NUMERICAL SOLUTION OF THREE-DIMENSIONAL STEADY FLOW IN AXISYMMETRIC PIPES

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SUBSCRIPTS

E
N
P
S'
W
P'

m
n

\{ u \}
\{ v \}
\{ w \}

velocity components in \( \{ r, e, z \} \) directions

\{ r \}
\{ e \}
\{ z \}

indices for grid points in the \( r \) and \( \theta' \) directions

corresponding points in Figs. 2-3

correction for the pressure

correction for the variable

prediction for the variable

SUPERSCRIPTS

p

pressure

u
v
w

corresponding velocity components
1. INTRODUCTION.

Three-dimensional viscous flows occur frequently in practical aerodynamics for example, in flows through inlets, curved ducts, corners, etc. Such flows may be predicted by solution of the parabolized three-dimensional Navier-Stokes equations. The finite difference equations are solved by marching integration in the direction of the main flow.

Parabolization of the Navier-Stokes equations is possible when:

(a) there is a dominant direction for the flow;
(b) the diffusion of momentum and mass is negligible in this direction;
(c) the downstream pressure field has little influence on the upstream field.

Some numerical techniques exist for three-dimensional flow in rectangular ducts, e.g. Patankar and Spalding (1) or Briley (2). In the present work these two methods are extended to the use of non-symmetrical flow in round pipes, with a constant cross section. Briley uses a regular mesh, while Patankar and Spalding utilize a "staggered grid" in the lateral xy-plane. We use a staggered grid in all three directions, z, r and θ. The order of the operations on the velocity components is changed: first, estimates of the cross-stream velocity components u and v are found, later the main-stream velocity w and the correction for it are calculated and finally the corrections for u and v are computed.

Another novel point is the treatment of the duct center, which requires a special arrangement of the grid.

Stability and second order accuracy are maintained through the use of ADI techniques.

The parabolized Navier-Stokes equations may be referred to as three-dimensional boundary layer equations.

The description of a numerical procedure for their solution consists of two parts:

(a) discussion of the mathematical foundation, Chapter 2;
(b) description of the numerical method used to solve the equations, Chapter 3.
2. FORMULATION OF THE EQUATIONS

2.1 The Elliptic Differential Equations

The differential equations for steady state, incompressible three-dimensional fluid flow in cylindrical coordinates are:

I  Momentum equations:

\[
\begin{align*}
\frac{\partial u}{\partial r} + \frac{u}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} &= - \frac{1}{\rho} \frac{\partial p}{\partial r} + \\
+ v \left[ \frac{\partial^2 u}{\partial z^2} + \frac{1}{\rho} \left( \frac{\partial}{\partial \theta} \left( \frac{\partial (ru)}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right] \tag{1}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} + w \frac{\partial v}{\partial z} &= - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \\
+ v \left[ \frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho} \left( \frac{\partial}{\partial \theta} \left( \frac{\partial (rv)}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right] \tag{2}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} &= - \frac{1}{\rho \partial z} + \\
+ v \left[ \frac{\partial^2 w}{\partial z^2} + \frac{1}{\rho} \frac{\partial w}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right] \tag{3}
\end{align*}
\]

II  Continuity equation:

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \ . \tag{4}
\]

In these equations \( u, v, w \) are velocity components in the \( r, \theta, z \) directions respectively, \( \nu \) is the kinematic viscosity, \( \rho \) is the density. The pressure \( p \) can be written as:

\[ p = p_0 (z) + p(r, \theta, z) \]

where:

\( p_0 \) - is the inviscid contribution to the pressure, and

\( p \) - is the viscous contribution to the pressure.
The geometry of the problem is shown in Figure 1 below.

The primary flow is in the z-direction and the secondary flow in the r-θ plane.

2.2 Parabolization of the Equations

Boundary layer simplification gives:

\[ \frac{r}{z} << 1 \]

and dimensional analysis of equations (1)-(4) gives the following conditions:

\[ \frac{u}{w} << 1 \]
\[ \frac{v}{w} << 1 \]
\[ \frac{\partial p}{\partial r} \cdot \frac{1}{r} \frac{\partial p}{\partial \theta} << \frac{\partial p}{\partial z} \]

and

\[ \frac{p(r, \theta, z)}{p_0(z)} << 1 \]

Application of this approximation to the Navier-Stokes equations results in the following set of equations:
The equations to be solved then are (4)-(7). As can be seen easily for (5)-(7) the second derivatives of \( u, v, w \) with respect to \( z \) disappear altogether and only those with respect to \( r \) and \( \theta \) remain. As such, these equations look like parabolic equations in which \( z \) is the direction of marching. In fact, the solution of these equations is achieved by marching and this, of course, is much cheaper than the solution of the full Navier-Stokes equations (1)-(4).

Integration of the continuity equation (4) over the cross section of the duct yields the integral mass flow equation

\[
\int_0^R \int_0^{2\pi} w r dr d\theta = \frac{\dot{m}}{\rho} \tag{8}
\]

where \( \dot{m} \) is the mass flow rate through the duct.

Finite difference solution of Eqs. (4)-(7) does not guarantee that Eq.(8) is satisfied, and therefore this equation must be checked after every integration step.
3. THE NUMERICAL METHOD

In this section a finite-difference procedure for solving Eqs. (4)-(7) based on the ADI method is discussed. In this method, like in Patankar-Spalding (1) or Briley (2) the equations are linearized. The velocity and pressure distributions in the secondary flow are calculated simultaneously by iteratively solving a Poisson equation as in Briley's method and a Poisson-like equation as in Patankar-Spalding method.

We realize that Equations (4)-(7) have a singularity at \( r=0 \). This singularity is not physical since the flow has no singularity whatsoever at \( r=0 \), but is comes up due to the transformation \((x,y) \rightarrow (r,\theta)\) being singular at \( r=0 \). This singularity may cause a drop in the order of accuracy of the finite difference equations if a uniform grid in the \( r \)-direction is used. One way of overcoming this problem is explained in the sequel.

We note that in our numerical solution we make use of a staggered grid in all three coordinates \( r, \theta, z \).

For solving the velocity components we multiply Equations (4)-(7) by \( r^2 \).

Momentum equations:

\[
\begin{align*}
0_r^2 \frac{\partial u}{\partial z} &= - \frac{r^2}{\rho} \frac{\partial p}{\partial r} - u r^2 \frac{\partial u}{\partial r} - v r \frac{\partial u}{\partial \theta} + \nu^2 r \\
&+ \nu \left[ \frac{r^2}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru) \right) + \frac{\nu^2}{\theta^2} - 2 \frac{\partial v}{\partial \theta} \right] \\
0_r^2 \frac{\partial v}{\partial z} &= - \frac{r}{\rho} \frac{\partial p}{\partial \theta} - u r^2 \frac{\partial v}{\partial r} - v r \frac{\partial v}{\partial \theta} - u v r + \\
&+ \nu \left[ \frac{r^2}{r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rv) \right) + \frac{\nu^2}{\theta^2} + 2 \frac{\partial v}{\partial \theta} \right] \\
0_r^2 \frac{\partial w}{\partial z} &= - \frac{r^2}{\rho} \frac{\partial p}{\partial z} - u r^2 \frac{\partial w}{\partial r} - v r \frac{\partial w}{\partial \theta} + \nu \left[ \frac{r}{r} \frac{\partial}{\partial r} (r \frac{\partial w}{\partial r}) + \frac{\nu^2}{\theta^2} \right].
\end{align*}
\]  

Continuity equation:

\[
r \frac{\partial (ru)}{\partial r} + r \frac{\partial v}{\partial \theta} + r^2 \frac{\partial w}{\partial z} = 0 .
\]
In addition, initial conditions for all velocity components and pressure must be specified. The boundary condition for the velocity is:

\[ u = \dot{v} = w = 0 \]

at the wall of the duct.

3.1 Mesh Staggering

The method of solution of this problem utilizes a staggered grid in all three directions \( r, \theta, z \). Figs. 2 and 3 show how the points are arranged in the \( r-\theta \) and \( r-\theta-z \) planes, respectively.

**Figure 2**

**Figure 3**
The points $\alpha$, $\beta$, $\gamma$ are the mid-points of the segments $\overline{PS}$, $\overline{PW}$ and $\overline{PP'}$ respectively.

The continuity equation is discretized at the point $P$, and the momentum equations in the $r$, $\theta$, $z$ directions are discretized at the points $\alpha$, $\beta$ and $\gamma$ respectively.

The first point in the $r$-direction is taken to be $r = h_r/2$.

3.2 The Finite Difference Equations

To implement the procedure, the flow region is discretized by grid points having equal spacings $h_r$ and $h_\theta$ in the $r$-$\theta$ directions, respectively, and a variable axial step size $h_z$. The subscripts $m$ and $n$, and the superscript $k$ denote the location of the grid point in the $r$, $\theta$, $z$ directions, respectively. The following shorthand difference operator notation is used for finite difference formulae for the derivatives:

\[
\delta_\theta f^k_{m,n} = \frac{f^k_{m,n+1} - f^k_{m,n-1}}{2h_\theta} - \left(\frac{\partial f}{\partial \theta}\right)_{m,n} + O(h_\theta^2) \tag{13}
\]

\[
\delta^2_\theta f^k_{m,n} = \frac{f^k_{m,n+1} - 2f^k_{m,n} + f^k_{m,n-1}}{h_\theta^2} - \left(\frac{\partial^2 f}{\partial \theta^2}\right)_{m,n} + O(h_\theta^2) \tag{14}
\]

\[
r_m \delta_r (r_m^k f^k_{m,n}) = \frac{r_m}{h_r^2} \left[ (r_{m+1/2}^k f^k_{m+1,n} - f^k_{m,n}) - (r_{m-1/2}^k f^k_{m,n} - f^k_{m-1,n}) \right] \]

\[
= r \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right)_{m,n} + O(h_r^2) \tag{15}
\]

\[
\delta_r \left( \frac{1}{r_m} \right)_{m,n} (r_m^k f^k_{m,n}) =
\]

\[
\left[ \left( \frac{1}{r} \right)_{m+1/2} (r_{m+1} f^k_{m+1,n} - r_m f^k_{m,n}) - \frac{1}{4} \left( \frac{1}{r} \right)_{m-1/2} (r_m f^k_{m,n} - r_{m-1} f^k_{m-1,n}) \right]/h_r^2 =
\]

\[
= \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial f}{\partial r} \right)_{m,n} + O(h_r^2) \tag{16}
\]

\[
\delta_z f^k_{m,n} = \frac{f^k_{m,n+1} - f^k_{m,n}}{h_z} - \left(\frac{\partial f}{\partial z}\right)_{m,n} + O(h_z) \tag{17}
\]
where \( f_{m,n}^k \) is a dummy symbol representing any one of the dependent variables, and \( r_m, \theta_n, z^k \) are the coordinates at which \( f \) is evaluated, actually:

\[
\begin{align*}
  r_m &= (m-1)h_r & \text{for } f = u \\
  r_m &= (m-k)h_r & \text{otherwise,} \\
  m &= 1, 2, \ldots.
\end{align*}
\]

It is assumed that the solution in the cross section of the duct at \( z = z^k \) is known, and the values for \( w^{k+1}, u^{k+1} \) and \( v^{k+1} \) are to be computed. It should be noted that this formulation together with the movement of beginning of the grid from the center to \( h_r/2 \) ensures second order accuracy. This will be illustrated in the following:

\[
\begin{align*}
  r \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) &= r_m \frac{\left( r \frac{\partial f}{\partial r} \right)^{k}_{m+\frac{1}{2}, n} - \left( r \frac{\partial f}{\partial r} \right)^{k}_{m-\frac{1}{2}, n}}{h_r} + O(h_r^2) = \\
  &= r_m \frac{f_{m+\frac{1}{2}, n}^k - f_{m-\frac{1}{2}, n}^k}{h_r} + O(h_r^2) = \\
  &= \frac{r_m}{h_r^2} \left[ f_{m+\frac{1}{2}, n}^k - f_{m, n}^k \right] - \frac{r_m}{h_r^2} \left[ f_{m-\frac{1}{2}, n}^k - f_{m-1, n}^k \right] + O(h_r^2).
\end{align*}
\]

At the first grid point we have:

\[
\begin{align*}
  r_1 &= h_r/2 \\
  r_{1+\frac{1}{2}} &= h_r \\
  r_{1-\frac{1}{2}} &= 0
\end{align*}
\]

and

\[
\begin{align*}
  \left[ r \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) \right]_{1, \theta} = \frac{f_{2, n}^k - f_{1, n}^k}{2} + O(h_r^2)
\end{align*}
\]

while for \( f = v \) and \( f = w \) we have a second order accurate discretization procedure allowed by the choice \( r_m = (m-\frac{k}{2})h_r \), the problem of accuracy remains for \( f = u \). The solution to this problem will be discussed in Section 3.2.1.
As stated above we solved the three-dimensional boundary layer problem by both Patankar and Spalding's method and by Briley's method.

The similarities and differences between these two methods are described below:

Application of Briley's method to the present equations gives the following procedure:

1. $v_p^{k+1}$ and $u_p^{k+1}$ are computed from the momentum equations (9) and (10);
2. $w^{k+1}$ is computed from the axial momentum equation (11), with $p_m^{k+1}$ determinated implicity to ensure that the axial mass flow relation (8) is satisfied;
3. $v_c^{k+1}$ and $u_c^{k+1}$ are computed from the equation (12), using a velocity potential. The problem that one has to solve is a Neumann problem for the Poisson equation;
4. $(\frac{\partial p}{\partial r})^{k+1}$ and $(\frac{\partial p}{\partial \theta})^{k+1}$ are computed from Equations (9) and (10).

(At this point all the velocity components at $k+1$ are known.)

Application of the Patankar and Spalding method to the present equations gives the following procedure:

1. $v_p^{k+1}$ and $u_p^{k+1}$ are computed from the momentum equations (9)-(10);
2. An estimate of $w^{k+1}$ is computed from Equation (11) with $\frac{dP_0}{dz}$ guessed. Then the correct velocity component in the $z$-direction and $\frac{dP_0}{dz}$ are determined from the axial mass flow relation (8).
3. Corrections for $v_c^{k+1}$ and $u_c^{k+1}$ and the pressure are computed from equation (12), which is formulated as a difference equation that resembles a Poisson equation with Neumann boundary conditions.

The difference between these two methods is in the calculation of the correction for the velocity components $u$ and $v$. In Briley's method the correction is assumed to be the gradient of potential $\phi$, while Patankar and Spalding correct the pressure which is then used to correct the velocity components $u$ and $v$. 
Both in Briley's and Patankar and Spalding's methods the velocity components, u, v and w are obtained by using the ADI method in the r and θ directions, and the pressure gradient is calculated from the upstream pressure.

3.2.1 Calculation of u and v

It is noted that Equations (9) and (10) can be written in the form:

\[ r^2 w \frac{\partial f}{\partial z} = L^2 f + S_f \]  

(18)

where

f - denotes any of the dependent variables (u) or (v);

\[ L^2 \] - is a second-order differential operator, defined as:

\[ L^2 f = -ur^2 \frac{\partial^2 f}{\partial r^2} - vr \frac{\partial^2 f}{\partial \theta^2} + v \left( r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial \theta^2} \right) \]

(19)

\[ S_f \] - is source term, given by:

for f = u: \[ S_f = -\frac{r^2}{\rho} \frac{\partial p}{\partial r} - 2v \frac{\partial v}{\partial \theta} + v^2 r \]

(20)

for f = v: \[ S_f = -\frac{r}{\rho} \frac{\partial p}{\partial \theta} + 2v \frac{\partial u}{\partial \theta} - uv_r \]

(21)

Now, Equation (18) can be written in the following form:

\[ \frac{\partial f}{\partial z} = \frac{L^2(f)}{r^2 w} + \frac{S_f}{r^2 w} \]

or in finite-difference form:

\[ f_p = A^f_{N} f_N + A^f_{S} f_S + A^f_{E} f_E + A^f_{W} f_W + C^f \]

(22)

where the A-coefficients originate from the operator \( L^2 \) and \( C^f \) is the "discretized" source term.

In Briley's method the source terms in (20) and (21) are discretized using finite differences, where as in Patankar and Spalding's method the "discretized" source terms are given as:
\[ C^u = B^u + D^u(p_p - p_S) \]  
\[ C^v = B^v + D^v(p_p - p_w) \]  
where the coefficients \( B^u \) and \( B^v \) express the effect of convection from upstream and of source terms; and \( D^u \) and \( D^v \) involve areas, mass flow rates, and other quantities.

Equation (22) is solved separately for velocity components \( u \) and \( v \) with the \( A's, B's, C's \) and \( D's \) known from upstream.

For \( f = u \) the points with \( r = 0 \) are all grid points which are on the boundary of the mapping of the circle (cross section of the duct); and since \( f \) at \( r = 0 \) has only one value which is not known, second order accurate difference formulas on the mapping of the cross section fail at the grid points with \( r = h_w \). In order to overcome this problem we resort to the following method: A function \( u(x,y) \) which is analytic at \( x = y = 0 \) has a Fourier series of the form \( u(r,\theta) = \sum_{n=-\infty}^{\infty} a_n r^n \sin n\theta \) for \( 0 < r < R \) for some \( R > 0 \) and \( 0 < \theta < 2\pi \) from which we can easily deduce that \( u(-r, \theta \pm \pi) = u(r, \theta) \). This shows that \( u(r, \theta) \) can also be defined for \( -R < r < R \) and \( 0 < \theta < \pi \). If this is done, then the points with \( r = 0 \) become interior points of the region defined by \( S = \{ -R < r < R, 0 < \theta < \pi \} \), hence we can now consider the problem in \( S \) with the boundary conditions given on \( r = \pm R \) and \( u(-r, \theta) = u(r, \theta - \pi) \).

3.2.2 Calculation of \( w \)

The axial momentum equation (11) can be written in the form:

\[ r^2 w \frac{\partial w}{\partial z} = M^2 w + T_w \]  

where
\( M^2 \) is a second-order differential operator such that

\[
M^2 w = -ur^2 \frac{\partial w}{\partial r} - vr \frac{\partial w}{\partial \theta} + \nu \left[ r \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial \theta^2} \right]
\]  

(26)

\( T_w \) is a source term given by.

\[
T_w = - \frac{r^2}{\rho} \frac{dp_0}{dz}.
\]

(27)

The finite-difference form of Eq. (25) is

\[
w_p = A_{NN} w_N + A_{WS} w_S + A_{WE} w_E + A_{EW} w_W + C_w
\]

(28)

where

\[
C_w = - \frac{1}{w} \frac{dp_0}{dz}.
\]

(29)

There is not much difference between the methods of Patankar and Spalding and Briley as far as the solution for \( w \) is concerned. In Briley's method one solves (28) with \( C_w \) as given in (29), whereby in Patankar and Spalding's method \( C_w \) is discretized as:

\[
C_w = B_w + D_w \frac{dp_0}{dz}
\]

(30)

where \( B_w \) and \( D_w \) are terms that depend on the upstream flow.

Equation (28) is solved when the \( A's \) and \( C's \) are calculated upstream and the velocity components \( u' \) and \( v' \) are approximated by the predicted \( u_p' \) and \( v_p \) described above.

The difference between the two methods is in the correction for the velocity component \( w \):

I. Briley's method

When we solve Eq. (28) the pressure gradient is unknown and we approximate it by the upstream pressure gradient. Therefore, the value of the velocity component \( w \) is not exact. Correction for this value can be obtained from the integral mass conservation equation (8):
\[ \int_0^{2\pi} \int_0^R w_i r dr d\theta - \frac{m}{\rho} = \Delta m_i \]  

(31)

where

- \( w_i \) is the i-th iteration for \( w \), based on \( \frac{dP_0}{dz} \) at \( i \),

- \( \Delta m_i \) - the added mass, should be theoretically zero.

Because of the linear character of the difference equations, we can find

\[ \frac{dP_0}{dz} \] for zero added mass flow; \( \Delta m = 0 \)

\[ \frac{dP_0}{dz} \bigg|_3 = \frac{\Delta m_1 \frac{dP_0}{dz} \bigg|_2 - \Delta m_2 \frac{dP_0}{dz} \bigg|_1}{\Delta m_1 - \Delta m_2} \]  

(32)

Figure 4: The pressure gradient correction
Now, once we have found the pressure gradient $\frac{dP}{dz}$, that anihilates $\Delta m$, the velocity component $w$ can be found from Eq. (21).

II. The Patankar-Spalding method. We calculate $w$ from the momentum equation with estimated $\left(\frac{dP}{dz}\right)_{\text{est}}$, which is taken to be $\left(\frac{dP}{dz}\right)_k$.

The total mass-flow rate must be found by $\Sigma \rho D W r r h_0$, which will, in general, differ from the true mass-flow rate through the duct $m$, which can be calculated from the boundary and initial conditions. The difference between the two terms is used to find the correction for the pressure gradient

$$\frac{dP}{dz}\bigg|_c = \frac{\dot{m} - \Sigma \rho \omega r r h_0}{\Sigma \rho D W r r h_0}$$  \hspace{1cm} (33)

where

$$\frac{dP}{dz}\bigg|_{k+1} = \left(\frac{dP}{dz}\right)_{\text{est}} + \left(\frac{dP}{dz}\right)_k.$$  \hspace{1cm} (34)

The corrected axial velocity is:

$$w_{k+1} = w_p + D_w \left(\frac{dP}{dz}\right)_k.$$  \hspace{1cm} (35)

3.2.3 The correction of the lateral velocity components

The next step in the two methods is to compute the correct values of the lateral velocity components: $u_{k+1}$ and $v_{k+1}$, which satisfy the transverse momentum equations (9) and (10) and continuity equation (12).

The correct values are assumed to be given by the following equations:

$$u_{k+1} = u_p + u_c$$

$$v_{k+1} = v_p + v_c,$$  \hspace{1cm} (36)

where:

$u_p$ and $v_p$ are the predictions of $u_{k+1}$ and $v_{k+1}$ computed from the secondary flow momentum equations (9) and (10), as explained in Section 3.2.1;
$u_c$ and $v_c$ are corrections to $u_p$ and $v_p$ computed so as to satisfy the continuity equation (12).

By substitution of (36) in (12) we obtain:

$$r \frac{\partial}{\partial r} (r u_c) + r \frac{\partial v_c}{\partial \theta} = \left[ r^2 \frac{\partial w}{\partial z} + r \frac{\partial v_p}{\partial \theta} + r \frac{\partial}{\partial r} (r u_p) \right].$$

(37)

The solution of (37) is different in the two methods:

I. Briley's method: It is assumed that the velocity corrections are irrotational, and a velocity potential $\phi$ introduced is such that

$$u_c = \frac{\partial \phi}{\partial r},$$

$$v_c = \frac{1}{r} \frac{\partial \phi}{\partial \theta}.$$

(38)

The boundary conditions require that the normal velocity vanishes at the walls. The boundary condition for the potential $\phi$ is:

$$\frac{\partial \phi}{\partial r} \bigg|_{r=R} = 0.$$  

(39)

Equation (38) is substituted in Eq. (37). The resulting difference equation is:

$$r_m \delta_r (r_m \delta_r (\phi)) + \delta_\theta^2 (\phi) = - r^2 \delta_z (w) + r \delta_\theta (v_p) + r \delta_r (r u_p)$$

or

$$\phi_{m+1,n} \left( \frac{r_{m+1} \delta_{r}}{h_r^2} \right) - \phi_{m,n} \left( \frac{r_m \delta_{r} + r_{m-1} \delta_{r}}{h_r^2} \right) + \phi_{m-1,n} \left( \frac{r_{m-1} \delta_{r}}{h_r^2} \right) +$$

$$+ \phi_{m,n+1} \left( \frac{1}{h_\theta^2} \right) - \phi_{m,n} \left( \frac{2}{h_\theta^2} \right) + \phi_{m,n} \left( \frac{1}{h_\theta^2} \right) =$$

$$= - \left[ r^2 \frac{w^{k+1} - w^k}{h_z} + \frac{r_m}{h_\theta} (v_{m,n+1} - v_{m,n}) + \frac{r_m}{h_r} ((ur)_{m+1,n} - (ur)_{m,n}) \right].$$

(40)
II. Patankar and Spalding's Method: The velocity corrections in this method are given by:

\[ v^{k+1} = v_p + D^v(p'_p - p'_w) \]
\[ u^{k+1} = u_p + D^u(p'_p - p'_s) \]  \hspace{1cm} (41)

This is an approximation for the momentum equations with the pressure corrections \( p'_p \), given by:

\[ p^{k+1} = p_p + p'_p \]  \hspace{1cm} (42)

Now we must solve a Poisson-like difference equation for \( p'_p \), where this one can be viewed as being obtained by substituting (42) and (41) in Equation (37).

\[ p'_p = A_p p'_p + A_{p p} p'_p + A_{p p} p'_p + A_{p p} p'_p + B_p \]

or

\[ p^{m-1}_{m,n} \left( \frac{r_m^D u_{m,n}}{h_r^D} \right) - p_{m,n} \left( \frac{r_m^D u_{m+1,n}}{h_r^D} + \frac{r_m^D u_{m,n}}{h_r^D} \right) + p_{m+1,n} \left( \frac{r_m^D u_{m+1,n}}{h_r^D} \right) + p_{m,n-1} \left( \frac{r_m^D u_{m,n-1}}{h_r^D} \right) - p_{m,n} \left( \frac{r_m^D u_{m+1,n}}{h_r^D} + \frac{r_m^D u_{m,n}}{h_r^D} \right) + p_{m,n+1} \left( \frac{r_m^D u_{m,n+1}}{h_r^D} \right) = 0 \]  \hspace{1cm} (43)

The boundary conditions for the pressure correction at a wall is derived from the condition that the velocity correction vanishes there, and so the normal derivative at the boundary must be zero:

\[ \frac{\partial p'_p}{\partial r} \bigg|_{r=R} = 0. \]  \hspace{1cm} (44)

Thus, we can see that the solution of the velocity correction in Briley's method with Eqs. (39)-(40) is similar to the solution of the velocity correction in Patankar and Spalding's method with Eqs. (43)-(44).
In both cases, the Poisson's equation has a Neumann boundary condition, with all the quantities on the right-hand side of the equation known. The equation is solved by the ADI method. Next, the velocity components are calculated by Eq. (36) in Briley's method or Eq. (41) in Patankar-Spalding's method.

3.3 The Solution of the Finite-Difference Equations

The equation

\[ \frac{\partial f}{\partial z} = A_r^2(f) + S_D + O(h^2) \]  

(45)

can be solved by the ADI method. To describe the method we replace Eq. (45) by its finite-difference analog:

\[ \frac{\partial f}{\partial z} = (A_r + A_\theta)f + S_D + O(h^2) \]  

(46)

where

- \(A_r, A_\theta\) - are matrix operators,
- \(f\) - represents \(u, v, w\),
- \(S_D\) - is a finite difference form of the source term.

The Peaceman-Rachford scheme for Eq. (46) is:

\[
(I - \frac{h_z}{2} A_r)f^* = (I + \frac{h_z}{2} A_\theta)f^k + \frac{h_z}{2} S_{D}^{k+\frac{1}{2}}
\]

(47)

\[
(I - \frac{h_z}{2} A_\theta)f^{k+\frac{1}{2}} = (I + \frac{h_z}{2} A_r)f^* + \frac{h_z}{2} S_{D}^{k+\frac{1}{2}}
\]

where

- \(I\) - denotes to the unit matrix.

The Poisson's equation is solved by the ADI method too, but in this case Equations (47) are repeated until a stationary value for \(f\) is obtained, with \(h_z\) understood as a time step.
3.4 Computational Details

3.4.1 Remarks on the solution of Neumann problem for Poisson's Equation

In the solution of Eq. (40) for \( \phi \) some attention must be given to the integral constraint which arises when solving the Poisson equation with normal derivative boundary conditions. The two-dimensional Poisson equation for \( \phi \) with source distribution \( f(r, \theta) \) and Neumann boundary conditions has a solution only if the following condition is satisfied.

\[
\int_A f(r, \theta) \, dA = \int_C \frac{\partial \phi}{\partial r} \, dS \quad (48)
\]

where

- \( A \) - is the area enclosed by \( C \)
- \( r \) - is the outward normal to \( C \)
- \( S \) - is a distance along \( C \).

In our problem \( f(r, \theta) \) is not known exactly by an approximation to it is obtained by discretization. Therefore, we do not expect (48) to be satisfied exactly. Hence we compute the total correction \( E \) defined by

\[
E = \int_A f(r, \theta) \, dA - \int_C \frac{\partial \phi}{\partial r} \, dS \quad (49)
\]

The average correction is then defined to be

\[
f_{\text{ave}} = \frac{E}{A} \quad (50)
\]

and the corrected source term then is

\[
f_{\text{correct}}(r, \theta) = f(r, \theta) - f_{\text{ave}} \quad (51)
\]

It is with this \( f_{\text{correct}}(r, \theta) \) that Poisson's equation for \( \phi \) in (40) is solved.
3.4.2 Treatment of \( w \) on the axis of the duct

As we have seen the numerical solution for \( w \) has been obtained at the grid points for which \( r^m = (m-k)\Delta r \). However, we are also interested in finding the value of \( w \) at \( r = 0 \). For this we start by writing Equation (41) in the form

\[
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{w}{r} \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{u}{r} \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta}.
\]

which is simply the two-dimensional Poisson's equation

\[ v^2 w = f \]

where \( f \) is the right-hand side of (52).

Using the divergence theorem again, we have

\[
\int_0^{2\pi} \left( \frac{\partial w}{\partial r} \right) r \, d\theta \bigg|_{r=h_r/2} = \int_0^{2\pi} \int_0^R f \, r \, dr \, d\theta.
\]

By approximating the derivative \( \frac{\partial w}{\partial r} \) in (53) using finite differences

\[
\left. \frac{\partial w}{\partial r} \right|_{r=h_r/2} = \frac{w_{3h_r/2} - w_{h_r/2}}{h_r} + O(h_r)
\]

we obtain an equation for \( w \) at \( r = 0 \) which can be solved easily.
4. THE COMPUTER PROGRAM

The procedure of computation by Briley's method
(staggered grad)

$K = 0$
Initial and boundary condition
SUBROUTINE DATA

Solution of $v_p^{k+1}$ and $u_p^{k+1}$, Eqs. (9)-(10)
SUBROUTINE VELV
SUBROUTINE VELU

Solution of $w^{k+1}$, Eqs. (11), (8)
SUBROUTINE VELW

Solution of velocity correction, Eq. (40)
SUBROUTINE CORVEL

Solution of corrected $u$, $v$, Eq. (36)
in MAIN.PROGRAM

Solution of corrected pressure gradient
SUBROUTINE CORPR

$K = K + 1$
The procedure of computation by Patankar-Spalding method (staggered grad)

\[ K = 0 \]

Initial and boundary condition

SUBROUTINE DATA

Solution of \( \nu_p^{k+1} \) and \( \nu_p^{k+1} \), Eqs. (9)-(10)

SUBROUTINE VELV

SUBROUTINE VELU

Solution of \( \nu_p^{k+1} \), Eq. (11)

SUBROUTINE VELW

Solution of velocity corrected \( \dot{w}_y \), Eq. (35)

SUBROUTINE CORR

Solution of pressure correction \( p' \), Eq. (43)

SUBROUTINE PRESS

Computation of corrected \( u, v, p \),

Eqs. (41)-(42)

in MAIN PROGRAM

\[ K = K + 1 \]
REFERENCES


