RESOLUTION OF SINGULARITIES AND STABILITY
OF WEAK SOLUTIONS FOR POLYNOMIAL NONLINEAR
PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

The weak solutions of polynomial nonlinear PDEs are defined as elements in algebras of classes of sequences of continuous functions on domains in Euclidean spaces. That approach offers two main advantages. First, the resolution of singularities concentrated on nowhere dense subsets and exhibited by weak solutions of polynomial nonlinear PDEs. Second, a direct and explicit answer to the stability problem of these weak solutions, a problem which is essential in the case of nonlinear PDEs, being however quite often overlooked.


Keywords and phrases. Resolution of singularities, stability of weak solutions, polynomial nonlinear PDEs,
CHAPTER 1 - ALGEBRAS OF WEAK SOLUTIONS CONTAINING THE DISTRIBUTIONS

§1. Introduction

A variety of important nonlinear PDEs are known to possess weak or distribution solutions. An extensively studied case is the one of the shock wave solutions of nonlinear hyperbolic conservation laws [30, 39, 40, 31-34, 26, 35, 1, 13-16, 18, 19, 25, 27-29, 38, 45, 46, 49, 74, 77, 78]. Recently, [36], certain nonconservative nonlinear hyperbolic equations were shown to possess shock wave type solutions. Another case is given by the distribution or singular solutions of nonlinear second order wave equations [5-9, 20, 41, 42, 50, 52-56, 75, 80]. Several additional classes of nonlinear PDEs encountered in applications lead also to solutions which are distributions [2, 3, 12, 17, 21, 22; 47-49, 65, 79].

The present work deals with the class of polynomial nonlinear PDEs:

(1) \[ \sum_{1 \leq i \leq k} c_i(x) \cdot \prod_{1 \leq j \leq k} D^{p_{ij}} u(x) = f(x), \quad x \in \Omega, \]

where \( u \) is the unknown function, \( \Omega \subset \mathbb{R}^n \) is nonvoid, open, \( c_i, f \in C^0(\Omega) \) and \( D^{p_{ij}} \), with \( p_{ij} \in \mathbb{N}^n \), denote the usual partial derivatives.

A method is presented which enables the resolution of nowhere dense singularities of solutions for PDEs of the form (1). That method also offers the possibility for an explicit stability analysis of the weak solutions for the PDEs in (1). It is important to mention that in the case of nonlinear PDEs, the stability of weak solutions becomes an essential and nontrivial problem, which is quite often overlooked. Namely, given a weak solution defined by a sequence of functions

(2) \[ s = (\psi_0, \psi_1, \ldots, \psi_k, \ldots), \quad \text{with} \quad \psi_k \in C^m(\Omega), \]
m being the order of the PDE, the question arises: which are all the sequences of functions

\[ t = s + v = (\psi_0 + x_0, \psi_1 + x_1, \ldots, \psi_v + x_v, \ldots), \quad \text{with} \quad v = (x_0, x_1, \ldots, x_v, \ldots) \]

and \( x_v \in C^m(\Omega) \),

such that \( t \) defines the same weak solution with \( s \). In the case of a linear PDE with \( C^\infty \) smooth coefficients, due to the continuity of the corresponding PDO, the answer is simple: one can take in (3) any sequence of functions \( v \) which is weakly convergent to zero in \( D'(\Omega) \). However, in the case of a nonlinear PDE, the answer is no more so simple and one cannot define a weak solution by only exhibiting one single sequence of functions (2) whose properties could happen to be a rather irrelevant accident, resulted from the discontinuity of the corresponding nonlinear PDO on a certain space of distributions.

The method presented in this work is based on the construction of embeddings of the distributions in \( D'(\Omega) \) into associative and commutative algebras

\[ D'(\Omega) \subset A^0 \rightarrow \ldots \rightarrow A^l \rightarrow \ldots \rightarrow A^\infty \]

of classes of sequences of functions

\[ A^l = A^l/I^l, \quad \text{with} \quad l \in \mathbb{N} = \mathbb{N} \cup \{\infty\} \]

where \( A^l \) is a subalgebra in the algebra \( (C^\infty(\Omega))^N \) of all the sequences of complex valued \( C^l \) smooth functions on \( \Omega \), considered with the term by term operations on sequences of functions, while \( I^l \) is an ideal \( A^l \).

The algebras (5) will possess partial derivative operators
(6) $D^p : A^k \rightarrow A^k$, for $k, l \in \mathbb{N}, \ k \leq l, \ p \in \mathbb{N}, \ |p| \leq 2 - k$ (with $-\infty - \infty \cdots$),

which are linear mappings, extend the usual ones and satisfy the Leibnitz rule for product derivative.

Therefore, the polynomial nonlinear PDO

(7) $T(D) = \sum c_i(x) \prod_{1 \leq i \leq h} D^p_{i j}$, $x \in \Omega$, $1 \leq j \leq k_i$

corresponding to the PDE in (1), will generate mappings

(8) $T(D) : A^m \rightarrow A^o$

where

(9) $m = \max\{|p_{i j}| \mid 1 \leq i \leq h, 1 \leq j \leq k_i\}$

is the order of the PDE in (1), respectively PDO in (7).

In that way, the PDE in (1) will be replaced by the PDE

(10) $T(D)S = F$, with given $F \in A^o$ and unknown $S \in A^m$,

in which case, the weak solutions of the PDE in (1) will be classes of sequences of functions

(11) $S = s + I^m \in A^m = A^m/I^m$, $s \in A^m$.

The resolution of singularities is obtained through the fact that the weak solutions (11) — in particular, distribution solutions $S \in D'(\Omega)$ with singularities on nowhere dense subsets $\Gamma \subset \Omega$ — will satisfy the PDE in (10) in the usual algebraic sense, with the multiplication in $A^o$ and the extended partial derivative operators $D^p : A^m \rightarrow A^o$, $p \in \mathbb{N}$, $|p| \leq m$, given in (6).
Now, the problem of stability of weak solutions obtains a direct and explicit answer: if a weak solution $S$ in (11) is defined by a certain sequence of functions $s \in A^m$, then all the sequences of functions $t = s + v$, with $v \in I^m$, will define the same weak solution $S$. Therefore, the stability of any such weak solution is explicitly given by the size of the ideals $I^m$ which can be effectively constructed. It follows that the maximal stability means ideals $I^m$ which are maximal within the conditions they will have to satisfy.

Finally, another advantage of the method is the resulting exactness property of the weak solutions $S \in A^m$ for the PDE in (10). Namely, given $S = s + I^m$, with $s \in A^m$ and $F = r + I^0$, with $r \in A^0$, the resulting sequence of functions

$$w = T(D)s - r$$

satisfies the exactness condition

$$w \cdot z \in I^0, \forall z \in A^0,$$

since $w \in I^0$ and $I^0$ is an ideal in $A^0$. Therefore, the "error" $w$ caused by the nonclassical weak solution $S$, is given an explicit algebraic test (13), prior to other, topological ones. Obviously, the smaller $I^0$ and the larger $A^0$, the better exactness one obtains.

It is important to notice that maximal stability, meaning maximal $I^m$, will conflict with maximal exactness, requiring minimal $I^0$. In this respect, the proper framework in (4) will result from the specific interplay of the stability and exactness demands in each particular case treated. Therefore, the need for various embeddings of type (4).
Concerning the embeddings in (4) and the way partial derivative operators are defined in (6), on the algebras, it is particularly important to mention that, due to very simple reasons (see [61], p.28), it is not possible to construct embeddings 
\[ D^1(\hat{\Omega}) \ni A, \text{ where } A \text{ is an algebra with arbitrary partial derivative operators } D^p: A \rightarrow A, p \in \mathbb{N}. \]
However, the framework given by (4) and (6), which offers the extension (8) of the polynomial nonlinear PDO in (7), proves to be satisfactory, when dealing with resolution of singularities and stability of weak solutions.

As a general remark, the approach of the algebraic aspects underlying the weak solutions presented here, gives a possibility for the study of the polynomial nonlinear PDEs with continuous coefficients, a class of nonlinear PDEs sufficiently wide and important in applications. The method employed does not belong to the usual linear or nonlinear functional analysis and points towards the theory of algebras of continuous functions. In this respect, the method is not elementary, employing Zorn's lemma, the Baire category argument and cardinality properties of spaces of continuous functions.

It is worth mentioning that even in the case of linear problems, the algebraic aspects involved have not always been studied and understood sufficiently, basic facts and properties being only pointed out recently [66-71].

And now, some notations. For \( s \in (C^0(\Omega))^N \), a sequence of complex valued continuous functions on \( \Omega \), we denote by \( s_{v} \in C^0(\Omega) \), with \( v \in \mathbb{N} \), the function which is the \( v \)-th term in \( s \). Given \( \psi \in C^0(\Omega) \), denote by \( u(\psi) \in (C^0(\Omega))^N \) the sequence of functions with all the terms \( \psi \). Denote also
\[ U^\psi = \{ u(\psi) \mid \psi \in C^0(\Omega) \}, \text{ with } \psi \in \mathbb{N}, \]
which is a subalgebra of \((\mathcal{C}^\infty(\Omega))^N\) and it is isomorphic to \(\mathcal{C}^\infty(\Omega)\), through the mapping \(\psi \mapsto u(\psi)\). Obviously, \(u(1)\) is the unit element, while \(\mathcal{O} = \{u(0)\}\) is the null ideal in each of the algebras \((\mathcal{C}^\infty(\Omega))^N\), with \(\mathcal{L} \in \mathbb{N}\).

For \(\mathcal{L} \in \mathbb{N}\) define \(D^\mathcal{L}: (\mathcal{C}^{|\mathcal{L}|}(\Omega))^N \to (\mathcal{C}^0(\Omega))^N\) as the term by term \(\mathcal{L}\)-order partial derivative of sequences of functions, that is

\[
(D^\mathcal{L}s)_v(x) = (D^\mathcal{L}s)_v(x), \forall v \in (\mathcal{C}^{|\mathcal{L}|}(\Omega))^N, \mathcal{L} \in \mathbb{N}, x \in \Omega.
\]

Now, one can define \(T(D): (\mathcal{C}^\mathcal{M}(\Omega))^N \to (\mathcal{C}^0(\Omega))^N\) by

\[
T(D)s = \sum_{1 \leq j \leq k} \left( \sum_{1 \leq i \leq k'} u(c_{ij}) \right) P_j D^L s_i, \forall s \in (\mathcal{C}^\mathcal{M}(\Omega))^N.
\]

§2. Regularizations and Algebras Containing the Distributions

The algebras \((4)\) containing the distributions will be constructed by an adaptation of the method in [61], Chap. 1, §§4-7.

Given \(\mathcal{L} \in \mathbb{N}\), denote by \(S^\mathcal{L}\) and \(V^\mathcal{L}\) the set of all sequences of functions in \(\mathcal{C}^\infty(\Omega)\) which converge weakly to distributions, respectively to zero in \(D^1(\Omega)\). Then, the mapping:

\[S^\mathcal{L} \ni s \longrightarrow \langle s, \cdot \rangle \in D^1(\Omega)\]

where

\[
\langle s, \psi \rangle = \lim_{\mathcal{L} \to \infty} \int_{\Omega} s_v(x)\psi(x)dx, \forall \psi \in D(\Omega),
\]

is a linear surjection, with the kernel \(V^\mathcal{L}\), therefore the mapping
I is a vector space isomorphism.

Given a subalgebra $A \subset (\mathcal{C}^0(\Omega))^N$ and a vector subspace $V \subset A$, denote by

$$I(V, A)$$

the ideal in $A$ generated by $V$. If $u(1) \in A$, then $I(V, A)$ is obviously the vector subspace in $A$ generated by $V \cdot A$. In case $A = (\mathcal{C}^0(\Omega))^N$, we shall simply denote $I(V)$ instead of $I(V, A)$.

The basic notion is introduced now. Given $\lambda \in \bar{N}$, a pair $(V, S)$ of vector subspaces in $V^\lambda$, respectively $S^\lambda$, is called a $C^\lambda$-regularization, only if

$$I(V) \cap S = \emptyset$$

$S^\lambda = V^\lambda \oplus S$

$u^\lambda \subset V^\lambda \oplus S$

where

$$V^\lambda = \{v \in V \mid \exists p \in V, \forall p \in \bar{N}^2, |p| \leq \lambda\}$$

The existence of $C^\lambda$-regularizations, for any $\lambda \in \bar{N}$, will be proved in Theorem 4, §3. Meanwhile, we show the way they are instrumental in constructing embeddings of type (4). First, a few notations. Call a subalgebra $A \subset (\mathcal{C}^{2+}[|p|]\Omega)^N$, with $\lambda \in \bar{N}$, derivative invariant, only if

$$D^p(A \cap (\mathcal{C}^{2+}[|p|]\Omega)^N) \subset A, \forall p \in \bar{N}^2.$$
Given a vector subspace $H \subset (C^\infty(\Omega))^\mathbb{N}$ and $\ell \in \mathbb{N}$, denote by

$$A^\ell(H)$$

the derivative invariant subalgebra in $(C^\infty(\Omega))^\mathbb{N}$ generated by $H$ and denote by

$$I^\ell(V, H)$$

the ideal in $A^\ell(V_k + H)$ generated by $V_k$.

Theorem 1: If $(V; S)$ is a $C^\infty$-regularization, with $\ell \in \mathbb{N}$ given, for each $k \in \mathbb{N}$, $k \leq \ell$, one obtains the diagram of inclusions

\[
\begin{array}{ccc}
I_k^k(V, S) & \longrightarrow & A^k(V_k \bigoplus S) \\
\downarrow & & \downarrow \\
V_k & \longrightarrow & V_k \bigoplus S \leftarrow S^k \\
\downarrow & & \downarrow \\
V^k & \longrightarrow & U^k
\end{array}
\]

and

$$I_k^k(V; S) \cap (V_k \bigoplus S) = V_k$$

or, equivalently

$$I_k^k(V; S) \cap S = \emptyset.$$ 

Proof. The inclusions in (19) result easily and we shall only prove (19.1) obviously, $I_k^k(V; S) \subset I(V)$, hence $I_k^k(V; S) \cap (V_k \bigoplus S) \subset I(V) \cap (V_k \bigoplus S) \subset V_k$ the last inclusion resulting from (16.1) and the fact that $V_k \subset I(V)$. Thus, the inclusion $\subset$, in (19.1) is proved. The converse one follows from (19). □
Based on Theorem 1, the algebras in (4) can be obtained as follows. Suppose $(V, S)$ is a $\mathcal{C}^\ell$-regularization, with $k \in \bar{N}$ given, then, one can define

$$A^k(V, S) = A^k(V, S)/I^k(V, S), \quad \text{with } k \in \bar{N}, \ k \leq \ell.$$  

(20) 

The reason $\mathcal{C}^\ell$-regularization $(V, S)$ are needed is that, as shown in [61], pp. 10-13, it is not possible to construct diagrams of inclusions

$$I \xrightarrow{\iota} A \xrightarrow{\pi} (\mathcal{C}^\ell(\Omega)),$$

with $I \cap S^\ell = V^\ell$,

which would generate embeddings of distributions into algebras

$$D^\ell(\Omega) = S^\ell/V^\ell \subset A = A/I.$$  

However, using a $\mathcal{C}^\ell$-regularization $(V, S)$ in order to restrict $V^\ell$ to $V_k$ and $S^\ell$ to $V_k \mp S$, one can, due to (19) and (20), construct such embeddings, namely

$$D^\ell(\Omega) \subset A^k(V, S), \quad \text{with } k \in \bar{N}, \ h \leq \ell$$

as shown next in Theorem 2.

It follows that for any $\mathcal{C}^\ell$-regularization $(V, S)$

(21) 

$$V \not\subset V^o$$  

(21.1) 

$$V_k \not\subset V^k, \ \forall k \in \bar{N}, \ k \leq \ell$$

(21.2)
As the ideals $I^k(V,S)$ in (20) depend directly on $V_k$, being the ideals generated by $V_k$ in $A^k(V_k \oplus S)$, the stability of weak solutions will be directly connected with the size of $V_k$, that is $V$, in a given $C^2$-regularization $(V,S)$. In view of the strict inclusion in (21), the problem of maximal stability of weak solutions needs a special study (see [61], Chap. 8).

The next two theorems present the main properties of the above algebras, which enable us to obtain the results mentioned in §1.

**Theorem 2:** If $(V,S)$ is a $C^2$-regularization, with $\lambda \in \bar{N}$ given, then

1) $A^k(V,S)$, with $k \in \bar{N}$, $k \leq \lambda$, are associative and commutative algebras of classes of sequences of $C^k$-smooth functions, with the unit element $u(1) + I^k(V,S)$, and the multiplication in $A^k(V,S)$ induces on $C^k(\Omega)$ the usual multiplication of functions.

2) The diagrams are commutative for each $j, h, k \in \bar{N}$, $j \leq h \leq k \leq \lambda$:

\[
\begin{array}{ccc}
A^k(V,S) & \xrightarrow{\gamma_{k,j}} & A^j(V,S) \\
\uparrow{\gamma_{k,h}} & & \uparrow{\gamma_{h,j}} \\
A^h(V,S) & \xrightarrow{\gamma_{h,j}} & A^j(V,S) \\
\end{array}
\]

where $\gamma_{k,h}$ are algebra homomorphisms, and $\varepsilon_k = \beta_k \circ \alpha_k^{-1} \circ \omega_k^{-1}$ are linear, injective with $\omega_k$ from (14) and

\[
S^k/V^k \xleftarrow{\alpha_k} (V_k \oplus S)/V_k \xrightarrow{\beta_k} A^k(V,S)
\]

given by
\[ \alpha_k(s+V_k) = s + V^k \text{ linear, bijective} \]

\[ \beta_k(s+V_k) = s + I^k(V,S) \text{ linear, injective.} \]

**Proof.** It follows from Theorem 1 and (16.3). \( \square \)

**Theorem 3:** If \((V,S)\) is a \(C^\lambda\)-regularization, with \( \lambda \in \bar{N} \) given, then

1) The extended partial derivative operators are defined by the linear mappings

\[ D^p; A^k(V,S) \rightarrow A^h(V,S) \]

with

\[ D^p(s + I^k(V,S)) = D^p s + A^h(V,S) \]

for any \( h, k \in \bar{N}, h < k \leq \lambda, p \in \mathbb{N}^n, |p| \leq k-h \) (with \( \infty - \infty = \infty \)).

Moreover, they coincide on \( C^k(\Omega) \) with the usual partial derivatives of functions.

2) The extended partial derivatives satisfy the Leibnitz rule for product derivative.

**Proof.**

1) Obviously

\[ (22) \quad D^p V_k \subset V_h \]

Moreover

\[ (23) \quad D^p A^k(V_k \oplus S) \subset A^h(V_h \oplus S) \]

since the subalgebras in (18.1) are derivative invariant. It follows that
Indeed, (16.3) implies that \( u(1) \in \mathcal{U}^k \subseteq V_k \oplus S \subseteq \mathcal{A}^k(V_k \oplus S) \) hence \( \mathcal{I}^k(V,S) \) is the vector subspace generated by \( V_k \mathcal{A}^k(V_k \oplus S) \). Now, (24) will result from (22) and (23) as well as the fact that \( \mathcal{D}^P \) applied to sequences of functions satisfies the Leibnitz rule for product derivatives.

2) results easily from 1).

Based on Theorem 2 and 3 above, we can associate to polynomial nonlinear PDOs mappings acting between the algebras containing the distributions. Indeed, suppose given the \( m \)-order PDO in (7), with continuous coefficients. If \( (V,S) \) is a \( C^m \)-regularization, one can obviously define the mapping

\[
T(D) : \mathcal{A}^m(V,S) \rightarrow \mathcal{A}^0(V,S)
\]

by

\[
T(D)S = \sum_{1 \leq i \leq h} \prod_{1 \leq j \leq k_i} \mathcal{D}^{p_{ij}}S, \quad \forall S \in \mathcal{A}^m(V,S).
\]

§3. The Construction of Regularizations

A useful and simple method for constructing regularizations \( (V,S) \) is presented in this section. The regularizations obtained will define algebras (20) used in Chapter 2, for the resolution of singularities of weak solutions of polynomial nonlinear PDEs.

Denote by \( \mathcal{I}_{nd} \) the set of all sequences of continuous functions \( \mathcal{W} \in (C^0(\Omega))^\mathbb{N} \) which admit subsets \( B \subseteq \Omega \) with nowhere dense complementary, so that \( w\_v \) vanishes at each point in \( B \), for sufficiently large \( v \in \mathbb{N} \),
that is,

\[ \exists B \subseteq \Omega, \text{ with } \Omega \setminus B \text{ nowhere dense in } \Omega: \]
\[ \forall x \in B: \]
\[ (26) \exists \mu \in N: \]
\[ \forall \nu \in N, \nu \geq \mu: \]
\[ w(\nu)(x) = 0. \]

It is easy to see that \( I_{nd} \) is an ideal in \( (C^0(\Omega))^N \). We shall call \( I_{nd} \) the nowhere dense ideal in \( (C^0(\Omega))^N \).

The essential property of \( I_{nd} \) is presented in:

**Proposition 1:** For each \( w \in I_{nd} \), there exists an open, dense subset \( \Omega' \subseteq \Omega \), such that

\[ \forall x \in \Omega': \exists V \text{ neig. of } x, \nu \in N: \forall \nu \in N, \nu \geq \mu: \]
\[ (27) \quad w(\nu) = 0 \text{ on } V. \]

If \( w \in I_{nd} \cap B^0 \) then

\[ (28) \quad \supp \langle w, \cdot \rangle \text{ is nowhere dense in } \Omega. \]

If \( w \in I_{nd} \cap (C^0(\Omega))^N \) with \( \ell \in \overline{N} \) given, then

\[ (29) \quad \delta_p w \in I_{nd} \setminus \forall p \in \mathbb{N}^N, |p| \leq \ell. \]

**Proof.** Assume \( w \in I_{nd} \) and \( B \subseteq \Omega \) given in (26). Then, the interior \( B_1 \) of \( B \) is dense in \( \Omega \), since \( \Omega \setminus B \) is nowhere dense in \( \Omega \). Moreover

\[ \forall x \in B_1: \exists \mu \in N: \forall \nu \in N, \nu \geq \mu: w(\nu)(x) = 0. \]
Now, Lemma 1 below, applied to each nonvoid closed subset \( H \subseteq B_1 \) will give a nonvoid, relatively open subset \( G \subseteq H \) on which \( w_v \) vanishes at each point, for sufficiently large \( v \in N \). It follows that the union \( \Omega' \) of all nonvoid, open subsets \( G \subseteq B_1 \) on which \( w_v \) vanishes at each point, for sufficiently large \( v \in N \), is dense in \( B_1 \). Therefore, \( \Omega' \) is dense in \( \Omega \), as \( \Omega \setminus B_1 \) is nowhere dense. The properties (28) and (29) follows easily from (27).

**Lemma 1:** Suppose \( E \) is a complete metric space, \( F \) is a topological space and the continuous functions \( f : E \to F \), \( f_v : E \to F \), with \( v \in N \), satisfy the condition

\[
\forall x \in E: \exists \mu \in N: \forall \nu \in N, \nu > \mu: f_v(x) = f(x).
\]

Then, for each nonvoid, closed subset \( H \subseteq E \), there exists a nonvoid relatively open subset \( G \subseteq H \) and \( \mu \in N \), such that

\[
f_v = f \text{ on } G, \ \forall \nu \in N, \nu > \mu.
\]

**Proof:** Given \( H \subseteq E \) nonvoid, closed and \( \mu \in N \), denote

\[
H_\mu = \{ x \in H \mid f_v(x) = f(x), \forall \nu \in N, \nu > \mu \}
\]

then \( H_\mu \) is closed and

\[
H = \bigcup_{\mu \in N} H_\mu.
\]

Since \( H \) is itself a complete metric space, the Baire category argument implies the existence of \( \mu \in N \), such that the relative interior \( G_{\mu} \) of \( H_{\mu} \) in \( H \) is not void. Now, \( G \) and \( \mu \) obviously satisfy the required relation. \( \square \)
An ideal $I \subseteq (C^0(\Omega))^N$ is called vanishing, only if it satisfies the condition

$$\forall w \in I, \, \mu \in \mathbb{N}:$$

$$\exists v \in \mathbb{N}, \, v \geq \mu, \, x \in \Omega:$$

$$w_v(x) = 0.$$  \hfill (30)

Obviously, the nowhere dense ideal $I_{nd}$ is vanishing.

Concerning the generality of vanishing ideals, it is worth noticing the following property. Call a subset $H \subseteq (C^0(\Omega))^N$ sectional invariant, only if

$$\forall w \in (C^0(\Omega))^N:$$

$$\exists w' \in H:$$

$$w, w' \text{ coincide, except}$$

$$\text{a finite number of terms}$$

\hfill (31)

Then, it is easy to prove:

**Lemma 2:** If $I \subseteq (C^0(\Omega))^N$ is a sectional invariant ideal, then $I$ is vanishing, only if $I \neq (C^0(\Omega))^N$.

And now, the basic property of the vanishing ideals.

**Proposition 2:** If $I$ is a vanishing ideal and $x \in \mathbb{N}$, then there exist vector subspaces $T \subseteq S^x$ satisfying the conditions

$$I \cap T = V^x \cap T = \emptyset.$$  \hfill (32)

$$V^x + I \cap S^x = V^x \oplus T.$$  \hfill (33)
Proof. Taking $E = S^k$, $A = V^k$ and $B = I \cap S^k$ in Lemma 3 below, it suffices to show that

$$\text{codim} \frac{I \cap V^k}{I \cap S^k} \leq \text{codim} \frac{I \cap V^k}{V^k}. \tag{34}$$

First, we notice that

$$\text{codim} \frac{I \cap V^k}{I \cap S^k} \leq \dim I \cap S^k \leq \text{car} \cdot I \cap S^k. \tag{35}$$

But, $I \cap S^k \subseteq (C^0(\Omega))^N$ and $\text{car} \cdot C^0(\Omega) = \text{car} \cdot R^1$, hence

$$\text{codim} \frac{I \cap V^k}{I \cap S^k} \leq (\text{car} \cdot R^1) \cdot \text{car} \cdot N = \text{car} \cdot R^1. \tag{36}$$

Now, in order to prove (34), it suffices to show that

$$\text{codim} \frac{I \cap V^k}{V^k} \geq \text{car} \cdot R^1. \tag{37}$$

In this respect, define $v_\alpha \in V^k$, with $\alpha \in (0,1)$, by

$$v_\alpha(x) = \alpha^\nu, \quad \forall \nu \in N, x \in \Omega. \tag{38}$$

It is easy to see that $\{v_\alpha \mid \alpha \in (0,1)\}$ is a linear independent family in $V^k$. Denote by $V'$ the vector subspace it generates. Then

$$I \cap V' = \emptyset. \tag{39}$$

Indeed, assume $w \in I \cap V'$, then $w \in V'$ implies

$$w = \sum_{i=1}^k \lambda_i v_i, \tag{40}$$

with $\lambda_i \in C^1$ and $v_i \in I(0,1)$, hence
(40) \[ w(x) = \sum_{1 \leq i \leq h} \lambda_i (\alpha_i(x))^\nu, \quad \forall \nu \in \mathbb{N}, \ x \in \Omega. \]

But \( w \in I \), thus (40) and (30) result in

(41) \[ \sum_{1 \leq i \leq h} \lambda_i (\alpha_i(x))^\nu = 0, \text{ for infinitely many } \nu \in \mathbb{N}. \]

Then, it follows that

(42) \[ \lambda_1 = \ldots = \lambda_h = 0. \]

Indeed, if \( h = 1 \), then (42) is obvious, since \( \alpha_1 \neq 0 \). In case \( h > 1 \), one can assume \( 0 < \alpha_1 \ldots < \alpha_h < 1 \) and \( \lambda_i \neq 0 \), with \( 1 \leq i \leq h \). Then, dividing in (41) with \( \alpha_h \), one obtains

\[ \lambda_1 (\alpha_1/\alpha_h)^\nu + \ldots + \lambda_{h-1} (\alpha_{h-1}/\alpha_h)^\nu + \lambda_h = 0, \text{ for infinitely many } \nu \in \mathbb{N}, \]

which implies \( \lambda_h = 0 \), since \( 0 < \alpha_1/\alpha_h < 1 \), with \( 1 \leq i \leq h-1 \).

Now, (38) follows from (39) and (42). And (38) obviously implies (36). \( \square \)

**Lemma 3:** If \( A, B \) are vector subspaces in \( E \) and

\[ \text{codim} A \cap B \leq \text{codim} A \cap B \]

\[ B \leq A \]

then, there exist vector subspaces \( C \) in \( E \), such that

\[ A \cap C = B \cap C = 0 \quad \text{(null subspace)} \]

\[ A + B = A \oplus C. \]
Proof. Assume \((a_i \mid i \in I)\) and \((b_j \mid j \in J)\) are algebraic bases in \(A\), respectively \(B\), such that \((c_k \mid k \in K)\), with \(K = I \cap J\) and \(c_k = a_k = b_k\), is an algebraic base in \(A \cap B\). In view of the hypothesis, there exists an injective mapping \(\alpha: (J \setminus K) \rightarrow (I \setminus K)\). Then, \(C\) can be chosen as the vector subspace having the algebraic base \((a_{\alpha(j)} + b_j \mid j \in J \setminus K)\). ∎

Now, the way \(C^k\)-regularizations with arbitrary \(k \in \mathbb{N}\), can be constructed with the help of the ideal \(I_{nd}\), is presented in:

**Theorem 4:** Given \(k \in \mathbb{N}\), there exist vector subspaces \(T\) and \(S'\) in \(S^k\), such that

\begin{align*}
(43) & \quad I_{nd} \cap T = V^k \cap T = 0 \\
(44) & \quad V^k + I_{nd} \cap S^k = V^k \bigoplus T \\
(45) & \quad S^k = V^k \bigoplus T \bigoplus S' \\
(46) & \quad V^k \subset S'
\end{align*}

in which case, for any vector subspace \(V \subset I_{nd} \cap \mathbb{V}\), the pair

\(\left(V, T \bigoplus S'\right)\)

will be a \(C^k\)-regularization.

**Proof.** The existence of \(T\) satisfying (43) and (44) follows from Proposition 2. Moreover, one obtains

\((V^k \bigoplus T) \cap U = 0\).

Indeed, assume \(u(\psi) \in V^k \bigoplus T\) for a certain \(\psi \in C^k(\mathbb{N})\). Then, (44) gives \(u(\psi) = v + \hat{w}\), with \(v \in V^k\) and \(\hat{w} \in I_{nd} \cap S^k\). Hence \(\psi = \langle \hat{w}, \cdot \rangle\),
which in view of (28) will imply $\psi = 0$ on $\Omega$.

Now, it is obvious that there exist vector subspaces $S'' \subset S'$ satisfying

\[(48) \quad S' = V^k \oplus T \oplus U^k \oplus S''.\]

Choosing, then

\[(49) \quad S' = V^k \oplus S''.\]

one obtains (45) and (46).

Assume now given a vector subspace $V \subset I_{nd} \cap V^0$. In order to prove (47), first we check (16.1). Obviously $\mathcal{I}(V) \subset I_{nd}$ since $V \subset I_{nd}$ and $I_{nd}$ is an ideal in $(\mathcal{C}^0(\Omega))^N$. Therefore

\[\mathcal{I}(V) \cap (T \oplus S') \subset I_{nd} \cap (T \oplus S') = (V^k \oplus T) \cap (T \oplus S'),\]

the last inclusion being implied by (44). Now, in view of (48) and (49), one obtains

\[\mathcal{I}(V) \cap (T + S') \subset T',\]

which together with (43) will give (16.1). The relations (16.2) and (16.3) result from (49) and (48). \qed
CHAPTER 2 - RESOLUTION OF SINGULARITIES OF WEAK SOLUTIONS FOR POLYNOMIAL NONLINEAR PDEs

§1. The Case of Simple Polynomial Nonlinear PDEs

An m-order polynomial nonlinear PDO given in (7) is called simple only if it is of the form

\[ T(D)u(x) = \sum_{l \leq j < k} L_i(D)T_i u(x), \quad x \in \Omega, \]

where \( L_i(D) \) are linear PDOs with continuous coefficients, while \( T_i \) are polynomials of the form

\[ T_i u(x) = \sum_{l \leq j < k} c_{ij}(x)(u(x))^j, \quad x \in \Omega, \]

with \( c_{ij} \in C^0(\Omega) \).

The nonlinear hyperbolic conservation laws, as well as the nonlinear second order wave equations studied in §§2,3, are obviously of the above form (50).

In general, the following large class of quasilinear PDOs

\[ T(D)u(x) = \sum_{p \in \mathbb{N}^n} c_p(x)D^p u(x) + T'(\Omega)u(x) \]

where \( c_p \in C(\Omega) \) and \( T'(D) \) is an m-1-order, simple, polynomial nonlinear PDO, are also of the form (50).

A function \( u: \Omega \rightarrow C^1 \) is called a piecewise smooth weak solution of the simple polynomial PDE

\[ T(D)u(x) = f(x), \quad x \in \Omega, \text{ with given } f \in C^0(\Omega), \]

only if the following four conditions are satisfied.
There exists a family $\Gamma$ of smooth mappings $\gamma: \Omega \to \mathbb{R}$, such that the set

$$\tag{52.1} F_\Gamma = \{ x \in \Omega \mid \exists \gamma \in \Gamma: \gamma(x) = 0 \in \mathbb{R} \}$$

is closed, has zero Lebesgue measure and

$$u \in C^m(\Omega \setminus F_\Gamma).$$

If $k = \max \{ k_i \mid 1 \leq i \leq h \}$ then

$$\tag{53} u^{\kappa} \in L^1_{\text{loc}}(\Omega).$$

The weak solution property holds.

$$\tag{54} \int_{\Omega} \left( \sum_{1 \leq i \leq h} L_i^*(D) \psi(x) - f(x) \psi(x) \right) dx = 0, \quad \forall \psi \in D(\Omega),$$

where $L_i^*(D)$ is the formal adjoint of $L_i(D)$.

Finally, for each $\gamma \in \Gamma$, there exists a bounded, balanced neighbourhood $B^\gamma_0$ of $0 \in \mathbb{R}^m$, such that

$$\tag{55} \{ \gamma^{-1}(B^\gamma_0) \mid \gamma \in \Gamma \} \text{ locally finite in } \Omega.$$

Solutions of an important class of nonlinear hyperbolic conservation laws as well as nonlinear second order wave equations are known to be piecewise smooth weak solutions in the above sense [27, 62, 19, 54-57].

One can notice that the set $F_\Gamma \subset \Omega$ in (52.1), being closed and with zero Lebesgue measure, it is nowhere dense in $\Omega$. However, there exist closed, nowhere dense subsets with positive Lebesgue measure.

A first result on resolution of singularities of weak solutions for
nonlinear PDEs, is presented in:

**Theorem 1:** Suppose \( u : \Omega \rightarrow C^1 \) is a piecewise smooth weak solution of the \( m \)-order simple polynomial nonlinear PDE in (51). Then, one can construct \( C^m \)-regularizations \((V,S)\), such that

1) \( u \in A^m(V,S) \)

2) \( u \) satisfies the PDE in (51) in the usual algebraic sense, with the multiplication in \( A^0(V,S) \).

**Proof.** Obviously, one can assume that \( u \in C^m(\Omega) \).

Assume given \( \alpha_\gamma : \mathbb{R} \rightarrow [0,1] \), \( \alpha_\gamma \in C^m \), for each \( \gamma \in \Gamma \), in such a way that

\[
(56.1) \quad \alpha_\gamma = 0 \quad \text{on a neighbourhood } V_\gamma \quad \text{of } 0 \in \mathbb{R}^E \\
(56.2) \quad \alpha_\gamma = 1 \quad \text{on } \mathbb{R}^E \setminus B_\gamma \quad \text{(see (55)).}
\]

For each \( \nu \in \mathbb{N} \), define a regularization of \( u \), by

\[
s_\nu(x) = \begin{cases} 
    u(x) \cdot \prod_{\gamma \in \Gamma} \alpha_\gamma((\nu+1)\gamma(x)) & \text{if } x \in \Omega \setminus \mathbb{F}_\Gamma, \\
    0 & \text{if } x \in \mathbb{F}_\Gamma,
  \end{cases}
\]

then, each of these functions is \( C^m \) smooth, hence the resulting sequence of regularizing functions satisfies

\[
s = (s_\nu \mid \nu \in \mathbb{N}) \in (C^m(\Omega))^\mathbb{N}.
\]

Indeed, assume given \( \nu \in \mathbb{N} \) and \( x \in \Omega \). If \( x \in \Omega \setminus \mathbb{F}_\Gamma \), then

\[
\{ \gamma \in \Gamma \mid (\nu+1)\gamma(x) \in B_\gamma \} \quad \text{finite}
\]

as \( (\nu+1)\gamma(x) \in B_\gamma \iff x \in \gamma^{-1}(\frac{1}{\nu+1}B_\gamma) \).
Hence (59) results from (55) and the fact that \( B \) is balanced. Now, (59) and (56.2) imply that the product \( \prod_{\gamma \in \Gamma} ((\nu+1)\gamma(x)) \) in (57) contains only a finite number of factors \( \neq 1 \). Thus, \( s \) is well defined on \( \Omega \setminus \mathcal{F}_\Gamma \).

Since \( \Omega \setminus \mathcal{F}_\Gamma \) is open, one can take a compact neighbourhood \( V \) of \( x \), such that \( V \subset \Omega \setminus \mathcal{F}_\Gamma \). Then, the compactness of \( V \), the fact that the family in (59) is locally finite and finally, that \( B \) are balanced, will imply

\[
\{ \gamma \in \Gamma \mid (\nu+1)\gamma(V) \cap B \neq \emptyset \} \text{ finite}
\]

(60) due to (56.1) and (57). Thus, again \( s \in C^m \) at \( x \), and the proof of (58) is completed.

Define \( w \in (C^0(\mathbb{R}))^N \) by

\[
w = T(D)s - u(f).
\]

The sequence \( w \) of continuous functions is obviously measuring the error in (51) resulted by replacing \( u \) with its regularization \( s \) given in (57) and (58). The main point of the proof is to construct \( C^m \)-regularizations \( \{V, S\} \) so that

\[
w \in I^S(V, S)
\]

that is, \( w \) becomes a "negligible error" in the algebras \( A^0(V, S) \).

First, we need several properties of the regularizing and error sequences \( s \), respectively \( w \).
The sequences \( s \) and \( w \) satisfy the relations

\[ V \in \mathcal{F}, \quad \mathcal{K} \text{ compact:} \]

\[ \exists \mu \in \mathbb{N}: \]

\[ \forall \nu \in \mathbb{N}, \ \nu \geq \mu \]

\[ s_\nu = u, \ w_\nu = 0 \text{ on } \mathcal{K}. \]

Indeed, in view of (55)

\[ \Gamma_\mathcal{K} = \{ \gamma \in \Gamma \mid \gamma(\mathcal{K}) \cap B_\gamma \neq \emptyset \} \text{ finite,} \]

therefore

\[ a = \inf \{ \| y(x) \|_\gamma \mid \gamma \in \Gamma_\mathcal{K}, x \in \mathcal{K} \} > 0 \]

provided that \( \| \cdot \|_\gamma \), with \( \gamma \in \Gamma \), are norms on \( \mathbb{R}^\mathcal{K} \), so that

\[ \sup \{ \| x_\gamma \|_\gamma \mid \gamma \in \Gamma, x_\gamma \in B_\gamma \} < \infty. \]

Assume then \( \mu \in \mathbb{N} \), such that

\[ \sup \{ \| x_\gamma \|_\gamma \mid x_\gamma \in B_\gamma \} \leq \mu a, \ \forall \gamma \in \Gamma, \]

then

\[ (\nu + 1)\gamma(\mathcal{K}) \subset \mathbb{R} \setminus B_\gamma, \ \forall \gamma \in \Gamma, \ \nu \in \mathbb{N}, \ \nu \geq \mu. \]

Now, (62) results easily from (57), (58), (61) and (54).

Another relation satisfied by \( s \), given in

\[ s \in S^m, \ \langle s, s \rangle = u \]

follows obviously from (58), (62), (52.1) and (53), assuming \( k \geq 1 \) in
the last relation, since \( T(D) \) becomes trivial otherwise.

Finally, the essential property of the error sequence \( w \) is that

\[
(64) \quad w \in V^0.
\]

Indeed, assume \( \psi \in D(\Omega) \) and \( \nu \in \mathbb{N} \), then (61) and (54) imply

\[
| \int \omega_\nu(x) \psi(x) \, dx | = \left| \int_\Omega \left( \sum_{1 \leq i \leq h} T_i \hat{S}_\nu(x) \cdot L_i^*(D) \psi(x) - f(x) \psi(x) \right) \, dx \right| = 0.
\]

Hence,

\[
= \left| \int_\Omega \left( \sum_{1 \leq i \leq h} (T_i \hat{S}_\nu(x) \cdot T_i u(x)) L_i^*(D) \psi(x) + (T(D) u(x) \cdot f(x)) \psi(x) \right) \, dx \right| \leq \sum_{1 \leq i \leq h} \int_{\text{supp } \psi} \left| T_i \hat{S}_\nu(x) \cdot T_i u(x) \right| \cdot \left| L_i^*(D) \psi(x) \right| \, dx.
\]

Therefore, it suffices to prove that

\[
(65) \quad \lim_{\nu \to \infty} \int_{K} \left| T_i \hat{S}_\nu(x) \cdot T_i u(x) \right| \, dx = 0, \quad \forall 1 \leq i \leq h, \ K \subset \Omega, \ K \text{ compact.}
\]

First, we notice that (50.1) and (57) give for \( x \in \Omega \setminus F_1 \) the relation

\[
T_i \hat{S}_\nu(x) \cdot T_i u(x) = \sum_{1 \leq j \leq k_1} c_{ij}(\nu(x)) \gamma_j \left( \prod_{\gamma \in \Gamma} (\alpha_j ((\nu+1) \gamma(x)))^j - 1 \right), \quad \forall 1 \leq i \leq h, \ \nu \in \mathbb{N}.
\]

But, in view of (56), one obtains

\[
(66) \quad \left| \prod_{\gamma \in \Gamma} (\alpha_j ((\nu+1) \gamma(x)))^j - 1 \right| \leq 1, \quad \forall j \in \mathbb{N}, \ \nu \in \mathbb{N}, \ x \in \Omega.
\]

Further, the argument in the proof of (62) gives
Now, (66), (67) and (68), together with (53) and the fact — see (52.1) — that $F_\Gamma$ is of zero Lebesque measure, will imply (65), thus, completing the proof of (64). Based on the above results, we can proceed to construct the needed $C^m$-regularizations $(V,S)$. We shall choose $V$ as an arbitrary vector subspace. $V$ in $I_{nd}^m \cap V^O$ which satisfies

\begin{equation}
(69) \quad \omega \in V.
\end{equation}

That choice is obviously possible, since (62) and (64) imply $\omega \in I_{nd}^m \cap V^O$, as $F_\Gamma$ is nowhere dense in $\Omega$, according to (52.1).

On the other hand, Theorem 4 in Chap. 1, §3 grants the existence of vector subspaces $T$ and $S'$ in $S^m$ such that

\begin{equation}
(70) \quad V^m + I_{nd}^m \cap S^m = V^m \oplus T.
\end{equation}

\begin{equation}
(71) \quad S = V \oplus T \oplus S'.
\end{equation}

\begin{equation}
(72) \quad U \subseteq S'.
\end{equation}

But,

\begin{equation}
(73) \quad s \in V^m \oplus T \oplus U^m.
\end{equation}

Indeed, assume it is false, then (70) implies that $s = v + \omega + u(\psi)$, with $v \in V^m$, $\omega \in I_{nd}^m \cap S^m$ and $\psi \in C^m(\Omega)$. Thus, in view of (63)

\begin{equation}
(74) \quad u = <w,\gamma> + \psi.
\end{equation}
But $F = \text{supp} \langle \psi, \cdot \rangle$ is nowhere dense in $\Omega$, according to (28) in Proposition 1, Chap. 1, §3. Now, (52.2) and (74) imply that $u = \psi$ on $\Omega \setminus F$. Thus, $u = \psi$ in $D'(\Omega)$, as $F$ has zero Lebesgue measure, according to (52.1). In that way, we contradicted the initial assumption that $u \notin C^m(\Omega)$.

Now, in view of (73) and (63), one can assume that $S'$ also satisfies the condition.

(75) \[ s \in S' \]

Denoting,

(76) \[ S = T \odot S' \]

it follows, according to Theorem 4 in Chap. 1, §3, that $(V, S)$ is a $C^m$-regularization. The relations (75) and (76) will imply

\[ s' \in S \subseteq V_m \odot S \subseteq A^m(V_m \odot S) \]

therefore

\[ u = s + F^m(V, S) \subseteq A^m(V, S) \]

Finally, (69) will give

\[ w \in V = \tilde{V}_0 \subseteq L^0(V, S) \]

therefore

\[ T(D)u - f = T(D)s - u(f) = w \in L^0(V, S) \]

hence

\[ T(D)u - f = 0 \in A^0(V, S) \]
Remark 1. Suppose, the PDO in (50) is such that the linear PDOs $L_i(D)$ have $C^1$ smooth coefficients, while the polynomials $T_j$ have $C^{m+1}$ smooth coefficients, where $l_i \in \mathbb{N}$ is given. Suppose also, that the piecewise smooth weak solution $u : \mathbb{R} \cap C^1$ of the $m$-order, simple, polynomial nonlinear PDE in (51) is $C^2$ smooth, with $l_2 \in \mathbb{N}, l_2 > m$, in the sense that the mappings $\gamma \in \Gamma$ in (52) are $C^2$ smooth and $u \in C^2(\mathbb{N}, F)$. Then, the procedure in the proof of Theorem 1 above, leads to $C^2$-regularizations $(V, S)$. Moreover, one obtains $u \in A^2(V, S)$ and $u$ will satisfy the PDE in (51) in the usual algebraic sense, with the multiplication in any of the algebras $A^2(V, S)$, where $l \in \mathbb{N}$, $l \leq \min \{l_1, l_2 - m\}$.

2. Resolution of Singularities of Nonlinear Shock Waves

Consider the nonlinear hyperbolic conservation law

$$(77) \quad u_t(x, t) + a(u(x, t))u_x(x, t) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

with the initial condition

$$(77,1) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$

Suppose that the function $a : \mathbb{R} \rightarrow \mathbb{R}$ in (77) is a polynomial.

Then, (77) is a first-order, simple, polynomial nonlinear PDE on $\Omega = \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2$, since the left hand term of if can be written under the form in (50), provided that one takes $h = 2,$

$$L_1(D) = D_x, \quad L_2(D) = D_x^2, \quad T_1 u = u \quad \text{and} \quad T_2 u = b(u)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a primitive of $a : \mathbb{R} \rightarrow \mathbb{R}$. 
Under rather general conditions, [27, 62, 19], for $C^\infty$ smooth or piecewise smooth initial data $u_0$, the equation (77) possess shock wave solutions $u: \Omega \rightarrow \mathbb{R}^1$, with the properties:

There exists a finite set $\Gamma$ of $C^\infty$ smooth functions $\gamma: \Omega \rightarrow \mathbb{R}^1$ defining $C^\infty$ smooth curves $F_\gamma = \{ x \in \Omega \mid \gamma(x) = 0 \}$, such that

\[(78) \quad u \in C^\infty(\Omega \setminus F_\Gamma), \quad \text{where} \quad F_\Gamma = \bigcup_{\gamma \in \Gamma} F_\gamma.\]

\[(79) \quad u \text{ locally bounded on } \Omega.\]

\[(80) \quad \int_\Omega (u(x,t)\psi_t(x,t) + b(u(x,t))\psi_x(x,t))dxdt = 0, \quad \forall \psi \in D(\Omega),\]

where $b: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is a primitive of $a: \mathbb{R}^1 \rightarrow \mathbb{R}^1$.

Obviously, such a solution $u$ will be a piecewise smooth weak solution in the sense of the definition in §1.

**Theorem 2:** Suppose $u: \mathbb{R}^1 \times (0,\infty) \rightarrow \mathbb{R}^1$ is a shock wave solution of the nonlinear hyperbolic conservation law in (77), and the conditions (78-80) are satisfied. Then, one can construct $C^k$-regularizations $(V,S)$, for any $\ell \in \mathbb{N}$, $\ell \geq 1$, such that

1) $u = s + I^k(V,S) \in A^k(V,S)$, with $k \in \mathbb{N}$, $k \leq \ell$, where $s \in S$ does not depend on $k \in \mathbb{N}$.

2) $u$ satisfies the PDE in (77) in the usual algebraic sense, with the multiplication in any of the algebras $A^k(V,S)$, with $k \in \mathbb{N}$, $k \leq \ell-1$.

**Proof.** It follows from the proof of Theorem 1 as well as Remark 1 in §1.
3. Resolution of Singularities of Klein-Gordon Type Nonlinear Waves

Consider the Klein-Gordon type nonlinear wave equation

\[(81)\quad u_{tt}(x,t) - u_{xx}(x,t) = T(D)u(x,t), \quad x \in \mathbb{R}^1, \ t > 0,\]

with the initial conditions

\[(81.1)\quad u(x,0) = f(x), \quad x \in \mathbb{R}^1\]
\[(81.2)\quad u_t(x,0) = g(x), \quad x \in \mathbb{R}^1\]

and \(T(D)\) a first order, simple, polynomial nonlinear PDO, with \(C^\infty\)-smooth coefficients. Then, (81) obviously defines a second order, simple, polynomial nonlinear PDO on \(\mathbb{R}^1 \times (0,\infty) \subset \mathbb{R}^2\).

Under general conditions, [54-56], the equation (81) possesses local or global solutions \(u: \Omega \rightarrow \mathbb{R}^1, \quad \Omega = \mathbb{R}^1 \times (0,\infty), \quad \Omega\) open, with the properties:

There exists a finite number of points \(x_1, \ldots, x_\sigma \in \mathbb{R}^1\), originating light-cones with boundaries

\[F^-_\alpha = \{ (x,t) \in \Omega \mid x - x_\alpha - t = 0 \}, \quad F^+_\alpha = \{ (x,t) \in \Omega \mid x - x_\alpha + t = 0 \}, \quad 1 \leq \alpha \leq \sigma,\]

such that

\[(82)\quad u \in C^\infty(\Omega \setminus \mathcal{F}), \quad \text{where} \quad \mathcal{F} = \bigcup_{1 \leq \alpha \leq \sigma} (F^-_\alpha \cup F^+_\alpha)\]

\[(83)\quad u \text{ locally bounded on } \Omega\]

\[\int_\Omega u(x,t)(\psi_{tt}(x,t) - \psi_{xx}(x,t))dxdt =\]

\[\int_\Omega (\sum_{1 \leq i \leq h} T_i u(x,t) \cdot L_i^*(D)\psi(x,t))dxdt, \quad \forall \psi \in D(\Omega),\]
assuming that the first order, simple, polynomial nonlinear PDE $T(D)$ in
(81) has the form (50) and $L_1^*(D)$ is the formal adjoint of $L_1(D)$.

Obviously, such a solution $u$ is also a piece wise smooth weak
solution in the sense of §1, and in a way similar to Theorem 2, one obtains:

Theorem 3: Suppose $u: \Omega \to \mathbb{R}^1$, with $\Omega \subset \mathbb{R}^1 \times (0,\infty)$ nonvoid, open, is
solution of the Klein-Gordon type wave equation (81) and the conditions
(82-84) are satisfied. Then, one can construct $C^k$-regularizations $(V,S)$
for any $k \in \mathbb{N}$, $k \geq 2$, such that

1) $u = s + r^k(V,S) \in A^k(V,S)$, with $k \in \mathbb{N}$, $k \leq \lambda$, where $s \in S$ does
not depend on $k \in \mathbb{N}$.

2) $u$ satisfies the PDE in (81) in the usual algebraic sense, with the
multiplication in any of the algebras $A^k(V,S)$, with $k \in \mathbb{N}$, $k \leq \lambda-2$.

§4. Resolution of Singularities in the General Case

A general method for the resolution of singularities of weak solutions
for arbitrary polynomial nonlinear PDE of type (1) is presented now.

A distribution $S \in \mathcal{D}'(\Omega)$ is called a regular weak solution of the
$m$-order PDE (see (1), (7), and (9))

\begin{equation}
T(D)u(x) = f(x), \quad x \in \Omega,
\end{equation}

only if there exists a sequence of functions $s \in \mathcal{S}^m$ satisfying the
following four conditions:

\begin{align}
\text{(86)} & \quad s = \langle s, \cdot \rangle \\
\text{(87)} & \quad w = T(D)s - u(f) \in \mathcal{V}^0
\end{align}
\[(88) \quad I_w \cap U^m = 0\]

where \(I_w\) is the sectional invariant (see (31) in Chap.1, §3) ideal in \((C^0(\Omega))^N\) generated by \(w\), and finally,

\[(89) \quad \text{either } s \in U^m \text{ or } s \notin V^m + U^m + I_w.\]

It is easy to see that

\[(90) \quad I_w = \{w' \cdot t \mid \begin{align*} & (*) \ w', t \in (C^0(\Omega))^N \\ & **) \ w' \text{ coincides with } w, \text{ except a finite number of terms} \end{align*}\]

Examples of regular weak solutions are given in the next two theorems.

**Theorem 4:** A piecewise smooth weak solution of a simple, polynomial nonlinear PDE is a regular weak solution.

**Proof.** We shall use some of the relations obtained in the proof of Theorem 1, in §1.

We notice that (86) and (87) follow from (63) and (64). Further, (62) implies \(w \in I_{nd}\), hence

\[(91) \quad I_w \subset I_{nd}.\]

Now, in view of (27) in Proposition 1, Chap. 1, §3, one obtains (88).

Finally, (89) follows from (73), (70) and (91). \(\square\)

Denote by \(D'_{nd}(\Omega)\) the set of all distributions in \(D'_{nd}(\Omega)\) with nowhere dense support. A particular case of distributions in \(D'_{nd}(\Omega)\) are obviously, the linear combinations of Dirac delta distributions and their derivatives.
Theorem 5: A distribution $S \in D'(\Omega) \setminus (C^\infty(\Omega) + D'_\text{nd}(\Omega))$ which admits a representation $S = \langle s, \cdot \rangle$, with $s \in S^m$ satisfying

$$w = T(D)s - u(f) \in \mathcal{D}' \cap I'_{\text{nd}}$$

is a regular weak solution of the m-order, polynomial nonlinear PDE in (85).

Before proving the above statement, we can notice that, in view of (14) in Chap. 1, §2, any distribution $S \in D'(\Omega)$ admits a representation $S = \langle s, \cdot \rangle$, with $s \in S^m$.

Proof. The relation (92) gives

$$\mathcal{I}_w \subset \mathcal{I}'_{\text{nd}}$$

which in view of (27) in Proposition 1, Chap. 1, §3, will imply (88). On the other side, the relation $S' \in C(\Omega) + D'_\text{nd}(\Omega)$, as well as (93) together with (28) in Proposition 1, Chap. 1, §3, imply (89).

And now, the general result on the resolution of singularities of weak solutions for polynomial nonlinear PDEs.

Theorem 6: Suppose $S \in D'(\Omega)$ is a regular weak solution of the m-order, polynomial nonlinear PDE in (85). Then, one can construct $C^m$-regularizations $(V, S)$, such that:

1) $S \in A^m(V, S)$

2) $S$ satisfies the PDE in (85), in the usual algebraic sense, with the multiplication in $A^0(V, S)$.

Proof. We shall assume in (89) that $s \notin U^m$, since otherwise $S = \langle s, \cdot \rangle \in C^m(\Omega)$, and in view of (87), $S$ becomes a classical solution of the PDE in (85).
First, we show the way $V$ can be chosen. Take any sectional invariant ideal $I$ in $(C^0(N))^N$ such that

$$w \in I, \; I \cap U = Q, \; s \in V^m + U^m + I.$$  

Obviously, such ideals exist, since $I_w$ is one of them, according to (88) and (89). Moreover, the set of all such ideals is obviously chain complete. Therefore, Zorn's lemma will grant the existence of maximal sectional invariant ideals in $(C^0(N))^N$ satisfying (94). Now, $V$ can be taken any vector subspace in $(C^0(N))^N$, such that

$$w \in V \in I \cap V^0$$

that choice being possible due to (87) and the relation $w \in I$ in (94).

The main point of the proof is that, according to Proposition 1 below, the relation $I \cap U^m = 0$ in (94) together with the fact that $I$ is a sectional invariant ideal in $(C^0(N))^N$, grant the existence of vector subspaces $T$ and $S'$ in $S^m$, such that (99-102) hold for $\lambda = \omega$ and therefore, one obtains

$$\text{(96)} \quad (V, T \oplus S') \quad C^m\text{-regularization.}$$

We shall show now, that $S'$ can be chosen so that

$$\text{(97)} \quad s \in T \oplus S'.$$

Indeed, (89), (100), with $\lambda = m$, and the relation $s \in V^m + U^m + I$ in (94) imply $s \in V^m + T + U^m$.

But $V^m \cap (T + U^m) = 0$, in view of (101-102) applied for $\lambda = m$. Hence there exist vector subspaces $S''$ in $S^m$, such that $S^m = V^m \oplus (T + U^m) \oplus S''$ and $s \in S''$.\)
Assume now \( T + U^m = T \oplus U' \), for a suitable vector subspace \( U' \) in \( U^m \). Taking then \( S' = U' \oplus S'' \), one obtains (101).

Now, (97) will give the relation

\[
S = s + I^m(V, T \oplus S') \in A^m(V, T \oplus S')
\]

while (95) will result in

\[
T(0)S - f = w + I^o(V, T \oplus S') = 0 \in A^o(V, T \oplus S')
\]

The proof of Theorem 6 above, was based on:

**Proposition 1.** Suppose \( I \) is a sectional invariant ideal in \((C^o(\mathbb{N}))^N\)
and, for a certain \( k \in \mathbb{N} \), the relation holds

\[
I \cap V^k = 0
\]

Then, there exist vector subspaces \( T, S' \subseteq S^k \) such that

\[
I \cap T = V^k \cap T = 0
\]

\[
V^k + I \cap S^k = V^k \oplus T
\]

\[
S^k = V^k \oplus T \oplus S'
\]

\[
V^k \subseteq T \oplus S'
\]

in which case, for any vector subspace \( V \subseteq I \cap V^k \)

\[
(V, T \oplus S') \text{ is a } C^k\text{-regularization.}
\]

Conversely, the relations (99–102) imply (98).
Proof. Assume (98) holds. Then $I \neq (C^0(\mathfrak{a}))^N$, hence $I$ is a vanishing ideal, in view of Lemma 2 in Chap. 1, §3. Now, Proposition 2 in Chap. 1, §3, will grant the existence of vector subspaces $T \subseteq S^\wedge$ satisfying (99) and (100).

Assume for the time being that $T$ can be chosen satisfying also

$$ (104) \quad (V^\wedge \oplus T) \cap U^\wedge = T \cap U^\wedge $$

Take then a vector subspace $U' \subseteq U^\wedge$ satisfying

$$ (105) \quad U^\wedge = (T \cap U^\wedge) \oplus U'. $$

It follows that

$$ (106) \quad (V^\wedge \oplus T) \cap U' = 0 $$

since

$$ (V^\wedge \oplus T) \cap U' \subseteq ((V^\wedge \oplus T) \cap U^\wedge) \cap U' = (T \cap U^\wedge) \cap U' = 0, $$

the last two equalities being a consequence of (104) and (105).

In view of (106), one can choose vector subspaces $S' \subseteq S^\wedge$ satisfying (101) and (102).

We shall prove now that the resulting $(V, T + S')$ is a $C^m$-regularization, for any vector subspace $V \subseteq I \cap V^\wedge$. First, we check the condition (16.1) in Chap. 1, §2. Obviously, $I(V) \subseteq I$, since $V \subseteq I$ and $I$ is an ideal $(C^0(\mathfrak{a}))^N$. Therefore

$$ I(V) \cap (T \oplus S') \subseteq I \cap (T \oplus S') \subseteq (V^\wedge \oplus T) \cap (T \oplus S') $$

the last inclusion being implied by (100). Now, in view of (101), one obtains

$$ I(V) \cap (T \oplus S') \subseteq T $$
which together with (99) will give (16.1). The relations (16.2) and (16.3) follow from (101) and (102).

Now, it only remains to prove that the relation (104) can be obtained for a certain $T$. In this respect, we shall use Lemma 1 below, as well as Lemma 3 in Chap. 1, §3. Denote

$$E = S^l, \quad A = V^l, \quad B = U^l \text{ and } C = I \cap S^l$$

then

$$A \cap B = B \cap C = 0$$

the last equality being implied by (98).

Further, with the notations in Lemma 1 below, one obtains:

$$\bar{A} = V^l \bigoplus (U^l \cap (V^l + I \cap S^l))$$

(109)

$$\bar{B} = I \cap S^l \bigoplus (U^l \cap (V^l + I \cap S^l))$$

First, we shall prove the relation

$$\text{codim} \frac{\bar{A} \cap \bar{B}}{\bar{B}} \leq \text{codim} \frac{\bar{A} \cap \bar{B}}{\bar{A}}$$

(110)

An argument similar to the one in the proof of Proposition 2, Chap. 1, §3, used in order to establish (34), will give

$$\text{codim} \frac{\bar{A} \cap \bar{B}}{\bar{B}} \leq \text{dim} \frac{\bar{B}}{\bar{B}} \leq \text{car} \frac{\bar{B}}{\bar{B}} \leq \text{car} \frac{\bar{R}}{\bar{R}}.$$

Now, in order to obtain (110), it suffices to show that

$$\text{codim} \frac{\bar{A} \cap \bar{B}}{\bar{A}} \geq \text{car} \frac{\bar{R}}{\bar{R}}.$$  

(111)

We shall again use the sequences of functions $v_\alpha \in V^l$, with $\alpha \in (0,1)$,
defined in (37) and the vector subspace $V'$ generated by them, trying to prove that

$$(112) \quad \tilde{B} \cap V' = 0$$

Indeed, assume $\tilde{w} \in \tilde{B} \cap V'$, then $\tilde{w} \in V'$ implies

$$\tilde{w} = \sum_{1 \leq i \leq h} \lambda_i v_{a_i}$$

with $\lambda_i \in C^1$ and $a_i \in (0,1)$, hence

$$(113) \quad \tilde{w}_v(x) = \sum_{1 \leq i \leq h} \lambda_i(a_{i,v}) v, \quad \forall \nu \in \mathcal{N}, \ x \in \Omega.$$ But $\tilde{w} \in \tilde{B}$ and (109) give:

$$(114) \quad \tilde{w} = \tilde{w} + u(\psi)$$

with $w \in I \cap S^\perp$ and $\psi \in C^0(\Omega)$. We shall prove that

$$(115) \quad \psi(x) = 0, \ \forall x \in \Omega.$$ Indeed, assume (115) false and $\epsilon > 0$ and $\Omega' \subset \Omega$ nonvoid, open, are such that

$$\psi(x) < -2\epsilon, \ \forall x \in \Omega'.$$

Then, in view of (113) and (114), one obtains

$$\exists \mu \in \mathcal{N}; \ \forall \nu \in \mathcal{N}, \ \nu \geq \mu, \ x \in \Omega':$$

$$w_v(x) = \tilde{w}_v(x) - \psi(x) > \epsilon.$$
Define now $w' \in (C^k(\Omega))^N$ by

$$w'_{\nu} = \begin{cases} w_{\nu} & \text{if } \nu \geq \mu \\ 1 & \text{if } \nu < \mu \end{cases}$$

then, obviously

(116): $w' \in I$

since $w \in I$ and $I$ is sectional invariant. Assume $\chi \in C^k(\Omega)$, with supp $\chi \subset \Omega'$, and define $t \in (C^k(\Omega))^N$ by

$$t_{\nu} = \begin{cases} \chi/\nu & \text{if } \nu \geq \mu \\ \chi & \text{if } \nu < \mu \end{cases}$$

Then, in view of (116), one obtains

$$w' t = u(\chi) \in I \cap U$$

therefore, (98) implies

$$\chi(x) = 0, \forall x \in \Omega$$

which is absurd, since $\chi$ can be chosen arbitrarily. In that way, the proof of (114) is completed.

Now, (114) and (115) give

$$\bar{w} = w \in I \cap V' = 0$$

if taken into account (38) in the proof of Proposition 2, in Chap. 1, §3.

We can therefore conclude that (112) holds, since $V' \subset V^x \subset \bar{A}$ and
(v_\alpha | \alpha \in (0,1)) is an algebraic base in V. Then also (111) and finally, (110) are valid. Applying Lemma 3 in Chap. 1, §3, to \overline{A} and \overline{B} given in (109), one obtains a vector subspace \overline{C} in E = S^2', such that

\[ \overline{A} \cap \overline{C} = \overline{B} \cap \overline{C} = 0 \]
\[ \overline{A} + \overline{B} = \overline{A} \oplus \overline{C} \]

Then, (108) and Lemma 1 below, imply the existence of a vector subspace \[ D = I \cap E = S' \]

such that

\[ V^2 \cap I = I \cap V^2 = 0 \]
\[ V^2 + I \cap S^2 = V^2 \oplus I \]
\[ U^2 \cap (V^2 + I \cap S^2) = U^2 \cap I \]

which are obviously equivalent with (99), (100), and (104).

Conversely, assume (99-102) hold. Then

\[ I \cap U^2' \subset (V^2 \oplus I) \cap (I \oplus S') \subset I \]

the last inclusion being implied by (101). Now, (98) results in view of (99). \[ \square \]

Lemma 1: If A, B, C are vector subspaces in E and

\[ A \cap B = B \cap C = 0 \] (the null space of E)

then, the following two properties are equivalent

\[ \exists D \subset E \text{ vector subspace:} \]
\[ A \cap D = C \cap D = 0 \]
\[ A + C = A \oplus D \]
\[ B \cap (A + C) = B \cap D \]

(117)
and

\[ 3 \subset E \text{ vector subspace:} \]

\[ \bar{A} \cap \bar{C} = \bar{B} \cap \bar{C} = 0 \]

\[ \bar{A} + \bar{B} = \bar{A} \oplus \bar{C}, \]

where

\[ \bar{A} = A \oplus (B \cap (A + C)) \]

\[ \bar{B} = C \oplus (B \cap (A + C)). \]

Proof. It results from direct verification, if taken \( D = (B \cap (A+C)) \oplus \bar{C} \),

when (118) is assumed; when (117) is assumed, one can take any vector subspace \( \bar{C} \subset E \), such that \( D = (B \cap D) \oplus \bar{C} \). □

Remark 2. Obviously, the nowhere dense ideal \( I_{nd} \) is sectional invariant and satisfies (98). Therefore, Proposition 1 above is stronger than Theorem 4 in Chap. 1, §3, except for the relation (102).
CHAPTER 3 - STABILITY AND EXACTNESS OF WEAK SOLUTIONS FOR POLYNOMIAL NONLINEAR PDEs

§1. Stability of Weak Solutions

Suppose given the m-order, polynomial nonlinear PDE (see (1), (7) and (9)).

\( T(D)u(x) = f(x), \ x \in \Omega. \)

An element

\( S \in A^m(V_m \oplus S) = A^m(V_m \oplus S) / I^m(V_m \oplus S) \)

where \((V_m \oplus S)\) is a certain \(C^m\)-regularization, will be a solution of the PDE in (119), only if (see (25))

\( T(D)S - f = 0 \in A^0(V_m \oplus S). \)

As mentioned in Chap. 1, §1, the stability property of the weak solution \( S \) is directly expressed by the size of the ideal \( I^m(V_m \oplus S) \), which in view of (18) in Chap. 1, §2, is the vector subspace generated by

\( V_m = A^m(V_m \oplus S). \)

where (see (16.4))

\[ V_m = \{ v \in V \cap V^m \mid D^p v \in V, \forall p \in \mathbb{N}, |p| \leq m \} \]

while \( A^m(V_m \oplus S) \) is the derivative invariant subalgebra in \((C^m(\Omega))^N \) generated by \( V_m \oplus S \). It follows from (122), that the size of the ideal \( I^m(V_m \oplus S) \) will increase, whenever \( V \) or \( S \) are increased. However, as implied by the following lemma, \( S \) cannot be increased.
Lemma 1. If \((V_1, S_1)\) and \((V_2, S_2)\) are \(C^m\)-regularizations and \(S_1 \subseteq S_2\), then \(S_1 = S_2\).

Proof. It follows from (16.2) in Chap. 1, §2.

Therefore, we shall first concentrate on the possible variations of \(V\) only. In this respect, an approach is suggested by Theorem 4 in Chap. 1, §3 and Proposition 1 in Chap. 2, §4.

It is important to notice that the mentioned approach has in fact a necessary and sufficient character, as seen in:

Proposition 1. Suppose given \(l \in \mathbb{N}\). If \(I\) is an ideal in \((C^0(\Omega))^N\) which admits vector subspaces \(T, S' \subseteq S^0\) and \(V \subseteq I \cap V^0\) satisfying

\[
\begin{align*}
(123) & \quad I \cap T = V^0 \cap T = 0, \\
(124) & \quad V^0 + I \cap S' \subseteq V^0 \oplus T, \\
(125) & \quad S^0 = V^0 \oplus T \oplus S', \\
(126) & \quad U^0 \subseteq V^0 \oplus T \oplus S', \\
\end{align*}
\]

then

\[
(127) \quad (V, T \oplus S') \text{ is a } C^0\text{-regularization.}
\]

Conversely, any \(C^0\)-regularization can be written under the above form.

Proof. We shall use the notations in Chap. 1, §2.

Denote \(S = T \oplus S'\). Then, (16.2) and (16.3) follow respectively from (125) and (126). Now, we prove that (16.1) is also valid. Indeed, \(V \subseteq I\) implies
\[ I(V) \subseteq I, \text{ hence} \]

\[ I(V) \cap S \subseteq I \cap S \subseteq (I \cdot \cap S') \cap S \subseteq (V \oplus T) \cap (T \oplus S') \subseteq T, \]

the last inclusion being implied by (125). Therefore, in view of (123),

\[ I(V) \cap S \subseteq I \cap T = 0 \]

and the proof of (16.1) is completed, which means that \((V, S)\) is a \(C^\infty\)-regularization.

Conversely, assume given a \(C^\infty\)-regularization \((V, S)\). Take then \(I = I(V), T = S\) and \(S' = \emptyset\). Now, (123) will follow from (16.1) and (16.2). Further, (124) is implied by (16.2). Finally, (125) and (126) follow from respectively (16.2) and (16.3).

Call \(C^m\)-regular any sectional invariant ideal \(I\) in \((C^0(\Omega))^N\) which admits a \(C^m\)-regularization \((V, S)\), with

\[ V = I \cup V^0 \]

and denote by \(ID^m\) the set of all such ideals. In view of (128), we shall be interested in the maximal ideals in \(ID^m\), generating maximal \(Vs\).

First, we notice that Theorem 4 in Chap. 1, §3, implies

\[ I_{\text{nd}} \in ID^m \]

while Proposition 1 in Chap. 2, §4, gives the stronger property

\[ \forall I \text{ sectional invariant ideal in } (C^0(\Omega))^N: \]

\[ I \cap U^m = 0 \iff I \in ID^m. \]
The above result (130) can actually be improved, leading to a characterization of $C^m$-regular ideals, which will help proving the existence of maximal such ideals. Given an ideal $I$ in $(C^0(\Omega))^N$, denote by $I$ the ideal generated in $(C^0(\Omega))^N$ by $I \cap V^\circ$. Obviously,

(131) \[ I \cap V^\circ \subset I \subset I, \quad I \cap V^\circ = I \cap V^\circ, \quad I \text{-sectional invariant} \]

**Proposition 2.** A sectional invariant ideal $I$ in $(C^0(\Omega))^N$ is $C^m$-regular, only if

(132) \[ I \cap U^m = 0 \]

**Proof.** Assume $I \in IR^m$ and $(V, S)$, with $V = I \cap V^\circ$, is a $C^m$-regularization. Then, in view of the notation in (13) and the relation (16.1) in Chap. 1, 2, one obtains

(133) \[ I = I(V), \quad I \cap S = 0. \]

Assume now $w = u(\psi)$, with $w \in I$ and $\psi \in C^m(\Omega)$. Then (16.2) implies

\[ w = u(\psi) = v + s, \]

with $v \in V_m \subset V \subset I$ and $s \in S$, hence (133) gives

\[ w - v = s \in I \cap S = 0, \]

therefore $w = u(\psi) = v \in U_m \cap V_m = 0$, and the proof of (132) is completed.

Conversely, assume (132). Since $I$ is sectional invariant — see (131) — one can apply Proposition 1 in Chap. 2, §4, and obtain a $C^m$-regularization $(V, S)$, with $V = I \cap V^\circ$. But, (131) will give then $V = I \cap V^\circ$, hence it follows that $I$ is $C^m$-regular.  □
Theorem 1. For any $C^m$-regular ideal $I$, there exist maximal $C^m$-regular ideals $\overline{I}$, such that $I \subseteq \overline{I}$.

Proof. Due to Zorn's lemma, it suffices to show that $ID^m$ is chain complete relative to the inclusion relation. Assume then given a chain $(I_\lambda \mid \lambda \in \Lambda)$ of $C^m$-regular ideals and denote

$$I = \bigcup_{\lambda \in \Lambda} I_\lambda.$$

Obviously, will be a sectional invariant ideal in $(C^0(\Omega))^N$. Therefore, in view of Proposition 2, it suffices to show that

$$(134) \quad I \cap U^m = 0.$$

But, with the notation in (15), Chap. 1, §2, one obtains

$$I = I(I \cap \mathcal{V}^D) \subseteq \bigcup_{\lambda \in \Lambda} I(I_\lambda \cap \mathcal{V}^D) = \bigcup_{\lambda \in \Lambda} I_\lambda.$$

and

$$I_\lambda \cap U^m = 0, \forall \lambda \in \Lambda.$$

since, $I_\lambda$, with $\lambda \in \Lambda$, have been supposed $C^m$-regular. Therefore, the proof of (134) is completed. \(\square\)

Now, we can approach the problem of the dependence of the ideals $ID^m(V, S)$ in (120), on the vector subspaces $S \subseteq S^m$. Suppose given a vector subspace $S \subseteq S^m$ satisfying (see (16.2) and (16.3) in Chap. 1, §2)

$$(135) \quad S^m = V^m \oplus S, \quad U^m \subseteq S$$

and denote by $ID^m(S)$ the set of all $C^m$-regular ideals $I$ for which $(I \cap \mathcal{V}^D, S)$ is a $C^m$-regularization. Obviously, $ID^m(S)$ is not void,
since \( I = 0 \) is an \( C^m \)-regular ideal, as \((O,S)\) is a \( C^m \)-regularization. Nontrivial cases of ideals \( I \in ID^m(S) \) result easily from their characterization presented next in Proposition 3. It is important to notice that, in view of (102) in Proposition 1, Chap. 2, §4, it is not restrictive to assume \( U^m \subset S \), in (135), a condition evidently stronger than (16.3).

**Proposition 3.** A sectional invariant ideal \( I \) in \((C^0(\Omega))^N\) belongs to \( ID^m(S) \), only if:

\[
I \cap S = 0
\]

**Proof.** Assume (136) holds. Then, in view of (135), and (131) it is easy to see that \((I \cap V^0, S)\) is a \( C^m \)-regularization.

Conversely, if \((I \cap V^0, S)\) is a \( C^m \)-regularization, then (136) results from (16.1) in Chap. 1, §2 and (131).

**Theorem 2.** For any \( C^m \)-regular ideal \( I \in ID^m(S) \) there exist maximal \( C^m \)-regular ideals \( I \in ID^m(S) \), such that \( I \subset \overline{I} \).

**Proof.** Similar to the proof of Theorem 1, only that this time, the relation (136) is used, instead of (132).

**§2. Examples of Large \( C^m \)-regular Ideals**

The method presented in §1 above, reduced the problem of the maximal stability of weak solutions (120) and (121), to the problem of maximal ideals in \( IR^m \) and \( ID^m(S) \), with \( S \) satisfying (135).

One of the \( C^m \)-regular ideals, namely the nowhere dense ideal \( I_{\text{nd}} \) (see (129)) proved to be sufficiently large for handling the resolution.
of singularities of all the piece wise smooth weak solutions of simple, polynomial PDEs (see Theorem 1, Chap. 2, §1) as well as of a wide class of regular weak solutions of arbitrary polynomial nonlinear PDEs (see Theorems 5 and 6, Chap. 2, §1).

We shall now construct a class of $C^m$-regular ideals, which are strictly wider than $I_{nd}$.

Suppose given a dense sequence of points in $\Omega$

$$\xi = (\xi_v \mid v \in N), \text{ with } \xi_v \in \Omega \text{ for } v \in N,$$

then, denoting

$$B_\xi = \left\{ \{\xi_v \mid v \in N, \nu \geq \mu\} \cap \Omega \mid \nu \geq \mu \right\}$$

one obviously obtains a filter base of dense subsets in $\Omega$. Denote now, by $I_{\xi}$ the set of all sequences of functions $w \in (C^0(\Omega))^N$ which satisfy

$$\exists F \in B_\xi:$$

$$\forall x \in F:$$

$$\exists \mu \in N:$$

$$\forall w \in N, \nu \geq \mu:$$

$$w_{\nu}(x) = 0$$

Obviously, $I_{\xi}$ is a sectional invariant ideal in $(C^0(\Omega))^N$. It will be convenient to consider also the set $J_{\xi}$ of all sequences of functions $w \in (C^0(\Omega))^N$ satisfying
Then, it is easy to see that $J_{\xi}$ is again a sectional invariant ideal in $(C^0(\Omega))^N$ and

$$I_{nd} \cup J_{\xi} \subseteq I_{\xi}.$$  

**Proposition 4.** $I_{\xi} \subseteq ID^\xi$, with $\xi \in \bar{N}$, and $I_{nd} \not\subseteq I_{\xi}$.

**Proof.** The relation holds

$$I_{\xi} \n \ U \ = \ 0 .$$

Indeed, assume $u(\psi) \in I_{\xi}$, for a certain $\psi \in C^0(\Omega)$, then (137) implies that $\psi = 0$ on $F$, for a certain $F \in B_\xi$. But $F$ is dense in $\Omega$, therefore $\psi = 0$ on $\Omega$ and (140) is proved. Then, obviously $I_{\xi} \n \ U = 0$, $\forall \xi \in \bar{N}$ and in view of Proposition 2 in §1, it follows that $I_{\xi} \in ID^\xi$, $\forall \xi \in \bar{N}$.

In order to end the proof, it suffices to show that

$$J_{\xi} \n I_{nd} \neq \emptyset$$

and then, recall (139).

Assume thus, $\psi_v \in C^0(\Omega)$, with $v \in N$, such that

$$\forall x \in \Omega: \psi_v(x) = 0 \Leftrightarrow x \notin \xi_v$$

and define $w \in (C^0(\Omega))^N$ by

$$w(x) = \psi_0(x) \ldots \psi_v(x), \ \forall v \in N, \ x \in \Omega$$

then, obviously $w \in J_{\xi} \n I_{nd}$.
The ideals $I_\xi$ constructed above, are particular cases of the ideals introduced next.

Suppose $F$ and $M$ are filters on $\Omega$, respectively $N$ and denote by $I_{F,M}$ the set of all sequences of functions $w \in (C^0(\Omega))^N$ which satisfy

$$
\begin{align}
\exists F &\in F: \\
\forall x &\in F: \\
(142) &\exists M \in M: \\
\forall v &\in M \\
&v(x) = 0
\end{align}
$$

then $I_{F,M}$ is obviously an ideal in $(C^0(\Omega))^N$. Moreover, $I_{F,M}$ is sectional invariant, only if $M_f \subset M$, where $M_f$ is the Frechet filter of subsets in $N$ with finite complementary. And further, $I_{F,M} \cap U^o = \emptyset$, only if $F$ is a filter of dense subsets in $\Omega$. In view of Proposition 2 in §1, one obtains

$$
I_{F,M} \in ID^o \iff \begin{cases}
(*) & \forall F \in F: \; \forall \text{ dense in } \Omega \\
(**) & M_f \subset M
\end{cases}
$$

It is also easy to see that

$$
I_\xi = I_{F_\xi,M_f}
$$

where $F_\xi$ denotes the filter generated by $B_\xi$ on $\Omega$.

Denote by $FL_d(\Omega)$ the set of all filters of dense subsets in $\Omega$. It follows in particular, that $F_\xi \in FL_d(\Omega)$. It is easy to see that $FL_d(\Omega)$ is chain complete. Hence, due to Zorn's lemma, for each
\( F \in \mathcal{F}_d(\Omega) \), there exist maximal \( \overline{F} \in \mathcal{F}_d(\Omega) \), such that \( F \subseteq \overline{F} \). Denote by \( \overline{\mathcal{F}_d(\Omega)} \) the set of maximal \( \overline{F} \in \mathcal{F}_d(\Omega) \).

Denote now by \( \mathcal{U}_L(\mathbb{N}) \) the set of ultrafilters \( \overline{M} \) on \( \mathbb{N} \), for which \( M_{\overline{L}} \subseteq \overline{M} \).

Then, (143) will imply

\[
\forall \overline{F} \in \overline{\mathcal{F}_d(\Omega)} ; \quad \overline{M} \in \mathcal{U}_L(\mathbb{N}) ;
\]

\[
1 \in I_{L, M} \in I_{D^L, \mathbb{N}}, \quad \text{with} \quad L \in \overline{\mathbb{N}}.
\]

The question remains open, whether the above ideals \( I_{L, M} \) are maximal in \( I_{D^L, \mathbb{N}} \) and, whether in the affirmative case, they give all the maximal ideals in \( I_{D^L, \mathbb{N}} \).

3. Stability of Regular Weak Solutions

In the present section it will be shown that the regular weak solutions of polynomial nonlinear PDE (see Chapter 2, §4) are solutions with maximal stability in the sense specified below.

Suppose the distribution \( S \in D'(\mathbb{N}) \) is a regular weak solution of the \( m \)-order, polynomial nonlinear PDE in (85) and given a sequence of functions \( s \in S^m \) which satisfies (86-89). As mentioned in the proof of Theorem 6, Chap. 2, §4, we shall assume in (89) that \( s \notin U^m \), otherwise \( s = \langle s, * \rangle \in C^m(\mathbb{N}) \) being a classical solution of the PDE in (85).

Denote by \( I_{D_s^m} \) the set of all \( C^m \)-regular ideals \( I \), which satisfy

\[
(146) \quad w \in I, \quad s \notin V^m + U^m + I.
\]

Then, \( I_{\overline{w}} \in I_{D_s^m} \), according to (88-89). Moreover, \( I_{D_s^m} \) is chain complete,
hence, due to Zorn's lemma, for any \( \tilde{I} \in \mathcal{I}_m^r \), there exist maximal
\( \tilde{I} \in \mathcal{I}_m^r \) such that \( I \subset \tilde{I} \).

Suppose now given a maximal ideal \( \tilde{I} \in \mathcal{I}_m^r \). Then, in view of (132)
in Proposition 2 in §1, (146) above, as well as the proof of Theorem 6
in Chap. 2, §4, one can construct \( C^m \)-regularizations \((\tilde{I} \cap V^p, S)\)
such that

\[
\begin{align*}
(147) \quad S &= s + \tilde{I}^m(\tilde{I} \cap V^p, S) \in A^m(\tilde{I} \cap V^p, S) . \\
(148) \quad S &\text{ satisfies the PDE in (85), in the usual algebraic sense,} \\
&\text{with the multiplication in } A^0(\tilde{I} \cap V^p, S) . 
\end{align*}
\]

In that case, the stability property of the regular weak solution \( S \)
follows directly from the representation in (147) and is expressed by the
size of the ideal \( \tilde{I}^m(\tilde{I} \cap V^p, S) \). Therefore, the regular weak solution \( S \)
as represented in (147), possesses maximal stability among all possible
representations

\[
\begin{align*}
(149) \quad S &= s + \tilde{I}^m(\tilde{I} \cap V^p, S) \in A^m(\tilde{I} \cap V^p, S) , \text{ with } \tilde{I} \in \mathcal{I}_m^r \\
&\text{with the same, given } S ,
\end{align*}
\]

§4. Exactness of Weak Solutions

We recall the notations in §1. Given the \( m \)-order, polynomial non-linear
PDE

\[
(150) \quad T(D)u(x) = f(x) , \quad x \in \Omega ,
\]

an element
(151) \( S \in A^m(V, S) \)

where \((V, S)\) is a certain \(C^m\)-regularization, will be a solution of the PDE in (150), only if

\[ T(D)S - f = 0 \in A^O(V, S) = A^O(V \oplus S)/I^O(V, S). \]

As mentioned in Chap. 1, §1, the exactness property of the weak solution \( S \) is directly expressed by the size of \( A^O(V \oplus S) \) and \( I^O(V, S) \). In this respect, a maximal exactness requires maximal \( A^O(V \oplus S) \) and minimal \( I^O(V, S) \). Now, in view of (18) in Chap. 1, §2, \( A^O(V \oplus S) \) is the derivative invariant subalgebra in \( (C^O(\Omega))^N \) generated by \( V \oplus S \), while \( I^O(V, S) \) is the vector subspace generated by \( V \cdot A^O(V \oplus S) \). It follows that \( A^O(V \oplus S) \) is increasing in the same time with \( V \) and \( S \), while in a conflicting manner, \( I^O(V, S) \) is decreasing in the same time with \( V \) and \( S \).

A suitable exactness property of the weak solution is obtainable in each particular case through an appropriate settlement of the conflict mentioned above, as well as the one resulting from stability demands.
REFERENCES


REFERENCES (cont'd)


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