STABILITY AND ACCURACY IN FLUID-STRUCTURE PROBLEMS

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ABSTRACT

A mathematical model of fluid-structure interaction is formulated. It is considered as an initial boundary value problem, where the structure represents the boundary. A finite-difference method is used for time integration. By spectral method the necessary and sufficient stability conditions and accuracy estimates are obtained.
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1. INTRODUCTION

Dynamic interaction between structures and the surrounding (hydrodynamic) medium is an important problem in modern engineering.

Insufficiently studied is a class of problems where structures immersed in compressible medium are affected by large dynamic loads. In these problems the mass of displaced medium is comparable with the mass of the structure, and the approximation by structure-vacuum model is not consistent. As examples of applications we can mention: dynamic reactions of underwater or underground structures to shock waves, space vessels impact on water surface, dynamic behaviour of shells filled by liquid.

In the area of modeling large liquid-metal reactors (LMFBR) two problems are important: the dynamic response of fast-reactor core sub-assemblies with the role of the coolant included and the response of core grid support structures in LMFBR to nuclear excursions.

From the mathematical point of view all these problems belong to the dynamics of continuum and are of the most difficult in applied mathematics.

The interaction problems consists of solving two mutually dependent problems:
1) determination of external forces, i.e. the hydrodynamic forces on the structure (hydrodynamic problem);
2) determination of stresses and deformations in the shell (structure problem).

Today there are no general mathematical methods for treatment of structural-hydrodynamical interaction problems. There exists only a
small number of analytical solutions for linear two-dimensional and axisymmetric problems with weak shocks and simple geometry of shells. The situation with computational methods is slightly better, and there is some progress in the last few years.

Most of the computer codes for the time integration of the equations of the interaction problems are based on finite-difference or finite-element methods. The most popular is the approach, when the structure is integrated implicitly and the surrounding medium is integrated explicitly. The reason for this is that the implicit-explicit partition avoids the stability complications and restrictions, which are results of interaction between two different physical processes in the structure and in the surrounding medium. But such approach leads to undesirable computational expense, and there remain still stability restrictions which must be overcome by extra computational expense [1].

From an economical point of view the explicit-explicit approach is much more attractive. In this research, we make the first step in analyzing the stability of explicit-explicit finite difference time integration. For this purpose we formulate a very simple linear, one-dimensional model. A wide class of interaction problems may be analyzed by this model. We consider the equations of this model as an initial boundary value problem, where the structure represents an elastic boundary for the surrounding medium. All the assumptions and simplifications as well as the equations are given in Section II. In Section III the exact solution to the initial boundary value problem is given. The numerical representation to the above problem is developed in Section IV.

For initial boundary-value problems the spectral method is very useful for stability analysis. The theoretical background of spectral
method is given in Section V. We found out by this analysis that the stability of interaction problems can not be considered separately for the shell and for the medium. We formulate the necessary and sufficient stability condition for the interaction problem.

In Section VI we prove that the accuracy is not less important than the stability. We found out that the behaviour of the structure is very sensitive to the surrounding medium. We formulate necessary and sufficient conditions on the mesh not to destroy the qualitative behaviour of the structure.

More details on the accuracy and the computational results are now in preparation for publication. A part of them are based on computations experiments with the computer code "Discom", which is developed by J.Kivitiy specially for solving interaction problems. Since, the equations of code "Discom" may be reduced to the equations of the same type as used in our model, the results are of great interest.
2. FORMULATION OF THE PROBLEM. GENERAL ASSUMPTIONS AND EQUATIONS

The interaction between nonstationary pressure waves with deformable shells is a very complicated problem. The rapid change of the process parameters with time, existence of shock fronts, cavitation, development of plastic zones in the shell material, the complicated structure of reflected and refracted waves characterize interaction problems. This is a reason why we need simplifying assumptions and hypotheses and why we use various approximate theories. On the other hand, it is very important to choose satisfactory mathematical models for all components, or one may miss important physical effects.

2.1 Shell Model

To approximate the structure we shall use thin shell models. There are a variety of shell models in literature. The general approach is to reduce the three dimensional equations of elasto-dynamics to two-dimensional ones, due to the fact that one dimension of the shell is small compared with the others. There are different methods of 'reduction', but none of these methods produce an ideal shell model; for each such methods there are restrictions on applicability. To choose a proper shell-model is very difficult. Today the most popular is the classical method, based on the Kirchhoff-Love hypothesis, this method leads to equations of parabolic type (P-model). The transverse shear and inertia of rotation are not taken into account in the P-model.

The most simple equation of the linear, one-dimensional P-model shell may be given by:

$$\frac{\partial^2 w(t)}{\partial t^2} + \omega^2 w(t) = -\frac{P(t)}{M}$$  \hspace{1cm} (2.1)
where \( w \)-displacement, \( M \)-mass, \( \omega \)-frequency, \( P(t) \)-hydrodynamic pressure on the shell surface.

The frequency \( \omega \) is representative of the type of the shell and of the boundary conditions. For instance, for membrane shell \( \omega = \frac{a_s \Omega}{R} \), for bending shell \( \omega = \frac{a_s h \Omega}{R^2} \), where \( a_s \)-sound velocity in shell material, \( R \)-radius of the shell, \( h \)-thickness of the shell, \( \Omega \), \( \Omega_b \)-coefficients.

We shall use the equation (2.1) as an approximation to the structure in the interaction problem, since it is very simple and easy to manage and on the other hand it is still general enough for description of most elastic shells.

2.2 Models of the Medium

In our work under "medium" we mean water.

There are two basic models to approximate the properties of the medium—ideal and nonideal. In the ideal model the stress is independent of orientation, i.e. isotropic. The behaviour of the medium in such model is completely defined by the equation of state

\[
P = P(\rho, T),
\]

where \( P \)-is the pressure, \( \rho \)-density, \( T \)-temperature.

In nonideal models the stress depends not only on the \( \rho \)ads but also on their direction. In this case the behaviour is defined by a tensorial relationship.

For waves created by explosions, water may be considered to be an ideal fluid \([6]\). Moreover, experiments show that in water the propagation
of shock wave with pressure at the front up to 30000 atm may be considered as an isentropic process [4]. The isentropic Tait state equation is [9]:

\[
\frac{P + B}{B} = \left(\frac{\rho}{\rho_0}\right)^\gamma
\]

(2.3)

where \( B = 3000 \) atm, \( \gamma = 7.15 \), \( \rho_0 = 1000 \frac{kg}{m^3} \).

The equation does not depend on temperature. Relationship of the type

\[ P = f(\rho) \]

(2.4)

are called barotropic.

There are different state equations of type (2.4) for water under large loads, [6] and [9]. We shall use, in general, ideal models with an equation of state of type (2.4). For nonideal models see [6].

We assume the following assumptions for the medium which are usual in the literature [9]:

1. Compressible,
2. Inviscid,

With the previous mentioned isentropy the equations of motion are:

\[
\frac{D\rho}{Dt} + \rho \mathbf{v} \cdot \mathbf{v} = 0 ,
\]

(2.5)

\[
\rho \frac{D\mathbf{v}}{Dt} + \mathbf{v} \rho = 0 ,
\]

(2.6)

where \( \mathbf{v} \) is the velocity vector, \( \frac{D}{Dt} \) is the material derivative.

These equations are nonlinear of hyperbolic type.
The linear (acoustic) approximation is valid for relatively small loads. In [4] is shown that for one-dimensional water flow the acoustic approximation is satisfactory for pressure on the front up to \(1100 \text{ kg/cm}^2\).

The equations (2.4)-(2.6) with initial and boundary conditions give a closed system.

In one-dimensional case the system (2.5)-(2.6) may be written in the form [3]:

\[
\rho_o (1 + \frac{\partial u}{\partial x}) = \rho_o \tag{2.7}
\]

\[
\rho_o \frac{\partial^2 u}{\partial t^2} + \frac{\partial p_o}{\partial x} = 0 \tag{2.8}
\]

where (*) is used for disturbed and (o) for undisturbed fluid, u-displacement, x-Lagrangian coordinate.

From Equations (2.3), (2.7) the pressure may be written:

\[
P = (P_o + B)[(1 + \frac{\partial u}{\partial x})^{-\gamma} - 1]. \tag{2.9}
\]

For small displacements

\[
P = -\gamma (P_o + B) \frac{\partial u}{\partial x} \tag{2.10}
\]

From Equations (2.8), (2.10) we obtain

\[
\frac{\partial^2 u}{\partial t^2} - a_0^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{2.11}
\]

where \(a_0\) is the sound velocity of undisturbed fluid.
2.3. Structural-Hydrodynamic Interaction Problem

The structural-hydrodynamic interaction problem consists of solving together equations of shell and medium with corresponding contact, initial and boundary conditions, see Fig. 1. The equations of shell (2.1) and medium (2.11) are correspondingly

\[
\frac{\partial^2 w}{\partial t^2} + \omega^2 w = -\frac{P}{M}\bigg|_{x=0}
\]

\[
\frac{\partial^2 u}{\partial t^2} - a_0^2 \frac{\partial^2 u}{\partial x^2} = 0.
\]

For pressure \( P \) we use the equation (2.10).

Under the contact condition we understand the condition of joint motion of the shell and adjoining medium particles. The condition of nonpermeability of the shell is:

\[
w = u(0,t).
\]

The initial conditions for the shell at \( t = 0 \) are:

\[
w = \frac{\partial w}{\partial t} = 0.
\]

The initial conditions for the motion of the disturbed medium are
The system (2.12) may be simplified by the following definitions:

\[ x' = x/L, \quad t' = tw, \quad q = a_o/\omega L, \quad (2.16) \]

where \( L = \frac{M}{\rho_o} \) represents the length of the fluid column with the same mass as the corresponding column of the shell.

By (2.16) the system (2.12) takes the form

\[
\begin{align*}
\frac{\partial^2 u}{\partial t'^2} - q^2 \frac{\partial^2 u}{\partial x'^2} &= 0 \\
\frac{\partial^2 w}{\partial t'^2} + w &= q^2 \frac{\partial u}{\partial x'} \bigg|_{x'=0} \\
w &= u \bigg|_{x'=0}
\end{align*}
\]

(2.17)

For simplicity we shall drop from now on the prime notation.

The system (2.17) together with initial and boundary conditions will be the basic model for our stability analysis.
3. EXACT SOLUTION FOR THE INTERACTION PROBLEM

Let us define the pressure at $t = 0$ as

$$p = -q^2 f'(qt + x).$$

Then the general solution to system (2.17) takes the form

$$u(x,t) = f(q + tx) - f(qt - x) + w(t - \frac{x}{q}).$$

From (2.19) the hydrodynamic pressure on the shell is:

$$p = -q^2 \frac{\partial u}{\partial x} |_{x=0} = -2q^2 f'(qt) + qw'(t).$$

Now the equation for $w$ is

$$\frac{\partial^2 w}{\partial t^2} + q \frac{\partial w}{\partial t} + w = 2q^2 f'(qt).$$

The solution for (2.19) is

$$w = c_1 e^{\sigma_1 t} + c_2 e^{\sigma_2 t} + w_s,$$

where $w_s$ is a special solution of (2.19), $\sigma_{1,2}$ are roots of characteristic equation of the homogeneous part of (2.19)

$$\sigma_{1,2} = \frac{-q + \sqrt{q^2 - 4}}{2},$$

$c_1, c_2$ may be found from initial conditions. For $q > 2$ we have a damped shell, for $q < 2$ the shell is oscillatory-damped, for $q = 2$ there is so called "critical" damping.

The existence of exact solution is an advantage of choosing the system (2.17) as a model.
4. NUMERICAL FORMULATION

We shall approximate the system (2.14) by finite-difference explicit scheme. As shown in [8] the linearization of Lagrangian one-dimensional scheme for fluid leads to the so-called leap-frog scheme.

The finite-difference equations are:

\[
\begin{align*}
\frac{u_{j}^{n+1} - 2u_{j}^{n} + u_{j}^{n-1}}{(\Delta t)^{2}} &= q^{2} \frac{u_{j+1}^{n} - 2u_{j}^{n} + u_{j-1}^{n}}{(\Delta x)^{2}} \\
\frac{w_{j}^{n+1} - 2w_{j}^{n} + w_{j}^{n-1}}{(\Delta t)^{2}} &= w_{j}^{n} - q^{2} \frac{\partial u}{\partial x} \bigg|_{x=0} \quad (\text{t=n\Delta t}) \quad (4.1)
\end{align*}
\]

where \( n,j \) - time and next point reference indices.

For \( \frac{\partial u}{\partial x} \bigg|_{x=0} \) we take the first-order approximation in \( \Delta x \).

Analysis of second order approximations are in preparation.

For the interaction condition (2.13) we take:

\[ w_{j}^{n} = u_{0}^{n} \quad (4.3) \]

The equations (4.1)-(4.3) will be used for the stability analysis.
5. STABILITY ANALYSIS

5.1 Theoretical Background

The stability analysis of initial boundary-value problems like (4.1)-(4.2) may be carried out by spectral methods, where (4.1) is considered as a boundary condition to (4.2). The description of spectral methods may be found in [7], [8].

The spectral method makes the ansatz \( u_j^n = z^n k^j \) for appropriate complex scalars \( z \) and \( k \). This substitution is made in both the difference scheme in the interior and on the boundary. The basic difference scheme is assumed stable and so the Von-Neumann stability condition says that \(|z| < 1\) for \(|k| = 1\). It is then proven that the initial-boundary scheme is stable if there are no nontrivial solutions of these equations with \(|z| > 1\) and \(|k| < 1\). Conversely, the scheme is unstable if there exist solutions with \(|z| > 1\) but \(|k| < 1\). When \(|z| = 1\); \(|k| = 1\) there are two cases to consider. If \( z \neq 1 \) the stability is not assured. When \( z = 1 \) it is called a generalized eigenvalue and to establish stability one must analyze the rate of growth.

5.2 Stability Conditions

We look for \( u_j^n = U z^n k^j \), where \( U \) is a scalar.

For the equation (4.1) we obtain

\[
\frac{(z-1)^2}{z} = \gamma^2 \frac{(k-1)^2}{k}, \tag{5.1}
\]

where \( \gamma = q \frac{A_t}{\Delta x} \). (Note: \( \gamma \) is invariant under transformations (2.16).)
For the equation (4.2), we obtain

\[
\frac{(z-1)^2}{z(\Delta t)^2} + 1 = q^2 \frac{k-1}{\Delta x} .
\]  

(5.2)

The necessary condition that the scheme is stable is the stability of (4.1) for \(|k| = 1\). That gives us the condition [7]

\[
\gamma \leq 1.
\]  

(5.3)

Now we have to find the conditions that there are no solutions with \(|z| > 1\) and \(|k| < 1\). For this purpose we introduce new variables \(p\) and \(s\) by

\[
p = \frac{k+1}{k-1} \quad \text{and} \quad s = \frac{z+1}{z-1} .
\]  

(5.4)

With (5.4) we look for conditions that for \(p\) with \(\text{Re} \cdot p < 0\) there are no \(s\) with \(\text{Re} \cdot s > 0\).

Now let us consider (5.1) and (5.2) together. After transformations (5.4) we obtain

\[
\begin{align*}
\frac{4}{(s^2-1)(\Delta t)^2} &= \frac{4q^2}{(p^2-1)(\Delta x)^2} \\
\frac{4}{(s^2-1)(\Delta t)^2} &= -1 + \frac{2q^2}{(p-1)\Delta x} .
\end{align*}
\]  

(5.5) (5.6)

The equations (5.5) may be solved for \(p\), using the combined equation

\[
\frac{4q^2}{(p^2-1)(\Delta x)^2} = -1 + \frac{2q^2}{(p-1)(\Delta x)}
\]  

(5.6)

or

\[
p^2(\Delta x)^2 - 2q^2 p \Delta x + 4q^2 - 2q^2 \Delta x - \Delta x^2 = 0 .
\]  

(5.7)
The solution of (5.7) is

\[ p_{1,2} = \frac{1}{\Delta x} \left[ q^2 \pm \sqrt{(\Delta x + q^2)^2 - 4q^2} \right]. \]  

(5.8)

Now we can see that for \( q > 2 \), \( p_{1,2} \) are real for all \( \Delta x \), and for \( q < 2 \), \( p_{1,2} \) are complex for \( \Delta x < q(2-q) \). The same may be said about \( s \) since from (5.5)

\[ s^2 = 1 + \frac{1}{\gamma^2} (p^2-1). \]  

(5.9)

A very important value of \( \Delta x \) is

\[ \Delta x = -q^2 + \sqrt{4q^4 + 4q^2}. \]  

(5.10)

For \( \Delta x < \overline{\Delta x} \), \( \text{Re} \, p_{1,2} > 0 \), but for \( \Delta x > \overline{\Delta x} \), \( \text{Re} \, p_2 < 0 \). That means that for \( \Delta x < \overline{\Delta x} \), the necessary and sufficient condition for stability is \( \gamma < 1 \) or in other words, stability of the fluid only is necessary and sufficient. For \( \Delta x > \overline{\Delta x} \) we must consider the equation (5.9). Since, always \( \overline{\Delta x} > q(2-q) \), for \( \Delta x > \overline{\Delta x} \) we have \( \text{Im} \, p_2 = 0 \). And to assume the stability we must demand

\[ 1 + \frac{1}{\gamma^2} (p^2-1) \leq 0 \]

or

\[ p^2 + \gamma^2 \leq 1. \]  

(5.11)

Now we proved the following: the scheme (4.1)-(4.3) is stable iff

\[ \begin{align*}
\gamma &< 1, \quad \Delta x < \overline{\Delta x} \\
\gamma^2 + p^2 &\leq 1, \quad \Delta x > \overline{\Delta x}
\end{align*} \]  

(5.12)

On Fig. 2 we see the stability region of interaction problem.
The stability curve has an asymptotic $\Delta t = 2$ as $\Delta t \to 0$. This is equivalent to stability of shell without interaction ($q = 0$). From Fig. 2 we see that the stability is influenced both by the fluid and by the shell. This fact was noticed by some authors in their computational experiments [2], [5].
6. ACCURACY ANALYSIS

In this section we discuss how the mesh size $\Delta x$ effects the numerical solution. We prove that improper $\Delta x$ may cause the numerical solution to change its qualitative behaviour.

Let us consider the functions $p_{1,2}(\Delta x)$ from previous section.

The Figure 3 represents the functions $\text{Re} \ p_{1,2}(\Delta x)$ for different values of $q$.

![Figure 3](image-url)
From Figure 3 (a,b,c) we can conclude that for $\Delta x > \Delta x^*$ the functions $p_{1,2}$ lose their oscillatory properties. By eq. (5.9) the numerical solution itself loses its oscillatory behaviour for $\Delta x > \Delta x^*$. It is against the fact that the exact solution of the homogeneous equations with $q < 2$ is oscillatory-damped. In most severe case (c), where $2 > q > 1$, we have for $\Delta x^* < \Delta x < \Delta x$, $p_{1,2} > 1$. That means that the numerical solution is strictly damped.

For $q > 2$ the exact solution of the homogeneous equations is strictly damped, but from Fig. 3(d) we see that for $\Delta x > \Delta x$, $p_2 < 1$, that means $s < 1$, and $z < 0$. In this case there are numerical oscillations without any physical reason for this.

In all cases there is a critical value of $\Delta x$, which cannot be exceeded. For $q < 2$ it is $\Delta x^*$, for $q > 2$ it is $\Delta x$.

The values of $\Delta x$ and $\Delta x^*$ are

$$\Delta x = \frac{1}{q}$$

$$\Delta x^* = q(2-q), \text{ for } q < 2.$$  \hspace{1cm} (6.1)

These values are smaller than $\Delta x$, which separates the region of fluid stability from the region of interaction stability. And this means that for accurate calculations we have to take $\Delta t$ from the region of fluid stability. In other words there are very strong demands to the size mesh. In cases when the fluid may be computed with the density of mesh smaller than the density of mesh demanded by the interaction problem, our analysis shows that there is a waste of computational effort. This problem may be overcome by separating the meshes of fluid and shell, or by computing the fluid and shell by different time steps, or by two advises together. The version of "Discom" with option for different time step computing is now completed, and the results are now in preparation for
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