METHODS FOR THE ACCURATE EVALUATION OF
VERY OSCILLATORY INFINITE INTEGRALS

by

Avram Sidi

Technical Report #152

May 1979
ABSTRACT

Recently the present author has given some modifications of a non-linear method due to Levin and Sidi for computing certain infinite integrals whose integrands have a simple oscillatory behavior. In this paper these methods are extended to very oscillatory infinite integrals whose integrands have complicated oscillatory behavior at infinity. An elegant and efficient algorithm that enables one to implement the new methods is also developed. The methods are successfully applied to a number of complicated integrals among them the solution to a problem in viscoelasticity. Some powerful convergence results for these methods are also presented.
1. INTRODUCTION

Recently Levin and Sidi (1975) have given some non-linear transformations for accelerating the convergence of slowly converging infinite integrals and series, which have proved to be very efficient from the numerical point of view. Some convergence properties of these transformations have been analyzed in a series of papers by Sidi (1978, 1979a, 1979b) and some results that throw light on the way these transformations work have been proved.

Two useful modifications of the transformation of Levin and Sidi (1975) that simplify the computation of oscillatory infinite integrals, with special emphasis on Fourier and Hankel transforms, have been given by the present author, see Sidi (1979c). Also for this case some powerful convergence results have been proved.

The methods developed in Sidi (1979c) make extensive use of the simple oscillatory behavior of the integrand, which is merely a sine or a cosine (or a combination of both) of the integration variable. The purpose of this work is to give simple and efficient methods for accelerating the convergence of very oscillatory infinite integrals whose oscillations do not necessarily have a regular pattern and the distances between consecutive zeros of their integrands tend to zero at infinity.

In the next section we shall present an elegant and simple algorithm for solving a certain set of linear equations which comes up in extrapolations, in general, see Sidi (1979c), and for the case of very oscillatory integrals in particular. In Section 3 we shall give a method of computation for these integrals, In Section 4 we shall give some numerical examples. In Section 5 we shall supply some results on the convergence of the method developed in Section 3.
2. THE \( W \)-ALGORITHM

Following Levin and Sidi (1975), Sidi (1979b, c) we give the following definition:

**Definition 2.1:** We shall say that a function \( a(x) \), defined for \( x > a > 0 \), belongs to the set \( A(\gamma) \), if it is infinitely differentiable for all \( x > a \) and if, as \( x \to \infty \), it has a Poincaré-type asymptotic expansion of the form

\[
(2.1) \quad a(x) \sim x^\gamma \sum_{i=0}^{\infty} a_i / x^i,
\]

and all its derivatives, as \( x \to \infty \), have Poincaré-type asymptotic expansions which are obtained by differentiating the right hand side of (2.1) term by term.

From this definition it follows that \( A(\gamma) \supset A(\gamma-1) \supset A(\gamma-2) \supset \ldots \).

**Remark:** If \( a \in A(0) \), then \( a(x) \) is infinitely differentiable for all \( x > a \), including \( x = \infty \) (but not necessarily analytic at \( x = \infty \)).

Let \( x_\lambda \), \( \lambda = 0,1,\ldots \), be such that \( a < x_0 < x_1 < \ldots \) for some \( a \geq 0 \) and \( \lim_{\lambda \to \infty} x_\lambda = \infty \). Assume now that the functions \( F(x) \), \( \psi(x) \), and \( \beta(x) \) are defined for \( x \geq x_0 \) and are such that \( \beta \in A(0) \) and

\[
(2.2) \quad L = F(x_\lambda) + \psi(x_\lambda) \beta(x_\lambda), \quad \lambda = 0,1,\ldots,
\]

for some fixed number \( L \), where \( L = \lim_{x \to \infty} F(x) \) whenever this limit exists. If \( \lim_{x \to \infty} F(x) \) does not exist, then \( L \) is said to be the anti-limit of \( F(x) \), and for this case \( \lim_{x \to \infty} \psi(x) \) does not exist.

The problem now is to find \( L \) whether it is the limit or anti-limit of \( F(x) \).
Definition 2.2: The approximation $\tilde{W}_n^{(j)}$ to $L$ and the parameters $\tilde{\beta}_i$, $i = 0, 1, \ldots, n$, are defined as the solution of the set of linear equations

\begin{equation}
\tilde{W}_n^{(j)} = F(x_j) + \psi(x_j) \sum_{i=0}^{n} \frac{\tilde{\beta}_i}{x_j^i}, \quad \tilde{\beta}_i = j, j+1, \ldots, j+n+1,
\end{equation}

provided the matrix of these equations is non-singular.

The solution of the equations in (2.3) can be achieved very simply as the following theorem shows:

Theorem 2.1: (The $W$-algorithm). Let $D_n^{(j)}$ denote the divided difference operator over the set of points $x_j, x_{j+1}, \ldots, x_{j+n+1}$, where for any function $g(x)$

\begin{equation}
D_n^{(j)}[g(x)] = g[x_j, x_{j+1}, \ldots, x_{j+n+1}],
\end{equation}

see Hildebrand (1956, Chapter 2). Then $\tilde{W}_n^{(j)}$ and the $\tilde{\beta}_i$ can be computed recursively from

\begin{equation}
D_n^{(j)} \left\{ \frac{\tilde{W}_n^{(j)} - F(x)}{\psi(x)} \right\} = D_n^{(j)} \sum_{i=0}^{n} \frac{\tilde{\beta}_i}{x_j^i} (n+p+1),
\end{equation}

in this order.

Proof. Let

\begin{equation}
A_{n,l}^{(j)} = \prod_{k=j}^{j+n+1} \frac{x_k - x_l}{x_k - x_j}, \quad l = j, j+1, \ldots, j+n+1.
\end{equation}
Then, see Hildebrand (1956, Chapter 2)

\[ (2.7) \quad D_n^{(j)}(g(x)) = \sum_{k=j}^{j+n+1} A_{n,k}^{(j)} g(k) \]

i.e., \( D_n^{(j)}(g(x)) \) is a linear combination of \( g(x) \), \( k = j, j+1, \ldots, j+n+1 \).

Multiplying then the equations in (2.3) by \( x^{n+p+1} A_{n,k}^{(j)} \), \( k = j, j+1, \ldots, j+n+1 \), respectively and adding all, we obtain (2.5).

Now if \( p = -1 \), then \( \sum_{i=0}^{n} \beta_i x^{n-i} \) is a polynomial of degree at most \( n \).

But

\[ (2.8) \quad D_n^{(j)}(g(x)) = 0, \quad g(x) \text{ a polynomial of degree } \leq n, \]

see Hildebrand (1956, Chapter 2). Therefore, the right-hand side of (2.5) vanishes when \( p = -1 \). Using this together with the fact that \( D_n^{(j)} \) is a linear operator, we obtain from (2.5),

\[ (2.9) \quad w_n^{(j)} = \frac{D_n^{(j)}(x^n p(x)/\psi(x))}{D_n^{(j)}(x^n/\psi(x))} \]

Next let us put \( p = 0 \). Then

\[ \sum_{i=0}^{n} \beta_i x^{n+1-i} = \beta_0 x^{n+1} + \sum_{i=1}^{n} \beta_i x^{n+1-i}, \]

the summation on the right-hand side of this equality being a polynomial of degree at most \( n \). Therefore, by (2.8) again we have

\[ D_n^{(j)} \left( \sum_{i=0}^{n} \beta_i x^{n+1-i} \right) = \beta_0 D_n^{(j)}(x^{n+1}); \quad \text{Hence from (2.5) we obtain} \]

\[ (2.10) \quad \beta_0 = \frac{D_n^{(j)}(x^{n+1}[w_n^{(j)} - p(x)]/\psi(x))}{D_n^{(j)}(x^{n+1})} \]

with \( w_n^{(j)} \) already computed in (2.9). Continuing this way we compute
\[ \bar{\beta}_1, \ldots, \bar{\beta}_{m-1}, \bar{\beta}_m \text{ then can be computed from the formula} \]

\[ \bar{\beta}_m = \frac{D^{(j)}_{m}[x^{n+m+1}[W^{(j)}F(x)]/\psi(x)] - \sum_{i=0}^{m-1} \bar{\beta}_i x^{n+m+1-i}}{D^{(j)}_{m}[x^{n+1}]} \]  

This completes the proof of the theorem.

The formulas given in (2.9), (2.10) and (2.11) can be very easily implemented since the computation of the divided differences \( D^{(j)}_{n}(g(x)) \) can be done in a recursive manner, as follows:

\[ D^{(k)}_{m}(g(x)) = \frac{D^{(k+1)}_{m}(g(x)) - D^{(k)}_{m}(g(x))}{x^{k+m+1} - x^{k}}, \quad m = 0, 1, \ldots \]

with

\[ D^{(k)}_{-1}(g(x)) = g(x_k), \quad k = 0, 1, \ldots \]

Previously the author has considered two kinds of limiting processes, see Sidi (1978, 1979a, 1979b, 1979c):

a) Process I; \( n \) is fixed, \( j \to \infty \).

b) Process II; \( j \) is fixed, \( n \to \infty \).

For Process I, the computation of the \( W^{(k)}_n \), \( k = j, j+1, \ldots \), is done as follows: \( D^{(k)}_{m}(g(x)) \), for \( k = j, j+1, \ldots \), and \( m = 0, 1, \ldots n \), are computed for the two functions \( x^n F(x)/\psi(x) \) and \( x^n/\psi(x) \). Therefore, once \( W^{(j)}_n \) has been computed, the computation of \( W^{(j+1)}_n, W^{(j+2)}_n, \ldots \), requires only a small amount of additional arithmetic.

For Process II, however, since the functions \( x^n F(x)/\psi(x) \) and \( x^n/\psi(x) \) depend on \( n \), for each \( n \) the divided difference tables for these two functions have to be computed from the beginning. But the amount of computation to be done in order to compute \( W^{(j)}_n \) is much
smaller compared to that for the direct solution of equations (2.3),
for example, by Gaussian elimination, which can be seen by counting the
number of multiplications, divisions, and additions.

Actually the W-algorithm achieves the computation of $m_n^{(1)}$ in $p^2 + o(n)$ divisions and $n^2 + o(n)$ additions. In Gaussian elimination, on the other hand the number of multiplications is $n^2/3 + o(n^2)$, and so is the number of additions; and the number of divisions is $n^2/2 + o(n)$. 
3. ASYMPTOTIC EXPANSIONS AND THE W-ALGORITHM FOR VERY OSCILLATORY INTEGRALS

We shall now consider the asymptotic behavior of the integral
\[ \int_{\infty}^{\infty} f(t) dt \]
whose integrand \( f(x) \) can be expressed in the form
\[ f(x) = u(\theta(x)) e^{\phi(x)} h(x), \]
where \( u(t) \) denotes either \( \sin t \) or \( \cos t \), \( \theta \in A^{(m)} \) for some positive integer \( m \), \( \phi \in A^{(k)} \) for some non-negative integer \( k \), and \( h \in A^{(\gamma)} \) for some \( \gamma \), such that \( f(x) \) is integrable at infinity. We assume that \( \theta(x) \) and \( \phi(x) \) are real.

Consider now the function
\[ g(x) = \exp \{ i\theta(x) + \phi(x) + \log h(x) \}, \]
whose real or imaginary part is just \( f(x) \) in (3.1). Now \( g(x) \) satisfies the homogeneous first order differential equation \( g(x) = p(x)g'(x) \), where
\[ p(x) = [i\theta'(x) + \phi'(x) + h'(x)/h(x)]^{-1}. \]
Since \( \theta' \in A^{(m-1)} \), \( \phi' \in A^{(k-1)} \), \( h'/h \in A^{(-1)} \) and \( m \geq 1 \), \( k \geq 0 \), we see that \( (i\theta' + \phi' + h'/h)^{-1} \in A^{(j)} \), where
\[ j = \min \{-m+1, -k+1\}. \]
Since \( j \leq 0 \) the conditions of a theorem of Levin and Sidi (1975), (see also Sidi (1979b, 1979c)), are satisfied, hence we have
\[ \int_{\infty}^{\infty} g(t) dt = xg(x) \beta(x), \]
where \( \beta \in A(0) \). Now \( \theta(x) \) is of the form \( \theta(x) = \overline{\theta}(x) + \Delta(x) \), where \( \overline{\theta}(x) \) is a polynomial in \( x \) of degree \( m \) and \( \Delta \in A^{(0)} \), hence \( \Delta(x) = o(1) \) as \( x \to \infty \). Similarly \( \phi(x) \) is of the form \( \phi(x) = \overline{\phi}(x) + \Lambda(x) \), where \( \overline{\phi}(x) \) is a polynomial in \( x \) of degree \( k \) and \( \Lambda \in A^{(0)} \), hence \( \Lambda(x) = o(1) \) as \( x \to \infty \). In particular, if

\[
\theta(x) \sim x^m \sum_{i=0}^{\infty} \frac{\theta_i}{x^i}, \quad \text{as} \quad x \to \infty,
\]

then

\[
(3.6) \quad \overline{\theta}(x) = \sum_{i=0}^{m-1} \theta_i x^{m-1},
\]

and similarly for \( \phi(x) \) and \( \overline{\phi}(x) \).

Therefore,

\[
(3.8) \quad e^{i\theta(x)} = e^{i\overline{\theta}(x)} e^{i\delta(x)},
\]

where \( \delta(x) = e^{i\Delta(x)} \) and can be seen easily to belong to \( A^{(0)} \).

Similarly,

\[
(3.9) \quad e^{i\phi(x)} = e^{i\overline{\phi}(x)} e^{i\lambda(x)},
\]

where \( \lambda(x) = e^{i\Lambda(x)} \) and \( \lambda \in A^{(0)} \), too. Hence the function \( g(x) \),

\[
(3.10) \quad g(x) = e^{i\overline{\theta}(x)} e^{i\overline{\phi}(x)} \overline{h}(x),
\]

where \( \overline{h} \in A^{(\gamma)} \). Substituting \( (3.10) \) in \( (3.5) \) it follows that

\[
(3.11) \quad \int_{x}^{\infty} g(t) dt = x^{i+\gamma} e^{i\overline{\theta}(x)} e^{i\overline{\phi}(x)} \beta^*(x),
\]

where \( \beta^* \in A^{(0)} \).
By taking the real or imaginary part of (3.11), we obtain

\begin{equation}
(3.12) \quad \int_a^\infty f(t)dt = x_1^{i+\gamma} e^{\Phi(x)} \{u(\theta(x))b_0(x) + u'(\theta(x))b_1(x)\},
\end{equation}

where \( f(x) \) is as in (3.1), \( u'(t) \) is the derivative of \( u(t) \), and \( b_0, b_1 \in A(0) \).

Remark. If \( \theta(x) \) is a complicated function, then so is \( u(\theta(x)) \).

On the other hand, \( u(\theta(x)) \) is always a simple function since \( \theta(x) \) is a polynomial. The advantage of this and hence of (3.12) will become clear below.

Let us consider the integral \( \int_a^\infty f(t)dt \) for \( a > 0 \). Let also \( x_0 \)

be the smallest zero of \( u(\theta(x)) \) greater than \( a \), and let \( x_0, x_1, x_2, \ldots \)

be the consecutive zeros of \( u(\theta(x)) \). Approximately then \( x_\varepsilon = 0(\varepsilon^{1/m}) \) as \( \varepsilon \to 0 \). Therefore from (3.12) it follows that

\begin{equation}
(3.13) \quad \int_a^\infty f(t)dt = \int_a^{x_0} f(t)dt + u'(\theta(x))e^{\Phi(x)} x_\varepsilon^{i+\gamma} b_1(x),
\end{equation}

\( \varepsilon = 0, 1, \ldots \).

This is exactly the situation described in Section 2, equation (2.2), with \( L = \int_a^\infty f(t)dt \), \( F(x) = \int_a^x f(t)dt \), \( \psi(x) = u'(\theta(x))e^{\Phi(x)} x^{i+\gamma} \),

and \( \beta(x) = b_1(x) \). Hence the \( W \)-algorithm can be applied to this case to obtain approximations \( \{W_n(x)\} \) to \( \int_a^\infty f(t)dt \).
From what has been said above we see that one has to do the following in order to obtain the approximations $w_n^{(j)}$ to $\int_0^\infty f(t)dt$:

1) Analyze $\theta(x)$, $\phi(x)$, and $h(x)$ asymptotically and determine $x_0, x_1, \ldots, x_j$, $k$ and $\phi(x)$, and $y$ using the values of $m$, $k$, and $y$ computed in equation (3.4). (As a matter of fact, we do not even have to know $\phi(x)$, but only knowledge of $k$ is enough. We can then replace $\theta(x)$ in (3.13) by $\phi(x)$ and $h^{-1}(x)$ by a different function which is also in $A(0)$.)

2) Compute the $x_\ell$, $\ell = 0, 1, \ldots$, by solving the equation $u'(\theta(x)) = 0$. This is equivalent to solving the polynomial equation $\bar{\theta}(x) = (q + \frac{1}{2})\pi$ or $\bar{\theta}(x) = q\pi$, where $q$ takes on integer values such that a sequence of solutions $x_\ell$, $\ell = 0, 1, \ldots$, with the property $a < x_0 < x_1 < \ldots$, is obtained.

3) Once the $x_\ell$ have been computed, evaluate numerically the integrals $F(x_\ell)$, $\ell \leq j+n+1$, preferably by using a Gaussian integration rule to obtain high accuracy with a small number of abscissas. Compute also $\psi(x_\ell)$, $\ell \leq j+n+1$, letting $\psi(x) = u'(\theta(x)) x^{\frac{1}{2}} e^{\phi(x)}$ or $\psi(x) = u'(\theta(x)) x^{\frac{1}{2}} e^{\phi(x)}$. We note that $u'(\theta(x))$ takes on the value $+1$ or $-1$ alternately for $\ell = 0, 1, 2, \ldots$. Hence we can let $\psi(x_\ell) = (-1)^{\ell} x_\ell^{\frac{1}{2}} e^{\phi(x_\ell)}$ or $\psi(x_\ell) = (-1)^{\ell} x_\ell^{\frac{1}{2}} e^{\phi(x_\ell)}$ in (2.2), without affecting $W_n^{(j)}$ much.

4) Finally apply the W-algorithm to obtain the $w_s^{(j)}$, $s = 0, 1, \ldots, n$.

From our analysis above we can see that in order to be able to apply the W-algorithm, the integrand $f(x)$ does not have to be explicitly given as in (3.1). It only has to be expressible as in (3.1).
Besides, if \( f(x) \) is of the form

\[
 f(x) = \sum_{i=1}^{p} \alpha_i f_i(x),
\]

where each of the \( f_i(x) \) is as in (3.1), i.e.,

\[
 f_i(x) = u_i(t) e^{\theta_i(x)} h_i(x), \quad i = 1, \ldots, p,
\]

such that 1) \( u_i(t) \) is either \( \sin t \) or \( \cos t \), 2) each of the \( \theta_i(x) \) is in \( A^{(m)} \) for a certain positive integer \( m \) and such that \( \theta_i(x) = \theta_j(x) \) for \( i \neq j \), 3) each of the \( \phi_i(x) \) is in \( A^{(k)} \) for a certain non-negative integer \( k \) and such that \( \phi_i(x) = \phi_j(x) \) for \( i \neq j \), 4) each of the \( h_i(x) \) is in \( A^{(q_j)} \) for a certain \( \gamma \) (this can be the case, for instance, if \( h_j \in A^{(j)} \), \( j = 1, \ldots, p \), but \( \gamma_j \gamma_j = \text{integer for } i \neq j \), and \( \gamma = \max\{\gamma_1, \ldots, \gamma_p\} \); then \( f(x) \) satisfies (3.12) with \( j \) as given by (3.4) and \( \theta(x) = \theta_i(x), \quad \phi(x) = \phi_j(x), \quad \) and with \( u(t) \) denoting either \( \sin t \) or \( \cos t \). For our purposes it is not important whether \( u(t) \) is \( \sin t \) or \( \cos t \), we can decide that it is either one of them. Once we have decided what to take for \( u(t) \) we go ahead with the four steps that lead to the computation of \( W_n^{(j)} \).

As an example consider

\[
 f(x) = \begin{cases} 
 J_{\gamma}(\theta(x)) \phi(x) h(x), \\
 Y_{\gamma}(\theta(x)) \end{cases},
\]

where \( \theta(x), \phi(x), \) and \( h(x) \) are as described in the first paragraph of this section and \( J_{\gamma}(t) \) and \( Y_{\gamma}(t) \) are the Bessel functions of
order \( v \) of the first and second kinds respectively. Let \( v(t) \) denote either \( J_v(t) \) or \( Y_v(t) \). Then, as \( t \to \infty \)

\[
(3.17) \quad v(t) = \cos t \eta_1(t) + \sin t \eta_2(t);
\]

where \( \eta_1, \eta_2 \in A^{(-\frac{1}{2})} \). Since \( \theta(x) \to \infty \) as \( x \to \infty \), it is easy to see that

\[
(3.18) \quad v(\theta(x)) = \cos(\theta(x))\eta_1(\theta(x)) + \sin(\theta(x))\eta_2(\theta(x))
\]

where it can be seen easily that \( \bar{\eta}_1, \bar{\eta}_2 \in A^{(-m/2)} \).

Hence we have shown that \( f(x) \) in (3.16) is expressible in the form (3.14) with \( p = 2 \) and \( \theta_1(x) = \theta_2(x) = \theta(x) \) and \( \phi_1(x) = \phi_2(x) = \phi(x) \) and \( h_1, h_2 \in A^{(\gamma-m/2)} \) with \( u_1(t) \) being \( \cos t \) and \( u_2(t) \) being \( \sin t \) or vice versa. For an application of these ideas see Example 4.4.
4. NUMERICAL EXAMPLES

In this section we shall give four numerical examples that show the accuracy of the method presented in the previous section when applied to very oscillatory integrals. All the results have been obtained by using the \( W \)-algorithm of Section 2, for Process II using \( j=0, \) since Process II is the more efficient of the two processes. (See also Theorem 5.1.)

Example 4.1 \( \int_{0}^{\infty} \sin \left( \frac{\pi}{2} t^2 \right) dt = \frac{\pi}{4} \)

For this case \( u(t) = \sin t, \theta(x) = \delta(x) = (\pi/2)x^2, \phi(x) = \) constant, and \( \gamma = 0. \) Hence \( x_\ell, \ell = 0, 1, \ldots, \) are roots of the equation \( (\pi/2)x^2 = (2\ell+1)^2, \ell = 0, 1, \ldots, \) i.e., \( x_\ell = \sqrt{2(2\ell+1)}, \ell = 0, 1, \ldots. \) Since \( m = 2 \) and \( k = 0, \) we have \( j = -1. \) Therefore, \( \psi(x) = \cos(\pi x^2/2)/x \) and \( \psi(x_\ell) = (-1)^{\ell+1}/x_\ell, \ell = 0, 1, \ldots. \) Table 4.1 contains some of the results of the computations for this integral.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( W_n^{(0)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4997</td>
</tr>
<tr>
<td>3</td>
<td>0.500002</td>
</tr>
<tr>
<td>5</td>
<td>0.4999999991</td>
</tr>
<tr>
<td>7</td>
<td>0.500000000004</td>
</tr>
<tr>
<td>9</td>
<td>0.499999999998</td>
</tr>
<tr>
<td>11</td>
<td>0.500000000000009</td>
</tr>
<tr>
<td>Exact</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 4.1 - Approximations \( W_n^{(0)} \) for the integral \( \int_{0}^{\infty} \sin(\pi t^2/2) dt = \frac{\pi}{4}. \)
Example 4.2

\( I(x,t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp \left\{ -\xi (1+\xi^{-2})^{-1} \sin \left( \frac{\tanh^{-1} \xi}{2} \right) \right\} \times \sin \left\{ \xi t - \xi x (1 + \xi^2)^{-1/2} \cos \left( \frac{\tanh^{-1} \xi}{2} \right) \right\} \, d\xi, \quad \forall x, t \geq 0. \)

\( I(x,t) \) is an integral representation for the solution to a problem of wave propagation in a viscoelastic medium \( \text{Longman (1979)} \) for which approximate solutions have been obtained by \( \text{Longman (1972, 1973)} \) by using approximate Laplace transform inversion through rational approximations and accurate approximations have been obtained by \( \text{Levin (1975)} \) by using a method to accelerate the convergence of Bromwich's integral for Laplace transform inversion which has been shown to be a special case of the D-transformations of \( \text{Levin and Sidi (1975)} \).

The integrand of the integral on the right-hand side of (4.1) is not of the form which suits our purposes as a simple analysis shows. However, by making the change of variables \( \xi = z^2 \) we can put the integral in the form \( \int_0^\infty f(z) dz \), where

\( f(z) = e^{\phi(z)} \sin(\theta(z)) h(z) \)

where \( \phi \in A^{(1)} \) with \( \phi(z) = xz/\sqrt{2} \), \( \theta \in A^{(2)} \) with \( \theta(z) = tz^2 - xz/\sqrt{2} \), and \( h(z) = 2/z \) hence \( h \in A^{(-1)} \), i.e., \( k = 1, m = 2, \gamma = -1 \). Therefore, \( j = -1 \). Now we compute the zeros of \( \sin(\theta(z)) \) by solving the quadratic equation \( tz^2 - xz/\sqrt{2} = q\pi, q \) integer. We considered the cases for which \( x = 1, \) and \( t = 0.1, 0.5, 1, 100 \). It is easy to see that \( x_1 \) is the positive solution of the quadratic equation above for \( q = 0, 1, \ldots \).
Taking then \( \psi(z) = e^{\phi(z)} \cos(\phi(z))/z^2 \) and using the \( W \)-algorithm with the \( x_k \) as determined above we obtain the results given in Table 4.2. The results for \( x = 1, t = 0.1, 0.5, 1.0, \) are much more accurate than those given in Levin (1975) although they have been obtained with much less labor than those of Levin.

**Example 4.3**

\[
(4.3) \quad I = \int_0^\infty \sin \left( \frac{\pi^2}{t^2} \right) \cos \left( b^2 t^2 \right) \frac{dt}{t^2} = \frac{\sqrt{\pi}}{4\sqrt{a}} \\
\times \left[ \sin(2ab) + \cos(2ab) + e^{-2ab} \right], \quad a > 0, b > 0.
\]

The integrand in this example has an infinite number of oscillations both as \( t \to \infty \) and as \( t \to 0 \). Therefore we divide the range of integration into two: \((0,T)\) and \((T,\infty)\). We then map the interval \((0,T)\) to \((1,\infty)\) by the change of variable \( t = T/t \), hence obtaining two infinite integrals whose integrands oscillate at infinity an infinite number of times. For this example we take \( a = \sqrt{\pi}, b = \sqrt{\pi}/2 \) and \( T = 1 \). With this choice of \( a, b \) and \( T \) we have

\[
(4.4) \quad I = \int_0^1 f(t)dt + \int_1^{\infty} f_1(t)dt,
\]

where

\[
f(x) = \sin \left( \frac{\pi^2}{x^2} \right) \cos \left( \frac{\pi x^2}{4} \right)x^{-2}
\]

\[
(4.5) \quad f_1(x) = \sin(\pi x^2) \cos \left( \frac{\pi}{4x^2} \right).
\]
<table>
<thead>
<tr>
<th>n</th>
<th>$\frac{1}{2} + \frac{1}{\pi} w_n^{(0)} (x=1, t=0.1)$</th>
<th>$\frac{1}{2} + \frac{1}{\pi} w_n^{(0)} (x=1, t=0.5)$</th>
<th>$\frac{1}{2} + \frac{1}{\pi} w_n^{(0)} (x=1, t=1)$</th>
<th>$\frac{1}{2} + \frac{1}{\pi} w_n^{(0)} (x=1, t=100)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0273612</td>
<td>0.3984</td>
<td>0.65069</td>
<td>0.997</td>
</tr>
<tr>
<td>3</td>
<td>0.0273610735</td>
<td>0.3983862</td>
<td>0.6506313</td>
<td>0.99997</td>
</tr>
<tr>
<td>5</td>
<td>0.0273610736805</td>
<td>0.398386354</td>
<td>0.650631565</td>
<td>0.99999997</td>
</tr>
<tr>
<td>7</td>
<td>0.027361073680255</td>
<td>0.3983863538565</td>
<td>0.650631563258</td>
<td>1.000000006</td>
</tr>
<tr>
<td>9</td>
<td>0.02736107368025598</td>
<td>0.39838635385809</td>
<td>0.6506315632650</td>
<td>1.000000002</td>
</tr>
<tr>
<td>11</td>
<td>0.02736107368025598</td>
<td>0.3983863538580801</td>
<td>0.650631563264991</td>
<td>0.99999999991</td>
</tr>
</tbody>
</table>

*Table 4.2 - Approximations $\frac{1}{2} + (1/\pi) w_n^{(0)}$ for $I(x,t)$ in (4.1).*
The function \( f(x) \) is of the form (3.1) with \( u(t) = \cos t \),
\( \theta(x) = \theta(t) = \pi x^2/4 \), \( \phi(x) \equiv \text{constant} \), and \( h \in A(-4) \), i.e., \( \gamma = -4 \).
Hence \( x \xi = 2\sqrt{l}+1, \ l = 0,1,\ldots \). Since \( m=2 \) and \( k=0 \), we have \( j=-1 \).
Therefore, \( \psi(x) = \cos(\pi x^2/4)/x^5 \) and \( \psi(x \xi) = (-1)^{l+1}/x^5, \ l = 0,1,\ldots \).

The function \( f_1(x) \) is also of the form (3.1) with \( u(t) = \sin t \),
\( \theta(x) = \theta(t) = \pi x^2 \), \( \phi(x) \equiv \text{constant} \), and \( h \in A(0) \), i.e., \( \gamma = 0 \). Hence \( x \xi = \sqrt{\xi}+2, \ l = 0,1,\ldots \). Again \( j=-1 \). Therefore, \( \psi(x) = \cos(\pi x^2)/x \),
and \( \psi(x \xi) = (-1)^{l}/x, \ l = 0,1,\ldots \).

Let us denote the \( \mathcal{W} \)-approximations for the integrals \( \int_1^\infty f(t)dt \) and \( \int_1^\infty f_1(t)dt \) by \( \mathcal{W}^{(j)}_n[f] \) and \( \mathcal{W}^{(j)}_n[f_1] \) respectively. In Table 4.3 we give the approximations to \( I \) obtained as \( \mathcal{W}^{(0)}_n[f] + \mathcal{W}^{(0)}_n[f_1] \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \mathcal{W}^{(0)}_n[f] + \mathcal{W}^{(0)}_n[f_1] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.16899</td>
</tr>
<tr>
<td>3</td>
<td>-0.1691378</td>
</tr>
<tr>
<td>5</td>
<td>-0.1691374808</td>
</tr>
<tr>
<td>7</td>
<td>-0.1691374816345</td>
</tr>
<tr>
<td>9</td>
<td>-0.16913748163516</td>
</tr>
<tr>
<td>11</td>
<td>-0.1691374816351481</td>
</tr>
<tr>
<td>Exact</td>
<td>-0.1691374816351482</td>
</tr>
</tbody>
</table>

Table 4.3 - Approximations \( \mathcal{W}^{(0)}_n[f] + \mathcal{W}^{(0)}_n[f_1] \) to the integral \( I = \int_0^\infty \sin(\pi/t^2)\cos(\pi t^2/4)dt/t^2 = 1/(4\sqrt{2})(e^{-1}-1) \).

\( f(x) \) and \( f_1(x) \) have been defined in (4.5).
Example 4.4

\[ I = \int_0^\infty \frac{(t^4 + 2t^2 + 5)}{t^2 + 4} \sqrt{t^2 + 9t + 20} \, dt. \]

The exact value of this integral is not known to the author. This integral can be dealt with by making use of the remarks at the end of Section 3.

First of all we have \( \theta(x) = \frac{x^4 + 2x^2 + 5}{x^2 + 4} \), \( \psi(x) \equiv \) constant, and \( h(x) = \sqrt{x^2 + 9 + 20} \) in (3.16). Therefore, \( \theta(x) = x^2 \), i.e., \( m = 2 \), \( k = 0 \), and \( \gamma = 1 \). Consequently, the integrand is of the form

\[ f(x) = \cos(x^2)h_1(x) + \sin(x^2)h_2(x) \]

where \( h_1, h_2 \in A(0) \). Letting \( x = \sqrt{\ell + 1} \), \( \ell = 0, 1, \ldots \), we have \( \psi(x) = \cos(x^2)/x \), hence \( \psi(x_\ell^k) = (-1)^{\ell+1}/x_\ell^k \), \( \ell = 0, 1, \ldots \). In Table 4.4 we give some of the results obtained by applying the \( W \)-algorithm to the integral \( I \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( W_n^{(0)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.61</td>
</tr>
<tr>
<td>3</td>
<td>2.6273</td>
</tr>
<tr>
<td>5</td>
<td>2.627159</td>
</tr>
<tr>
<td>7</td>
<td>2.627160408</td>
</tr>
<tr>
<td>9</td>
<td>2.62716040106</td>
</tr>
<tr>
<td>11</td>
<td>2.627160401844</td>
</tr>
</tbody>
</table>

Table 4.4 - Approximations \( W_n^{(0)} \) for the integral

\[ \int_0^\infty \frac{(t^4 + 2t^2 + 5)}{t^2 + 4} \sqrt{t^2 + 9t + 20} \, dt. \]
5. CONVERGENCE PROPERTIES

In Sidi (1979c) the following result has been proved:

**Theorem 5.1** Let \( a, x_0, x_1, \ldots, F(x), \psi(x), \beta(x) \) and \( L \) be as in Section 2. Suppose that \( \lim_{x \to \infty} F(x) = L \) so that \( \psi(x) = O(1) \) as \( x \to \infty \), hence \( \psi(x) = O(1) \) for \( x \geq x_0 \). If also

\[
(5.1) \quad \psi(x_j) \psi(x_{j+1}) < 0, \quad j = 0, 1, \ldots ,
\]

then, for Process I,

\[
(5.2) \quad L - W_n^{(j)} = o(x_j^{-n-1}) \quad \text{as} \quad j \to \infty, \quad n \text{ fixed},
\]

and

\[
(5.3) \quad \left| L - W_n^{(j)} \right| \leq \max_{j \leq k \leq j+n+1} |\psi(x_k)| \max_{j \leq k \leq j+n+1} |\beta(x_k) - \pi_n(x_k/x_j)|,
\]

where \( \pi_n(\xi) = \sum_{i=0}^{n} a_n i^i \) is the best polynomial approximation of degree \( n \) to \( \beta(x_j/\xi) \) on \( 0 < \xi < 1 \). Using in (5.3) the fact that \( \psi(x) = O(1) \) for \( x \geq x_0 \) and that \( \beta(x_j/\xi) \) is infinitely differentiable for \( 0 < \xi < 1 \), we have for Process II,

\[
(5.4) \quad L - W_n^{(j)} = O(n^{-\lambda}) \quad \text{as} \quad n \to \infty, \quad j \text{ fixed}, \quad \lambda > 0.
\]

For the case of the very oscillatory integrals that we treated in Section 3, \( F(x) = \int_a^x f(t) dt, \quad L = \int_a^\infty f(t) dt, \quad \psi(x) = u'(\theta(x)) e^{\phi(x) x^{1+\gamma}}, \) \( \beta(x) = b_{\xi}(x) \). Since the \( x_j \) are chosen as the consecutive zeros of \( u(\theta(x)) \), we have \( u'(\theta(x_j)) = (-1)^j \xi^j \) where \( \xi \) is a constant independent of \( \lambda \). Hence \( \psi(x) \) satisfies (5.1). The rest of the conditions of Theorem 5.1 have already been shown to hold in Section 3. Consequently,
(5.2), (5.3), and (5.4) hold, i.e., 
\[ \int_0^\infty f(t)dt - w_n^{(j)} \to 0 \quad \text{as} \quad j \to \infty, \]
\( n \) fixed, faster than \( x_j^{-n-1} \), \( 2 \) as \( n \to \infty, j \) fixed, faster than any negative power of \( n \).

As explained in Theorem 5.1, the result in (5.4) follows from (5.3) by making use of the facts that \( |\psi(x)| = O(1) \) for \( x \geq x_0 \) and that \( \max |\beta(x_j/\xi) - \pi_n(\xi)| \to 0 \) more quickly than any negative power of \( n \) as \( n \to \infty \), since \( \beta(x_j/\xi) \) is infinitely differentiable for \( 0 < \xi < 1 \).

(This is a standard result in approximation theory.) This general result can be sharpened in some cases as we show below.

In order to see what kind of convergence to expect in reality we go back to Example 4.1. Consider the function

(5.5) \( S(x) = \int_0^\infty e^{i\pi t^2/2} \, dt \).

By the change of variable \( \sqrt{\pi/2} t = \tau \), we can write

(5.6) \( S(x) = \sqrt{2/\pi} \int_0^\infty e^{i\tau^2} \, d\tau \).

Now the function \( \overline{S}(z) = \int_z^\infty e^{i\tau^2} \, d\tau \) is of the form

(5.7) \( \overline{S}(z) = \frac{e^{iz^2}}{z} b(z), \)

where \( b \in A(0) \), but \( \overline{b}(z) \) is not analytic at \( z = \infty \). Making the transformation \( \xi = \alpha/z \), for some \( \alpha > 0 \), we map the interval \( \alpha < z < \infty \) to \( 0 < \xi < 1 \). Then \( b(\alpha/\xi) \) can be expanded in a Chebyshev series on...
For $0 \leq \xi \leq 1$ in the form

$$b(\alpha/\xi) = \sum_{k=0}^{\infty} c_k T^*_k(\xi),$$

where $T^*_k(\xi)$ is the shifted Chebyshev polynomial of degree $k$. It is shown in Miller (1966) that

$$c_k = \mathcal{O}(\exp(-3\alpha^{1/3} k^{2/3})) \quad \text{as} \quad k \to \infty.$$ 

Hence

$$b(\alpha/\xi) = \sum_{k=0}^{n} c_k T^*_k(\xi) = \mathcal{O}(\exp(-3\alpha^{1/3} n^{2/3})) \quad \text{as} \quad n \to \infty,$$

for $0 \leq \xi \leq 1$. If the $n$-th partial sum of the Chebyshev series in (5.10) is replaced by the best polynomial approximation of degree $n$ to $b(\alpha/\xi)$ on $[0,1]$, then the right hand side stays the same, see Powell (1967). Hence in Example 4.1

$$\beta(x_j/\xi) - \pi_n(\xi) = \mathcal{O}\left(\exp\left[-3(\sqrt{\frac{\pi}{2}}, x_j)^{1/3} n^{2/3}\right]\right) \quad \text{as} \quad n \to \infty$$

for $0 \leq \xi \leq 1$. Substituting the result in (5.3), we have

$$\int_{0}^{\infty} \sin\left(\frac{\pi}{2}, t^2\right) dt - W_n^{(1)} = \mathcal{O}\left(\exp\left[-3(\sqrt{\frac{\pi}{2}}, x_j)^{1/3} n^{2/3}\right]\right) \quad \text{as} \quad n \to \infty$$

The result in (5.12) is a great improvement upon the general result of Theorem 5.1 in that it gives a much better bound on the rate of convergence of the $W_n^{(1)}$ approximations.
We note also that if \( \beta(x) \) is analytic for all \( x > x_0 \) up to and including \( x = \infty \), then 
\[
\beta(x_j/\xi) - \pi_n(\xi) = O(e^{-\omega n}) \quad \text{for some} \quad \omega > 0.
\]
Hence for these cases the estimate for the rate of convergence is even better, specifically 
\[
\int_{a}^{\infty} f(t) dt - \mu_n^{(j)} = O(e^{-\omega n}) \quad \text{as} \quad n \to \infty.
\]
REFERENCES


