NECESSARY AND SUFFICIENT CONDITION
FOR GLOBAL WEAK SOLVABILITY OF
NONLINEAR OPERATOR EQUATIONS

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A B S T R A C T

Within a wide framework, a basic necessary and sufficient condition for the existence of weak solutions for nonlinear operator, in particular partial differential equations is presented. The result is applied in order to prove the existence of weak solutions for several types of nonlinear PDEs.

Keywords and phases. Weak solutions for nonlinear PDEs, necessary and sufficient existence condition.
The aim of the present paper is to establish in a wide framework, a basic necessary and sufficient condition for the existence of global weak solutions for nonlinear operator equations and to show the way it can be applied to several types of linear and nonlinear PDEs. In [11-13], weak or distribution solutions of polynomial nonlinear PDEs were studied, based on embeddings of the distributions in $D'(\Omega)$, with $\Omega \subset \mathbb{R}^n$ nonvoid, open, into associative and commutative algebras of classes of sequences of continuous functions on $\Omega$. That study pointed out the special advantage of solving linear or nonlinear PDEs in algebras of classes of sequences of functions, which prove to provide a particularly flexible framework for the resolution of singularities of weak solutions, as well as for the explicit study of their stability and exactness properties. The results obtained in [11-13] lead naturally to the question: What is the ultimate framework the mentioned method can be used, when dealing with weak solutions of nonlinear PDEs?

The present paper gives an answer to that question by defining a fairly general notion of weak solution for a class of nonlinear operator equations, which contains most of the nonlinear PDEs encountered in applications. The resulting basic necessary and sufficient condition for the existence of global weak solutions — see Theorems 1 and 2 in §§4,5 — is then applied to nonlinear conservation laws and second order wave equations. The power of the method is illustrated by proving the existence of global weak solutions for the well known PDE of H. Lewy, [8], which — in distributional sense — does not possess even local weak solutions.

As a general remark, the method in the present paper constructs a framework which will grant the existence of global weak solutions for wide classes of nonlinear PDEs. After the existence of weak solutions is proved in specific instances, one can put forward the problems of stability, uniqueness, regularity, resolution of singularities, etc., as has partly been done for instance, in [11-13].
§2. ALGEBRAS OF CLASSES OF SEQUENCES OF FUNCTIONS

Given a nonvoid, open subset \( \Omega \subset \mathbb{R}^n \), and an arbitrary nonvoid set \( F \), we shall be interested in operators of the type

\[
T : F \to C^0(\Omega)
\]

which include wide classes of nonlinear PDEs. For instance, suppose given,

\[
F \in C^m(\Omega) \times \mathbb{R}^m
\]

where \( m \in \mathbb{N} \) is specified and \( m = \text{car} \{ p \in \mathbb{N}^n \mid |p| \leq m \} \). Assuming \( F = C^m(\Omega) \), one obtains an operator of type (1) by defining

\[
(T\psi)(x) = F(x, \ldots, D^p\psi(x), \ldots), \quad \forall \psi \in F, \quad x \in \Omega,
\]

where \( D^p \), with \( p \in \mathbb{N}^n \), \( |p| \leq m \), are the usual partial derivatives.

Our main problem will be to solve the equation

\[
T\psi = f, \quad \psi \in F
\]

where the operator \( T \) of type (1) and \( f \in C^0(\Omega) \) are given.

We shall be interested in the case when \( f \not\in TF \), therefore, the equation (4) does not possess 'classical solutions' \( \psi \in F \). In this respect, our aim is to define a general notion of 'weak solution' and establish a basic necessary and sufficient existence condition.

The usual way is to try with a 'weak solution' given by a sequence

\[
s = (\psi_0, \psi_1, \ldots, \psi_N, \ldots) \in F^N
\]

which substituted term by term in (4) gives the "error" sequence

\[
w_s = T\psi - u(f) \in (C^0(\Omega))^N
\]

where we denoted \( T\psi = (T\psi_0, T\psi_1, \ldots, T\psi_N, \ldots) \), \( u(f) = (f, f, \ldots, f, \ldots) \) and took into account that \((C^0(\Omega))^N\) is an associative and commutative algebra, if considered with the term-by-term operations on sequences of functions, the unit element being \( u(1) \), while \( C^0 = \{u(0)\} \) is the (null ideal).

Now, the problem is to properly define the notion

\[
w_s \text{ is "negligible"}
\]
based on appropriate algebraic and topological considerations. It turns out that the settlement of the algebraic problems involved - a task which is not elementary; employing for instance the Baire category argument - solves a good deal of the problems concerning the definition and existence of "weak solutions". That fact should not come as a surprise, since it seems that the study and understanding of the algebraic aspects involved has not been given sufficient attention even in the case of linear problems, basic facts and properties being only pointed out recently [14, 15].

From algebraic point of view, a natural way to define (7) is by requesting that \( w_s \) belongs to the zero class of a certain quotient vector space

\[
E = \mathcal{S}/V, \quad \text{with } V \subseteq \mathcal{S} \quad \text{vector subspace in } (\mathcal{C}^{\infty} (\Omega))^N,
\]

that is

\[
w_s \in V
\]

Obviously, in order that the "negligibility" condition (9) does not become trivial, \( V \) should not be too large. It turns out that it will be sufficient if \( V \) can distinguish between different continuous functions, that is

\[
V \psi, \chi \in \mathcal{C}^{\infty} (\Omega):
\]

\[
u(\psi) - u(\chi) \in V \Rightarrow \psi = \chi \text{ on } \Omega
\]

a condition equivalent with (see [17])

\[
V \cap U = \emptyset
\]

where we denoted

\[
U = \{u(\psi) \mid \psi \in \mathcal{C}^{\infty} (\Omega)\}
\]

However, in case the quotient space (8) has a vector space structure only, the conditions (9) and (10) will not be able to grant a meaningful notion of "weak solution". Indeed, in that case, any sequence \( s \in \mathcal{P}^N \) which satisfies the condition: \( w_s \notin U \), will give a "weak solution" of (4), as one can take in (8), \( V = \mathcal{L}^1_\mathcal{S} \) and \( S = (\mathcal{C}^{\infty} (\Omega))^N \), in which case (9) and (10) will obviously hold.

And here, we arrive to the main algebraic idea of the method presented in this paper, namely, to replace the quotient vector spaces (8) by quotient algebras (see also [11-13]).
where $A$ is a subalgebra in $(C^0(\Omega))^N$ and $I$ is an ideal in $A$, which satisfy

\[(11.1) \quad I \cap U = 0 \quad (\text{see } (10))\]

\[(11.2) \quad U \subset A\]

conditions equivalent to the existence of the algebra embedding

\[(12) \quad C^0(\Omega) \ni \psi \mapsto u(\psi) + I \in A\]

defined by the injective algebra homomorphism

\[(12.1) \quad C^0(\Omega) \ni \psi \mapsto u(\psi) + I \in A\]

The essential difference between the former "negligibility" condition (9) on $w_s$ and the corresponding new one

\[(13) \quad w_s \in I\]

is that $I$ being an ideal in $A$ not only $w_s$ will be "negligible" according to (13), but also each of its "projections" $w_s \cdot t$, with $t \in A$, that is

\[(14) \quad w_s \cdot t \in I, \quad \forall t \in A\]

The joint condition (13) and (14) proves to give a convenient, nontrivial control over the "weak solutions".

§3. DEFINING THE WEAK SOLUTIONS

Denote by $AL(\Omega)$ the set of all quotient algebras (11) which satisfy (11.1) and (11.2).

A sequence $\mathbf{s} \in \mathbb{P}^N$ is called a weak solution on $\Omega$ of the equation (4), only if for any subsequence $s' \in s$, there exists a quotient algebra $A' = A' / I' \in AL(\Omega)$, such that $w_s' \in I'$.

A useful, simple characterization of weak solutions is presented now. Given a sequence of functions $w \in (C^0(\Omega))^N$, define the quotient algebra $A_w = A_w / I_w$, where $A_w$ is the subalgebra in $(C^0(\Omega))^N$ generated by $U \cup \{w\}$, while $I_w$ is the ideal in $A_w$ generated by $\{w\}$. 

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Proposition 1

A sequence \( s \in \mathbb{E}^N \) is a weak solution on \( \Omega \), of the equation (4), only if \( A_w = A_w / I_w \in AL(\Omega) \), for any subsequence \( s' \) in \( s \).

Proof

It follows from Lemma 1, below.

Lemma 1

If \( w \in (C^0(\Omega))^N \) then \( A_w = A_w / I_w \in AL(\Omega) \), only if \( w \notin \mathcal{I} \), for a certain \( A_w = A / I_w \in AL(\Omega) \).

Proof

Assume \( A = A / I \in AL(\Omega) \) and \( w \notin \mathcal{I} \). Then \( U \cup \{w\} = A \cup I \in A \), in view of (11.2). Thus \( A_w \subseteq A \). Now, \( w \notin \mathcal{I} = w \in A = A_w \subseteq I \). Hence, \( I_w \cap U = I \cap U \neq 0 \), in view of (11.1), therefore \( A_w = A / I_w \in AL(\Omega) \). The converse is obvious, since \( w \notin \mathcal{I} \).

§4. NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF WEAK SOLUTIONS

Denote by \( R \) the set of all sequences of functions \( w \in (C^0(\Omega))^N \) which satisfy the condition:

\[
\forall \Omega^I \in \Omega \text{ non-void, open, } w \text{ subsequence in } w, \psi' \in C^0(\Omega') : \begin{align*}
&\{w^I = m(\psi'), \quad \Omega^I = \psi' = 0\} \quad \text{on } \Omega^I.
\end{align*}
\]

Obviously, \( w \in R \), only if \( w^I \in R \), for any subsequence \( w' \) in \( w \).

The basic result is the following characterization of weak solutions.
Theorem 1
A sequence $s \in F^N$ is a weak solution on $\Omega$, of the equation (4), only if
\begin{equation}
\tag{16}
w_s \in R
\end{equation}

Proof
it follows from Proposition 1 in §3 and Proposition 2 below.

Proposition 2
Given a sequence of functions $w \in (C^0(\Omega))^N$, then $A_{w^1} \in AL(\Omega)$, for each subsequence $w^1$ in $(w)$, only if $w \in R$.

Proof
Assume that $w \in R.$ Then, it suffices to show that
\begin{equation}
\tag{17}
I_w \cap U = 0
\end{equation}
In this respect, it is easy to see that an element of the intersection in (17) has the form
\begin{equation}
\tag{18}
u(\psi) = w \cdot u(\psi_0) + \ldots + w^{2+1} \cdot u(\psi_{2+1})
\end{equation}
where $\lambda \in \mathbb{N}$ and $\psi, \psi_0, \ldots, \psi_{2+1} \in C^0(\Omega).$ Therefore, in order to prove (17), it suffices to show that $\psi = 0$ on $\Omega$. 

Assume it is false and $\Omega' \subset \Omega$ is nonvoid, open, such that
\begin{equation}
\tag{19}
\psi(x) \not\equiv 0, \quad \forall x \in \Omega'
\end{equation}
Denoting by $w_v$, with $v \in \mathbb{N}$, the functions which are the terms in the sequence of functions $w$, the relation (18) written term by term, gives.
\begin{equation}
\tag{20}(w_v(x))^{2+1} = \psi_{2+1}(x) + \ldots + w_v(x) = \psi_0(x) + \psi(x) = 0,
\end{equation}

\begin{equation}
\psi \not\equiv 0, \quad \forall v \in \mathbb{N}, \quad x \in \Omega.
\end{equation}
Therefore, (19) will imply that the infinite matrix
\[
\begin{bmatrix}
(w_0(x))^{l+1} & \ldots & w_0(x) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
(w_0(x))^{l+1} & \ldots & w_0(x) & 1 \\
\end{bmatrix}, \quad \text{with } x \in \Omega'
\]

has rank at most \( l + 2 \), for any given \( x \in \Omega' \).

Now, a well-known property of the Vandermonde determinants implies that the infinite sequence of numbers

\[ w_0(x), \ldots, w_0(x), \ldots \]

contains at most \( l + 2 \) different terms, for any given \( x \in \Omega' \). Therefore, Lemma 2 below, will grant the existence of a closed, nowhere dense subset \( \Gamma' \subseteq \Omega' \), such that each \( x \in \Omega' \setminus \Gamma' \) possesses an open neighbourhood \( \Omega'' \), with the property that the infinite sequence of functions

\[ w_0, \ldots, w_0, \ldots \]

when restricted to \( \Omega'' \), contains only a finite number of different functions. In other words, there exists a subsequence \( w' \) in \( w \) and \( \psi' \in C^0(\Omega'') \), such that

\[ w'' = u(\psi') \text{ on } \Omega'' \]

Now, \( w \in R \) will imply that \( \psi'' = 0 \) on \( \Omega'' \). And then, (21) and (20) will contradict (19).

Conversely, assume that \( w \notin R \). Then

\[ \exists \Omega' \subseteq \Omega \text{ nonvoid, open, } w \text{ subsequence in } w, \ \psi' \in C^0(\Omega') : \]

\[ (22.1) \quad w' = u(\psi') \text{ on } \Omega' \]

\[ (22.2) \quad \psi' 
eq 0 \text{ on } \Omega' \]

We notice that in view of (22.1), one can assume \( \psi' \in C^0(\Omega) \), by taking \( \psi' \) one of the terms in \( w' \). Then, for any \( \psi \in C^0(\Omega) \), with \( \text{supp } \psi \subseteq \Omega' \), one obtains

\[ w' \cdot u(\psi) = u(\psi' \cdot \psi) \in \left( \bigcap_{w \in U} \right) \]

Since \( \psi \) was arbitrary and in view of (22.2), it follows that
Lemma 2

Suppose, the sequence of functions \( (w_n(x) \in C^0(\Omega)) \) is such that, for any given \( x \in \Omega \), the sequence of numbers \( w_0(x), w_1(x), \ldots, w_n(x), \ldots \) contains only a finite number of different terms. Then, there exists a closed, nowhere dense subset \( F \subset \Omega \), such that the sequence of functions \( w_0, w_1, \ldots, w_n, \ldots \) restricted to a suitable neighbourhood of any given \( x \in \Omega \), contains only a finite number of different terms.

Proof

Denote by \( \Gamma \) the set of all points \( x \in \Omega \), such that the sequence of functions \( w_0, w_1, \ldots, w_n, \ldots \) contains infinitely many different terms, when restricted to any neighbourhood of \( x \). It is easy to see that \( \Gamma \) is closed. Therefore, it only remains to prove that \( \Gamma \) has no interior. In this respect, it suffices to show that

\[
\Gamma \cap \Omega = \emptyset
\]

Indeed, if \( \Omega' = \text{int} \Gamma \neq \emptyset \), then \( \Gamma' \), corresponding as above to \( \Omega' \), will satisfy \( \Gamma' = \Omega' \), thus contradicting (23).

In order to obtain (23), the Baire category argument will be used in two successive iterations.

For \( \mu \in \mathbb{N} \), define the closed set

\[
\Delta_\mu = \{ x \in \Omega \mid \exists \lambda \in \mathbb{N}, \lambda \leq \mu, \text{such that } w_\lambda(x) = w_\mu(x) \}
\]
then, obviously
\[ \Omega = \bigsqcup_{\mu \in N} \Delta \mu \]
therefore, the Baire category argument implies that
\[ \Omega' = \text{int} \Delta \mu \neq 0 \]
for a certain \( \mu \in N \). We shall prove that
\[ \Omega \cap (\Omega \setminus \Omega') \neq 0 \]
In this respect, denote for \( \rho \in N \)
\[ \Delta^i = \left\{ x \in \Omega' \mid \forall \lambda, \tilde{\nu} \in N, \lambda < \nu \leq \mu : w_\lambda(x) \leq w_\nu(x) \Rightarrow |w_\lambda(x) - w_\nu(x)| \geq 1/(\rho+1) \right\} \]
then
\[ \Omega' = \bigcap_{\rho \in N} \Delta^i \]
Indeed, denote for \( x \in \Omega' \)
\[ M_x = \left\{ (\lambda, \nu) \in N \times N \mid \lambda < \nu \leq \mu, w_\lambda(x) \leq w_\nu(x) \right\} \]
and take \( \rho \in N \), such that
\[ 1/(\rho+1) \leq \min \left\{ |w_\lambda(x) - w_\nu(x)| \mid (\lambda, \nu) \in M_x \right\} \]
then, obviously \( x \in \Delta^i \). Moreover
\[ \Delta^i \text{ is closed for } \rho \in N \]
Indeed, denoting
\[ M = \left\{ (\lambda, \nu) \in N \times N \mid \lambda < \nu \leq \mu \right\} \]
one obtains:
\[ \Delta^i = \bigcap_{K \in M} \left( \bigcap_{\lambda, \nu \in K} \left\{ x \in \Omega' \mid |w_\lambda(x) - w_\nu(x)| \geq 1/(\rho+1) \right\} \right) \cap \left( \bigcap_{\lambda, \nu \in M \setminus K} \left\{ x \in \Omega' \mid w_\lambda(x) = w_\nu(x) \right\} \right) \]
Now, (25) and (26) together with the Baire category argument imply that
\[ \Omega' = \text{int} \Delta^i \neq 0 \]
for a certain \( \rho \in N \). The proof of (24) is completed if we show
that

\[ \Omega'' = \Omega \setminus \Gamma \]

Assume therefore \( x \in \Omega'' \) and \( V \subset \Omega'' \), an open, connected neighborhood of \( x \). We shall prove that the sequence of functions

\[ w_0, w_1, \ldots, w_N, \ldots \]

contains at most \( \mu + 1 \) different terms when restricted to \( V \).

Indeed, if \( \nu \in \mathbb{N} \), \( \nu \leq \mu + 1 \), then \( w_\nu(x) = w_\lambda(x) \), for a certain \( \lambda \in \mathbb{N} \), \( \lambda \leq \mu \), since \( x \in V \subset \Omega'' \subset \Delta^\nu \subset \Delta^\mu \). But then

\[ w_\nu = w_\lambda \text{ on } V \]

Assume indeed that (28) is false. Then \( w_\nu(y) \neq w_\lambda(y) \), for a certain \( y \in V \). Denote

\[ V' = \{ x' \in V \mid w_\nu(x') = w_\lambda(x') \} \]

\[ V'' = \{ x'' \in V \mid w_\nu(x'') \neq w_\lambda(x'') \} \]

then \( x \in V' \), \( y \in V'' \), \( V = V' \cup V'' \), \( V' \cap V'' = \emptyset \) and \( V' \) is obviously closed. But \( V'' \) is also closed, since:

\[ V'' = \{ x'' \in V \mid |w_\nu(x'') - w_\lambda(x'')| \geq 1/(\rho + 1) \} \]

the inclusion \( \supset \) being obvious, while the converse results as follows. Take \( x'' \in V'' \), then there exists \( \sigma \in \mathbb{N} \), \( \sigma \leq \mu \), such that \( w_\sigma(x'') = w_\sigma(x'') \), since \( \nu \geq \mu + 1 \) and \( x'' \in \Omega'' \subset V \subset \Omega^\nu \subset \Delta^\rho \subset \Delta^\mu \). Hence \( w_\sigma(x'') = w_\lambda(x'') \). Now, \( \sigma, \lambda \leq \mu \) and \( x'' \in \Omega'' \subset V \subset \Omega'' \subset \Delta^\rho \) imply that

\[ |w_\nu(x'') - w_\lambda(x'')| = |w_\sigma(x'')| \geq 1/(\rho + 1) \] and the proof of (29) is completed. As the decomposition \( V = V' \cup V'' \) contradicts the connectedness of \( V \), it follows that (28) holds.

Now, (28) implies (27), which implies (24). Thus, finally, (23) was proved.
§5. APPLICATIONS TO LINEAR AND NONLINEAR PDEs

It follows easily from Theorem 1 in §4, that

\[ \mathcal{T}^{-1}(u(f) + R) \subset F^N \]

is the set of all weak solutions on \( \Omega \), of the equation (4), therefore, the existence of weak solutions on \( \Omega \), for the equation (4) is equivalent to the condition

\[ \exists s' \in F^N : \]

(30) \( \forall \Omega' \subset \Omega \) nonvoid, open, a subsequence in \( s \), \( \psi' \in C^0(\Omega) : \)

\[ Ts' = u(f + \psi') \text{ on } \Omega' \Rightarrow \psi' = 0 \text{ on } \Omega' \]

An operator \( \mathcal{T} \) of type (1) is called expansive only if

\[ \exists s \in F^N : \]

(31) \( \forall s' \text{ subsequence in } s : \)

\[ \text{int} \left( \bigcup_{\nu, \mu \in N} Z(Ts'_{\nu} - Ts'_{\mu}) \right) = \emptyset \]

where \( Z(g) = \{ x \in \Omega \mid g(x) = 0 \} \) denotes the zero set of the continuous function \( g \in C^0(\Omega) \).

**Theorem 2**

If \( \mathcal{T} \) is expansive, the equation (4) possesses weak solutions on \( \Omega \), for any given \( f \in C^0(\Omega) \).

**Proof**

It follows easily from (31) and (30) □

First, we shall apply Theorem 2, to the well known PDO of H. Lewy, [8]:

(32) \[ L(D) = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} - 2i(x_1 + i x_2) \frac{\partial}{\partial x_3}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 \]

which for certain \( f \in C^0(\mathbb{R}^3) \), gives PDEs

(33) \[ L(D) \psi(x) = f(x), \quad x \in \mathbb{R}^3 \]

with not even local distribution solutions, [4].
We shall show that the PDEs in (33) possess weak solutions on \( \mathbb{R}^3 \), for any continuous function \( f \in C^0(\mathbb{R}^3) \). In this respect, in view of Theorem 2, it suffices to show that the operator

\[ L(D) : C^1(\mathbb{R}^3) \longrightarrow C^0(\mathbb{R}^3) \]

defined by (32), is expansive. Indeed, define \( s \in (C^1(\mathbb{R}^3))^N \) by

\[ s_v(x) = v(x_1 + x_2 + x_3), \quad \forall \, v \in N, \, x = (x_1, x_2, x_3) \in \mathbb{R}^3, \]

then, an easy computation gives

\[ Z(L(D)s_v - L(D)s_\mu) = \{ (\frac{1}{2}, -\frac{1}{2}, x_3) | x_3 \in \mathbb{R}^1 \} \]

\[ \forall \, v, \mu \in N, \, v \neq \mu, \]

therefore, the relation (31) holds.

As a second application of Theorem 2, we shall establish the existence of weak solutions on \( \Omega = \mathbb{R}^1 \times (0, \infty) \), for the nonlinear conservation law

\[ u_t(x,t) + a(u(x,t)) \cdot u_x(x,t) = 0, \quad (x,t) \in \Omega, \]

provided that

\[ a \in C^1(\mathbb{R}^1), \quad \text{a nonconstant on any interval in } \mathbb{R}^1. \]

In this respect, we shall show that the operator

\[ L(D) : C^1(\Omega) \longrightarrow C^0(\Omega) \]

defined by the left term of (32) is expansive, by constructing a sequence of functions \( s \in (C^1(\Omega))^N \) which satisfies (31). Define therefore

\[ s_v(x,t) = h_v x + k_v t, \quad \forall \, v \in N, \, (x,t) \in \Omega, \]

where \( h_v, k_v \in \mathbb{R}^1 \setminus \{0\} \), with \( v \in N \), satisfy the condition

\[ h_v \cdot k_\mu + h_\mu \cdot k_v, \quad \forall \, v, \mu \in N, \, v \neq \mu. \]

We shall show that

\[ \text{int} \, Z(L(D)s_v - L(D)s_\mu) = \emptyset, \quad \forall \, v, \mu \in N, \, v \neq \mu. \]

Indeed, assume that

\[ L(D)s_v = L(D)s_\mu \quad \text{on } \Omega^1 \]

for a certain \( \Omega^1 \subset \Omega \) nonvoid, open and \( v, \mu \in N, \, v \neq \mu \). Then, a direct computation gives
hence, applying the partial derivatives \( \partial / \partial t \), respectively \( \partial / \partial x \), one obtains

\[
\begin{align*}
\frac{h_v k_v a'(h_v x + k_v t)}{h_v k_v a'(h_v x + k_v t)} &= \frac{h_u k_u a'(h_u x + k_u t)}{h_u k_u a'(h_u x + k_u t)} \quad \forall (x,t) \in \Omega', \\
\frac{h_v^2 a'(h_v x + k_v t)}{h_v^2 a'(h_v x + k_v t)} &= \frac{h_u^2 a'(h_u x + k_u t)}{h_u^2 a'(h_u x + k_u t)} \quad \forall (x,t) \in \Omega'
\end{align*}
\]

which will obviously contradict (37) and (35), therefore ending the proof of (38). Now, (38) implies that the sequence of functions in (36) satisfies the condition (31).

As a final application of Theorem 2, we shall show that the second order nonlinear wave equation

\[
(39) \quad u_{tt}(x,t) - u_{xx}(x,t) + f(u(x,t), u_x(x,t), u_t(x,t)) = 0 \quad (x,t) \in \Omega = \mathbb{R}^1 \times (0,\infty),
\]

has weak solutions on \( \Omega \), provided that

\[
(40) \quad f \in C^1(\mathbb{R}^3), \text{ if nonconstant on any subset } (a,b) \times (c,d) \subset \Omega.
\]

Indeed, with the help of the sequence of functions in (36) satisfying (37), it is easy to show that the operator

\[
L(D) : C^2(\Omega) \longrightarrow C^0(\Omega)
\]

defined by the left term in (39) is expansive, under the condition (40).

Obviously, the nonlinear Klein-Gordon equation

\[
u_{tt} - u_{xx} + au^m = 0, \quad (x,t) \in \Omega.
\]

with \( m \neq 0 \), as well as the sine Gordon equation

\[
u_{tt} - u_{xx} + \sin u = 0, \quad (x,t) \in \Omega
\]

satisfy the condition (40).
REFERENCES


