(r)^n(r)/2 \quad \text{for large } r, \text{ by } (3.27).
\]

Thus, \( \limsup K_n(g)^{1/n} = \infty \) even though \( \int_a^b g(x)dx < \infty \); note too that \( g^{(k)}(x) > 0 \) in \((-y-1; y); (\ast)_{k=1}^k g^{(k)}(x) > 0 \) in \((y, y+1) \) for \( k = 0,1,2,\ldots \). By trivial modifications at (3.26), one could choose instead \( g(x) = |x-y|^{-\varepsilon} \) or \( g(x) = |\log|\log|x-y||| \). The above example also works for any quadrature rules (not necessarily of Gauss-Jacobi type) with abscissas dense in \( I \).

Remarks:

(a) In forming \( K_n^*(g) \), we omitted up to two abscissas per interior singularity of \( g \). Provided one assumes something about \( \alpha'(x) \), near the singularity, one need omit only the closest point to the singularity.
To illustrate, assume \( G(x) \) is as in the statement of Lemma 3.2, and assume \( y \in (x_{nt}, x_{nt+1}) \) \( (1 \leq t < n) \). Then using right continuity of \( \alpha \) and the Markov-Stieltjes inequality \( \lambda_{nt} \leq \alpha(x_{nt+1}) - \alpha(x_{nt-1}) \) (Freud [3, p.29]) and monotonicity of \( G \) in \((x_{nt}, y)\), we deduce

\[
\lambda_{nt} G(x_{nt}) \leq \frac{\alpha(x_{nt+1}) - \alpha(x_{nt-1})}{\alpha(y) - \alpha(x_{nt})} \int_y^{x_{nt}} G(x) \, d\alpha(x). \quad (3.28)
\]

Similarly

\[
\lambda_{n t+1} G(x_{t+1}) \leq \frac{\alpha(x_{t+2}) - \alpha(x_{nt})}{\alpha(x_{nt+1}) - \alpha(y)} \int_y^{x_{nt+1}} G(x) \, d\alpha(x). \quad (3.29)
\]

If \((a,b) = (-1,1)\) and

\[
0 < m < \frac{\alpha(x_2) - \alpha(x_1)}{x_2 - x_1} \leq M, \quad (3.30)
\]

for all \( x_1, x_2 \) in a neighbourhood of \( y \), then Theorem III.5.1 in Freud [3, p.111] shows \( c_1/n \leq x_{jn+1} - x_{jn} \leq c_2/n \) for \( j = t-1; t; t+1 \) with \( c_1, c_2 \) independent of \( n \). Hence if \( x_{nt+1} \) is closer to \( y \) than \( x_{nt} \), we have \( y - x_{nt} \geq (x_{nt+1} - x_{nt})/2 \geq c_1/(2n) \), so (3.30), (3.28) give

\[
\lambda_{nt} G(x_{nt}) \leq \frac{4MC_2}{mc_1} \int_y^{x_{nt}} G(x) \, d\alpha(x) \to 0 \quad \text{as} \quad n \to \infty.
\]

Similarly if \( x_{nt} \) is closer to \( y \) than \( x_{nt+1} \). Thus, provided \((a,b)\) is finite and (3.30) holds in a neighbourhood of each singularity of the monotone integrable function considered, we may replace (2.4A,B) by the simpler condition that the closest abscissa to each interior singularity is omitted. The same is true for the Hermite and Laguerre weights over unbounded intervals:
For the Hermite weight $\alpha'(x) = \exp(-x^2)$, Freud [4, p.180] shows that

$$\frac{\pi}{\sqrt{2n+1}} \leq x_{n+1} - x_n \leq \frac{\pi}{\sqrt{n+1}}$$

for all $j$ such that $x_{n_j}, |x_{n_j+1}| \leq \sqrt{n}$. Using this and differentiability of $\alpha$ in (3.28), (3.29), it is easy to see that $\lim_{n \to \infty} \lambda_n t! G(x_{nt}!)$ = 0 if $t!$ varies with $n$ in such a way that $x_{nt}!$ is the second closest abscissa to $y$. For the Laguerre weight $\alpha'(x) = x e^{-x}$ where $|\Delta| \leq 1/2$, a similar remark is possible. So for the classical weights (Jacobi, Hermite, Laguerre) it is possible to replace (2.4A.B) by the simpler condition that only the closest abscissa to each interior singularity is omitted.
4. CONVERGENCE OF PRODUCT INTEGRATION RULES

Theorem 3.5 enables one to prove convergence of product integration rules based on Gaussian quadratures. First, however, we need the following lemma:

Lemma 4.1:
(a) Assume $f$ is monotone integrable. Then if $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$, we have

$$\left| I_n^*[\beta; f] - \int_a^b f(x) d\beta(x) \right| \leq \| L_n^*[f] - f \|_{\alpha, p} \| \frac{d\beta}{d\alpha} \|_{\alpha, q}$$

provided the right member is finite.

(b) If $f^2$ is monotone integrable, and $P$ is a polynomial of degree $\leq n$, then

$$\left| I_n^*[\beta; f] - \int_a^b f(x) d|\beta|(x) \right|$$

$$\leq \| P - \frac{d\beta}{d\alpha} \|_{\alpha, 2} \left( \| K_n^*[f^2] \|_{1/2} + \| f \|_{\alpha, 2} \right)$$

$$+ \left| K_n^*[f|P|] - \int_a^b f(x)|P(x)| d\alpha(x) \right|$$

provided the right member is finite.

Proof:
(a) By definition of $I_n^*$, $L_n^*$ at (2.4A,B), (2.7), (2.20), we see

$$I_n^*[\beta; f] - \int_a^b f(x) d\beta(x) = \int_a^b (L_n^*[f] - f)(x) d\beta(x)$$

$$= \int_a^b (L_n^*[f] - f(x) \frac{d\beta}{d\alpha}(x) d\alpha(x)$$
(by (2.13)). Note that as \( f \) is monotone integrable, its points of discontinuity have \( \alpha \)-measure zero, and hence \( \beta \)-measure zero, by absolute continuity of \( \beta \) with respect to \( \alpha \). Hölder's inequality then gives (4.1) and that \( f \) is Lebesgue-Stieltjes integrable with respect to \( \beta(x) \), provided the norms in (4.1) are finite.

(b) The method of proof is very similar to that of Theorem 1 in Sloan and Smith [7]. Using (2.6), (2.15), (2.18), (2.21) and the fact that
\[ p(x_{nj}) = S_{n-1}[P](x_{nj}) = S_{n-1}[P](x_{nj}) \quad \text{(as degree \( (R) \leq n \)}
and \( \phi_n(x_{nj}) = 0 \), we see

\[
\left| J_n^* \beta; f \right| - \int_a^b f(x) \, d\beta(x) \leq \sum_{j \in \Gamma(n,f)} \lambda_n \left[ \left| S_{n-1} \left( \frac{d\beta}{d\alpha} \right) \right| - \left| S_{n-1}[P] \right| (x_{nj}) \right] \left| f(x_{nj}) \right|
\]

\[ + \sum_{j \in \Gamma(n,f)} \lambda_n \left( \left| P \right| f(x_{nj}) \right) - \int_a^b \left( \left| P \right| f(x) \right) \, d\alpha(x) \left| d\beta(x) \right|
\]

\[ + \int_a^b f(x) \left( \left| P \right| f(x) \right) - \left| \frac{d\beta}{d\alpha} (x) \right| \, d\alpha(x) \right| \right) . \quad (4.3)
\]

Now \( \| S_{n-1} \left( \frac{d\beta}{d\alpha} \right) - \left| S_{n-1}[P] \right| \right) (x_{nj}) \)

\[
\leq \left| S_{n-1} \left( \frac{d\beta}{d\alpha} \right) - \left| S_{n-1}[P] \right| \right| (x_{nj}) \]

\[ = \left| S_{n-1} \left( \frac{d\beta}{d\alpha} - P \right) \right| (x_{nj}) \right) .
\]

Then Hölder's inequality gives

\[
\left| \sum_{j \in \Gamma(n,f)} \lambda_n \left[ \left| S_{n-1} \left( \frac{d\beta}{d\alpha} \right) \right| - \left| S_{n-1}[P] \right| \right] (x_{nj}) \right| f(x_{nj}) \right| ^2
\]

\[
\leq \left\{ \sum_{j=1}^n \lambda_n \left[ S_{n-1} \left( \frac{d\beta}{d\alpha} - P \right) \right] (x_{nj}) \right\} \left\{ \sum_{j \in \Gamma(n,f)} \lambda_n f^2(x_{nj}) \right\}
\]

\[ = \left( \int_a^b S_{n-1} \left( \frac{d\beta}{d\alpha} - P \right)(x) \, d\alpha(x) \right) \left( K_n^* (f^2) \right) \]
We can now prove our main result. For the reader's convenience, the main assumptions on \( \alpha, \beta \) are restated.

**Theorem 4.2:** Assume that \( \alpha: (a, b) \to \mathbb{R} \) is right continuous, monotone increasing with infinitely many points of increase, and is the unique solution, apart from normalization, of the moment problem (2.1). Assume that \( \beta: (a, b) \to \mathbb{R} \) is right continuous, of bounded variation in \( (a, b) \), and is absolutely continuous with respect to \( \alpha \), as at (2.12). Suppose that the Radon-Nikodym derivative \( \frac{d\beta}{d\alpha} \) satisfies

\[
\| \frac{d\beta}{d\alpha} \|_{\alpha, 2} < \infty.
\]  

Then whenever \( f^2 \) is monotone integrable

\[
\lim_{n \to \infty} J_n^*[\beta; f] = \int_a^b f(x) d\beta(x) \quad (4.7)
\]

and

\[
\lim_{n \to \infty} J_n^*[\beta; f] = \int_a^b f(x) d|\beta|(x). \quad (4.8)
\]
Proof: Firstly, note that (2.1), (4.6) and Hölder's inequality imply that (2.11) holds. Further as in the proof of Lemma 4.1, \( \mathbf{f}^2 \) is Lebesgue-Stieltjes integrable with respect to \( d\beta(x) \). Then, Theorem 3.5(b), Lemma 4.1(a) with \( p=q=2 \) and (4.6) give (4.7).

Next, let \( \epsilon > 0 \). Since \( \frac{d\beta}{da} \) satisfies (4.6), M. Riesz's Theorem (Freud [3, p.75]) shows we can choose a polynomial \( P(x) \) such that

\[
\| P - \frac{d\beta}{da} \|_{L^2} < \epsilon, \quad \text{while as } \mathbf{f}^2 \text{ is monotone integrable, Theorem 3.5(a) gives}
\]

\[
\lim_{n \to \infty} K_n^\alpha (\mathbf{f}^2) = \int_a^b \mathbf{f}^2(x) d\alpha(x) = \| \mathbf{f} \|_{\mathbf{L}^2}^2.
\]

Hence the first term in the right member of (4.2) can be bounded by \( 3\epsilon \| \mathbf{f} \|_{\mathbf{L}^2}^2 \) for large \( n \). Finally, by Lemma 3.4(c), \( (f|P|)(x) \) is monotone integrable, so by Theorem 3.5(a), \( \lim_{n \to \infty} K_n^\alpha (f|P|) = \int_a^b (f|P|)(x) d\alpha(x) \) and this deals with the second term in the right member of (4.2).

Q.E.D.

For functions with at worst endpoint singularities, there is the following corollary:

**Theorem 4.3:** Assume that \( \alpha, \beta \) are as in Theorem 4.2, and (4.6) holds.

Let \( f \) be properly Riemann-Stieltjes integrable, with respect to \( d\alpha(x) \), in every compact subinterval of \( (a,b) \) and let

\[
\lim_{A \to a^+} \int_A^B \mathbf{f}^2(x) d\alpha(x) = \| \mathbf{f} \|_{\mathbf{L}^2}^2 < \infty.
\]

Suppose there exist functions \( G_0, G_1 \) satisfying (3.13), (3.15) and

\[
\lim_{x \to a^+} \sup f^2(x)/G_0(x) < \infty; \quad \lim_{x \to b^-} \sup f^2(x)/G_1(x) < \infty.
\]

Then

\[
\lim_{n \to \infty} \sum_{i=1}^n w_n_i f(x_{n_i}) = \int_a^b f(x) d\beta(x)
\]

\[
\lim_{n \to \infty} \sum_{i=1}^n \left| w_n_i f(x_{n_i}) \right| = \int_a^b f(x) d|\beta| (x).
\]
Proof: This follows from Theorem 4.2 as \( \tau(n,f) = \{1,2,\ldots, n\} \) all \( n \geq 1 \), so \( I_n^{[\beta;f]} = I_n[\beta;f] \) and so on.

Q.E.D.

For Hermite and Laguerre weights, Theorem 4.2 can be written in the following form:

**Theorem 4.4:**

(a) **(Laguerre Polynomials)** Suppose

\[
a'(x) = x^\Delta e^{-x} \quad \text{all } x \in [0,\infty) \text{ with } \Delta > -1
\]

Let \( k: (0,\infty) \to \mathbb{R} \) be (Lebesgue) measurable with

\[
\int_0^\infty k^2(x)x^{-\Delta}e^{-x}dx < \infty.
\]

Suppose \( f^2: (0,\infty) \to \mathbb{R} \) satisfies (3.11) in Definition 3.3(a) for some \( 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_\ell < \gamma_{\ell+1} = \infty \), as well as the following:

(i) \( \lim_{x \to 0^+} f(x)x^{-\delta} = 0 \)

some \( \delta \) such that either \( \delta < 0 \) or \( 2\delta \) is a non-negative integer, and such that \( 2\delta + \Delta > -1 \).

(ii) \( \lim_{x \to \gamma_i} |(x-\gamma_i)|^\varepsilon = 0, \quad i = 1,2,\ldots, \ell \), some \( 0 < \varepsilon < 1/2 \)

(iii) \( \lim_{x \to \infty} f(x)e^{-\eta x} = 0 \) some \( 0 < \eta < 1/2 \).

Then if \( d\beta(x) = k(x)dx \),

\[
\lim_{n \to \infty} I_n^{[\beta;f]} = \int_0^\infty f(x)k(x)dx \quad (4.14)
\]

\[
\lim_{n \to \infty} J_n^{[\beta;f]} = \int_0^\infty f(x)|k(x)|dx. \quad (4.15)
\]

If \( \ell = 0 \) so that \( f \) has no singularities interior to \( (0,\infty) \), we can
(b) (Hermite Polynomials) Suppose \( \alpha'(x) = \exp(-x^2) \) all \( x \in (0, \infty) \).

Let \( k: \mathbb{R} \to \mathbb{R} \) be measurable with

\[
\int_{-\infty}^{\infty} k^2(x) \exp(x^2) \, dx < \infty \tag{4.16}
\]

Suppose \( f^2: \mathbb{R} \to \mathbb{R} \) satisfies (3.11) in Definition 3.3(a) for some \( -\infty = y_0 < y_1 < \ldots < y_k < y_{k+1} = \infty \), as well as the following:

(i) \( \lim_{|x| \to \infty} f(x) \exp(-\delta x^2) = 0 \) some \( 0 < \delta < 1/2 \).

(ii) (4.12) holds.

Then (4.14), (4.15) hold with \( d\beta(x) = k(x) \, dx \) and \( -\infty \) replacing 0. If \( \varepsilon = 0 \) so that \( f \) has no singularities interior to \( \mathbb{R} \), we can replace \( l_n^*, J_n^* \) by \( l_n, J_n \) respectively in (4.14), (4.15).

Proof:

(a) We see \( \frac{d\beta}{dx}(x) = k(x)x^{-\frac{3}{2}}e^{-x} \), so (4.10) implies (4.6). To show \( f^2 \)
is monotone integrable, take \( G_0(x) = x^{2\delta}, \ G_1(x) = |x - y_i|^{-2\varepsilon} \), \( i = 1, 2, \ldots \ldots \), and \( G_{k+1}(x) = e^{2\varepsilon x} \). It is easy to see that these \( \{G_i\} \) satisfy the requirements of Definition 3.3, using (4.11), (4.12), (4.13), for \( g = f^2 \). Hence Theorem 4.2 can be applied. Note that there is no advantage in replacing the limits in (4.11), (4.12), (4.13) by the requirement that the corresponding lim sup's are finite.

(b) We see \( \frac{d\beta}{dx}(x) = k(x)\exp(x^2) \) and so (4.16) implies (4.6). To show \( f^2 \)
is monotone integrable, take \( G_0(x) = G_{k+1}(x) = \exp(2\delta x^2) \) and \( G_i(x) = |x - y_i|^{-2\varepsilon}, i = 1, 2, \ldots \).
REMARKS

(a) As in the remark after Example 3.6, when \((a, b)\) is bounded and
(3.30) holds in a neighbourhood of each singularity of \(f\), we may
replace (2.4A,B) by the simpler condition that we omit only the integer
\(j\), corresponding to the closest abscissa \(x_{nj}\) to each interior singularity of \(f\), in forming \(\tau(n, f)\). Hence in forming \(I_n[\beta; f]\), \(J_n[\beta; f]\) we
omit only one abscissa per interior singularity of \(f\) — namely the
closest abscissa. The same comment applies to the Hermite and
Laguerre weights over unbounded intervals.

(b) Theorem 4.3 contains Theorem 1 of Sloan and Smith [7] as the special
case, \(d\beta(x) = k(x)dx\); \((a, b) = (-1, 1)\); and \(f(x)\) bounded in \((a, b)\).
REFERENCES


