SOME PROPERTIES OF A GENERALIZATION OF THE
RICHARDSON EXTRAPOLATION PROCESS

by

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Technical Report #142
January 1979
ABSTRACT

The Richardson extrapolation process is generalized to cover a large class of sequences. Error bounds for the approximations are obtained and some convergence theorems for two different limiting processes are given. The results are illustrated by an oscillatory infinite integral.
1. INTRODUCTION

The purpose of this paper is to generalize the well known extrapolation process due to Richardson and to analyze in some detail the convergence properties of this generalization. In view of this analysis we shall also give some simple criteria for the efficient implementation of this "generalized Richardson extrapolation process" (GREP). An illustrative numerical example will also be appended.

Definition 1.1: We shall say that a function \( A(y) \), defined for \( 0 < y \leq b \), for some \( b > 0 \), where \( y \) can be a discrete or continuous variable, belongs to the set \( F^{(m)} \), for some integer \( m > 0 \), if there exist functions \( \phi_k(y), \beta_k(y), 0 \leq k \leq m-1 \), and a constant \( A \), such that

\[
A(y) = \lim_{y \to 0^+} A(y) + \sum_{k=0}^{m-1} \phi_k(y) \beta_k(y),
\]

where \( A = \lim_{y \to 0^+} A(y) \) whenever this limit exists, in which case \( \lim_{y \to 0^+} \phi_k(y) = 0 \), \( 0 \leq k \leq m-1 \), and \( \beta_k(\xi) \), as functions of the continuous variable \( \xi \), are continuous for \( 0 \leq \xi \leq b \), and for some constants \( r_k > 0 \), as \( \xi \to 0^+ \), have Poincaré-type asymptotic expansions of the form

\[
\beta_k(\xi) \sim \sum_{i=0}^{\infty} \beta_{k,i} \xi^{i/r_k}, \quad k = 0,1,\ldots,m-1.
\]

If, in addition, the functions \( B_k(t) = \beta_k(t^{1/r_k}) \), as functions of the continuous variable \( t \), are infinitely differentiable for \( 0 \leq t \leq r_k \), we shall say that \( A(y) \) belongs to the set \( F^{(m)}_{\infty} \).

Remark. If \( \lim_{y \to 0^+} A(y) \) does not exist, then in the nomenclature of...
Shanks (1955), $A$ is said to be the anti-limit of $A(y)$. In this case, for at least one $k$, $\lim_{k \to 0+} \phi_k(y)$ does not exist as is obvious from (1.1) and (1.2).

The problem is to find (or approximate) $A$ whether it is the limit or the anti-limit of $A(y)$ as $y \to Q+$.

**Definition 1.2 (GREP).** Let $A(y) \in F^{(m)}$, for some integer $m > 0$, with the same notation as in Definition 1.1. Denote the vector $(n_0, n_1, \ldots, n_{m-1})$ by $n$. Then $A_n^{(m,j)}$, the approximation to $A$, and the parameters $\delta_{k,i}$, $0 \leq i \leq n_k$, $0 \leq k \leq m-1$, are defined as the solution of the set of linear equations

\begin{equation}
A_n^{(m,j)} = A(y_\ell) + \sum_{k=0}^{m-1} \phi_k(y_\ell) \sum_{i=0}^{n_k} \delta_{k,i} y_\ell^i, \quad j \leq \ell \leq j+N,
\end{equation}

where $N = \sum_{k=0}^{m-1} (n_k+1)$ and $y_0 > y_1 > y_2 > \ldots$ such that $y_\ell > 0$ for all $\ell \geq 0$ and $\lim_{k \to 0} y_\ell = 0$, provided of course that the matrix of the coefficients of Equations (1.3) is non-singular.

**Remark.** The origin of this definition is in the work of Levin and Sidi (1975), which deals with the approximation of some infinite integrals and series. A brief outline of the important results of this work will be given in the next section.

We note that, in general, equations (1.3) have to be solved numerically on a computer by using a linear equation solver. (Only in a few cases can $A_n^{(m,j)}$ be computed in a simple manner and these are the T-transformations of Levin (1973) and ordinary Romberg integration.) In particular, we write equations (1.3) in the form:
(1.4) \[ Q \mathbf{c} = \mathbf{d}, \]

where \( Q \) is the matrix of the linear system (1.3) whose first column is the \((N+1)\)-dimensional vector \((1,1,\ldots,1)\) (\( T \) denotes transpose), \( \mathbf{c} \) is the \((N+1)\)-dimensional vector of unknowns whose first element is \( A_n^{(m,j)} \), and \( \mathbf{d} \) is the vector \((A(y_j), A(y_{j+1}), \ldots, A(y_{j+N}))^T \). Let the first row of \( Q^{-1} \), the inverse of \( Q \), be the vector \((\gamma_0, \gamma_1, \ldots, \gamma_N)\). Then \( \mathbf{c} = Q^{-1}\mathbf{d} \) implies

\[ A_n^{(m,j)} = \sum_{k=0}^{N} \gamma_k A(y_{j+k}), \]

and from \( Q^{-1}Q = I \) it follows that

\[ \sum_{k=0}^{N} \gamma_k = 1. \]

In view of (1.6) and (1.5), \( A_n^{(m,j)} \) seems to be some kind of an average of the \( A(y_i) \). But the weights \( \gamma_k \) of this average depend on the \( \phi_k(y_i) \) in a very complicated manner. In some cases, (see Levin (1973), Levin (1975), Levin and Sidi (1975)) the \( \phi_k(y) \) depend on \( A(y) \), hence the GREP can, in general, be viewed as a "non-linear summability method", (see Section 4).

In the next section we shall give examples of functions belonging to \( F(m) \) and we shall also summarize the basic points of the work of Levin and Sidi (1975) for later use. In Section 3 we shall derive some useful bounds on \( A_n^{(m,j)} \) for two different limiting processes and give some convergence theorems which are based in part on Sidi (1977, Chapter 5).

In Section 4 we shall comment on the two limiting processes of Section 3 in the light of the Silverman-Toeplitz theorem on summability. In Section 5 we shall illustrate the results of Sections 3 and 4 with a numerical example of the use of the results of Levin and Sidi (1975) on infinite integrals.
2. EXAMPLES OF FUNCTIONS IN $F^{(m)}$

Functions belonging to $F^{(m)}$ come up in a natural way in numerical integration through the Euler-MacLaurin sum formula and generalizations of it. In what follows we assume that the function $g(x)$ is infinitely differentiable on $[0,1]$ and define the "generalized trapezoidal rule" approximations to $I = \int_{0}^{1} G(x)dx$, where $G(x) = w(x)g(x)dx$ by the formula:

$$T(h) = h \sum_{j=1}^{n} G\left(\frac{2j-1+\epsilon}{2n}\right), \quad |\epsilon| < 1, \quad nh = 1.$$ 

The following generalizations of the Euler-Maclaurin sum formula are due to Navot (1961, 1962)

a) If $w(x) = x^{\beta}, -1 < \beta < 0$, then for $\alpha = 0$ for example (midpoint rule)

$$I \sim T(h) + h^2 \sum_{k=0}^{\infty} a_k h^{2k} + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k.$$ 

b) If $w(x) = x^{\beta} \log x, -1 < \beta < 0$, then again for $\alpha = 0$

$$I \sim T(h) + h^2 \sum_{k=0}^{\infty} a_k h^{2k} + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k + h^{1+\beta} \log h \sum_{k=0}^{\infty} c_k h^k.$$ 

Similar results for the case $\beta > 0$ for ordinary trapezoidal and Simpson rules have been given by Fox (1967). Fox has also used GREP (of low order) for approximating the singular integrals in (a) and (b) but has not gone as far as developing the method as generally as in Definition 1.2.

The results of Navot (1961, 1962) have been extended by Lyness and Ninham (1967) as follows:
c) If \( w(x) = x^\beta (1-x)^\delta \), \(-1 < \beta < 0\) and \(-1 < \delta < 0\), then

\[
I \sim T(h) + h^{1+\delta} \sum_{k=0}^{\infty} a_k h^k + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k.
\]

\[I \sim T(h) + h^{1+\delta} \sum_{k=0}^{\infty} a_k h^k + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k + h^{1+\beta} \log h \sum_{k=0}^{\infty} c_k h^k.
\]

d) If \( w(x) = x^\beta (1-x)^\delta \log x \), \(-1 < \beta < 0\) and \(-1 < \delta < 0\), then

Generalization of these results to multiple integrals on hypercubes and hyperspheres have been given by Lyness and McHugh (1970) and lately by Lyness (1976).

Recently, two other important examples connected with infinite integrals and series have been given by Levin and Sidi (1975). For future reference their results are summarized below:

**Definition 2.1:** We shall say that a function \( a(x) \), defined for \( x > a \), belongs to the set \( A(\gamma) \) if it is infinitely differentiable for all \( x > a \) and if, as \( x \to \infty \), it has a Poincaré-type asymptotic expansion of the form

\[
a(x) \sim x^\gamma \sum_{i=0}^{\infty} a_i / x^i,
\]

and all its derivatives, as \( x \to \infty \), have Poincaré-type asymptotic expansions which are obtained by differentiating the right hand side of (2.1) term by term.

From this definition it follows that \( A(\gamma) \supset A(\gamma-1) \supset A(\gamma-2) \supset \ldots \)

**Theorem 2.1** Let \( f(x) \) be defined for \( x > a > 0 \), and satisfy a homogeneous linear differential equation of order \( m \) of the form

\[
f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x),
\]
where \( p_k \in A \) but \( p_k \notin A \), such that \( i_k \) are integers satisfying \( i_k \leq k, 1 \leq k \leq m \). Let also

\[
(2.3) \quad \lim_{x \to \infty} p_k^{(i-1)}(x) f^{(k-1)}(x) = 0, \quad 1 \leq k \leq m, \quad 1 \leq i \leq m.
\]

If for any integer \( \ell = -1, 1, 2, 3, \ldots \),

\[
(2.4) \quad \sum_{k=1}^{m} \ell^{(i-1)} \ldots (\ell-k+1) p_k \neq 0,
\]

where

\[
(2.5) \quad p_k = \lim_{x \to \infty} x^{-k} p_k(x), \quad 1 \leq k \leq m,
\]

then

\[
(2.6) \quad \int_a^x f(t) dt = \int_a^x f(t) dt + \sum_{k=0}^{m-1} f^{(k)}(x) x^k \theta_k(x),
\]

where \( \theta_k \in A^{(0)} \) and \( \rho_k \) are integers satisfying

\[
(2.7) \quad \rho_k \leq \max (i_{k+1}, i_{k+2}, \ldots, i_{m+k+1}), \quad 0 \leq k \leq m-1.
\]

It also follows that \( \lim_{x \to \infty} f^{(k)}(x) x^\rho_k = 0, \quad 0 \leq k \leq m-1 \).

**Theorem 2.2** Let the elements of the sequence \( \{f_r\}_{r=1}^\infty \) satisfy a homogeneous linear difference equation of order \( m \) of the form

\[
(2.8) \quad f_r = \sum_{k=1}^{m} p_k(r) \Delta^k f_r,
\]

where \( \Delta \) is the forward difference operator operating on the index \( r \), and \( p_k(x) \), as functions of the continuous variable \( x \), are in \( A \).
but not in $A_{(i_k-1)}$ such that $i_k$ are integers satisfying $1 \leq k \leq m$. Let also $1 \leq k \leq m$. Let also

$$\lim_{r \to \infty} [\Delta^{k-1} P_k(r)] \Delta^{k-1} f_\infty = 0, \ i \leq k \leq m, \ 1 \leq i \leq m.$$ 

(2.9) If for every integer $s = -1,1,2,3,\ldots$, (2.4) holds together with (2.5) then

$$\lim_{R \to \infty} \sum_{k=0}^{m-1} \rho_k \psi_k(R),$$

(2.10) $\forall \ R \geq 1$.

where $\psi_k \in A^{(0)}$ and $\rho_k$ are integers satisfying (2.7). It also follows that $\lim_{R \to \infty} (\Delta^k f_\infty) = 0$, $0 \leq k \leq m-1$.

The proofs of both theorems are by construction and can be found in Levin and Sidi (1975), see also Sidi (1978) for the case $m=1$ of Theorem 2.2. Using these theorems the D- and D^\infty approximations are defined as in Definition 1.2.

**Definition 2.2** Let $f(t)$ be as in Theorem 2.1 with the same notation. The approximation $P_n^{(m,j)}$ to $\int f(t)dt$, where $n$ denotes the vector $(n_0,n_1,\ldots,n_{m-1})$, and the constants $\delta_{k,i}$, $0 \leq i \leq n_k$, $0 \leq k \leq m-1$, are defined as the solution of the linear equations

$$\sum_{k=0}^{m-1} \rho_k \sum_{i=0}^{n_k} \delta_{k,i} x_i = 0, \ j \leq x_i \leq j+N,$$

(2.11) where $N = \sum_{k=0}^{m-1} (n_k+1)$ and $a < x_0 < x_1 < \ldots$, such that $\lim_{x \to \infty} x_i = \infty$, provided the matrix of equations (2.11) is non-singular. (If the $\rho_k$ are not known exactly, then they can be replaced in (2.11) by $\sigma_k = \min(k+1,s_k)$, where $s_k = \max{s} \ | \ s \ integer, \lim_{x \to \infty} x^s f(k)(x) = 0}$.  

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Then \( \rho_k \leq \sigma_k \leq k+1 \) and \( \lim_{x \to \infty} f^{(k)}(x) x^k = 0, 0 \leq k \leq m-1 \). The finite integrals \( \int_a^b f(t) \, dt \) can be computed very accurately by using a low order Gaussian rule.

**Definition 2.3** Let the sequence \( \{f_r\}_{r=1}^\infty \) be as in Theorem 2.2 with the same notation. The approximation \( d_n^{(m,j)} \) to \( \sum_{r=1}^{\infty} f_r \), where \( n \) denotes the vector \( (n_0, n_1, \ldots, n_{m-1}) \), and the constants \( \psi_k, 0 \leq i \leq n_k, 0 \leq i \leq m-1 \), are defined as the solution of the linear equations

\[
(2.12) \quad d_n^{(m,j)} = \sum_{r=1}^{R_n-1} f_r + \sum_{k=0}^{m-1} (\Lambda^k f_{R_n}) R_n^k \sum_{i=0}^{n_k} \psi_{k,i} R_{j+k}^{1/4}, \quad j \leq \ell \leq j+N,
\]

where \( N = \sum_{k=0}^{m-1} (n_k + 1) \) and \( 1 \leq R_0 < R_1 < \ldots \), provided the matrix of equations (2.12) is non-singular. (If the \( \rho_k \) are not known exactly, then they can be replaced in (2.12) by \( \sigma_k = \min(k+1, s_k) \), where \( s_k = \max \{ s \mid s \text{ integer, } \lim_{R \to \infty} s R^k (\Lambda^k f_{R}) = 0 \} \). Then \( \rho_k \leq \sigma_k \leq k+1 \) and \( \lim_{R \to \infty} (\Lambda^k f_{R}) R_k = 0, 0 \leq k \leq m-1 \).)

It is obvious that the processes described by the approximations \( D_n^{(m,j)} \) and \( d_n^{(m,j)} \) are exactly the GREP defined in Definition 1.2, provided one lets \( y = 1/x \), \( A(y) \equiv \int_{-\infty}^{\infty} f(t) \, dt \), and \( \phi_k(y) = f^{(k)}(x) x^k \) in Def. 2.2 and \( y = 1/R \), \( A(y) \equiv \sum_{r=1}^{\infty} f_r \), and \( \phi_k(y) = (\Lambda^k f_{R}) R_k \), in Def. 2.3.

It is worth noting that the \( D- \) and \( d- \) approximations have proved to be extremely efficient for accelerating the convergence of infinite integrals and series of different kinds which could not be handled by the well known methods of Euler, see Bromwich (1942, p. 62), Shanks (1955), the G-transformations of Gray, Atchison and McWilliams (1971). For numerical
examples of varying degree of complexity, see Levin and Sidi (1975). The d-approximations for the case \( m = 1 \), are originally due to Levin (1973) and some aspects of their convergence theory have been analyzed in Sidi (1978, 1979). Also the case \( m = 1 \) of the D-approximations for Fourier integrals is due to Levin (1975).

3. ERROR BOUNDS AND CONVERGENCE THEOREMS

In this section we shall analyze the convergence properties of \( A_{m,j}^{(m,j)} \) for two kinds of limiting processes:

a) Process I: \( n \) fixed, \( j \to \infty \).

b) Process II: \( j \) fixed, \( n \to \infty \), i.e., \( n_k \to \infty \), \( k = 0, \ldots, m-1 \).

We shall be using the notation of Definitions 1.1 and 1.2, and for convenience we shall denote \( \mu = m-1 \).

If the equations in (1.3) are solved using Cramer's rule, then for \( A_{m,j}^{(m,j)} \) we obtain \( A_{m,j}^{(m,j)} = \frac{\text{det } M}{\text{det } K} \) where \( M \) and \( K \) are \( (N+1) \times (N+1) \) matrices. The \( (k+1) \text{th column of } M \) is the \( (N+1) \)-dimensional vector

\[
(A(y_{j+l}), \phi_0(y_{j+l})v_0^n(y_{j+l}), \phi_1(y_{j+l})v_1^n(y_{j+l}), \ldots)
\]

where \( T \) denotes transpose and \( v_k(y) \) are the \((s+1)\)-dimensional row vectors given by

\[
v_k(y) = (1, y, y^2, \ldots, y^s), \quad k = 0, \ldots, \mu.
\]
For example, for \( m = 2 \) \((\mu = 1)\), \( n_0 = 1, n_1 = 2 \), the matrix \( M \) takes the form

\[
M = \begin{bmatrix}
A(y_j) & A(y_{j+1}) & \cdots & A(y_{j+5}) \\
\phi_0(y_j) & \cdots & \cdots & \phi_0(y_{j+5}) \\
\phi_0(y_j)y_j & \cdots & \cdots & \phi_0(y_{j+5})y_{j+5} \\
\phi_1(y_j) & \cdots & \cdots & \phi_1(y_{j+5}) \\
\phi_1(y_j)y_j & \cdots & \cdots & \phi_1(y_{j+5})y_{j+5} \\
2r_1 & \cdots & \cdots & 2r_1
\end{bmatrix}
\]

The matrix \( K \) is obtained from \( M \) by replacing the first row of \( M \) by the \((N+1)\)-dimensional vector \((1,1,\ldots,1)\).

If we now denote the cofactor of \( A(y_{j+\xi}) \) in the first row of \( M \) by \( \delta_{\xi} \) and expand \( \det M \) and \( \det K \) with respect to their first rows, we can write

\[
A_{n}^{(m,j)} = \frac{\sum_{\xi=0}^{N} \delta_{\xi} A(y_{j+\xi})}{\sum_{\xi=0}^{N} \delta_{\xi}}
\]

From (3.3) and (1.5) it is clear that

\[
\gamma_{\xi} = \frac{\delta_{\xi}}{\sum_{i=0}^{N} \delta_{i}}, \quad \xi = 0, 1, \ldots, N,
\]

and (1.6) is again seen to be trivially satisfied.
Lemma 3.1 The error in the approximation $A_n^{(m,j)}$ satisfies the equality

\begin{equation}
A - A_n^{(m,j)} = \sum_{\ell=0}^{N} \gamma_{\ell} \sum_{k=0}^{\mu} \phi_{k}(y_{j+\ell}) \beta_{k}(y_{j+\ell}).
\end{equation}

Proof. The result follows by substituting (1.1) in (1.5) and using (1.6).

Corollary. With the help of (3.4), (3.5) can be re-expressed in the form

\begin{equation}
A - A_n^{(m,j)} = \frac{\sum_{\ell=0}^{N} \delta_{\ell}}{\sum_{\ell=0}^{N}} = \frac{\det M_1}{\det K},
\end{equation}

where $M_1$ is the matrix obtained from $M$ by replacing the first row of $M$ by the row vector

\begin{equation}
\left( \sum_{k=0}^{\mu} \phi_{k}(y_{j}) \beta_{k}(y_{j}), \ldots, \sum_{k=0}^{\mu} \phi_{k}(y_{j+N}) \beta_{k}(y_{j+N}) \right).
\end{equation}

For future reference we shall number the 2nd, ..., (N+1)th rows of the matrix $M_1$ (and/or $K$) with two indices as follows: We shall give the 2nd row the indices $(0,0)$, the 3rd row, the indices $(0,1)$, ..., the $(n_0+2)$th row the indices $(0,n_0)$, In the same manner we shall give the next $(n_1+1)$ rows the indices $(1,0)$, $(1,1)$, ..., $(1,n_1)$, etc. Then the last $(n+1)$ rows will have the indices $(\mu,0)$, ..., $(\mu,n_\mu)$. Thus the row $(\phi_k(y_{j})y_{j}^k, \ldots, \phi_k(y_{j+N})y_{j+N}^k)$ has the indices $(k, i)$.

Starting with (3.6) we shall now analyze the convergence properties of the two limiting processes defined in the beginning of this section.
a) Process I.

Theorem 3.1 The approximation $A_n^{(m,j)}$ satisfies

\begin{equation}
A - A_n^{(m,j)} = \sum_{l=0}^{N} \sum_{k=0}^{\mu} \phi_k(y_{j+l}) w_k^{k-1}(y_{j+l}), \tag{3.8}
\end{equation}

where

\begin{equation}
w_s^{k}(y) = \beta_k(y) - \sum_{i=0}^{s} \beta_{k,i} y^i, \quad k = 0, \ldots, \mu, \tag{3.9}
\end{equation}

with $\beta_{k,i}$ as defined in (1.2).

Proof. Let us subtract from the first row of $M_1$ the sum of the products of the rows $(k,i)$ by $\beta_{k,i}$, $i = 0, 1, \ldots, n_k$, $k = 0, 1, \ldots, \mu$, and leave the $2$nd, $\ldots$, $(N+1)$th rows unchanged. Let us denote the new matrix by $M'$. The first row of $M'$, by (3.7) and (3.9), is

\begin{equation}
\left( \sum_{k=0}^{\mu} \phi_k(y_j) w_k^{k-1}(y_j), \ldots, \sum_{k=0}^{\mu} \phi_k(y_{j+N}) w_k^{k-1}(y_{j+N}) \right) \tag{3.10}
\end{equation}

and furthermore $\det M' = \det M_1$, hence $A - A_n^{(m,j)} = \det M'/\det K$. If we now use the fact that the cofactors of the first row of $M'$ are still the $\delta_{k}$ and expand $\det M'$ and $\det K$ with respect to their first rows, (3.8) follows.

Remark. The assumption that $\beta_k(y)$ have Poincaré-type asymptotic expansions implies that

$$w_s^{k}(y) = 0(y^{s+1} x_k) \quad \text{as } y \to 0^+. \tag{3.8}$$

This together with (3.8) implies that
which shows that $A^{(m,j)}$ is indeed the generalization of the Richardson extrapolation process.

The following result can now be easily obtained from (3.8).

**Corollary 1.** The approximation $A^{(m,j)}_n$ satisfies the inequality

$$\frac{A^{(m,j)}_n}{A^{(m,j)}} \leq \sum_{k=0}^{\mu} \gamma_{j,k} \max_{j \leq i \leq j+\mu} |\phi_k(y_j)| \max_{j \leq i \leq j+\mu} \left| \frac{\partial}{\partial y_j} \phi_k(y_j) \right|.$$  

**Corollary 2.** As $j \to \infty$, hence as $y_j \to 0^+$, $A^{(m,j)}_n$ satisfies the inequality

$$\left| A^{(m,j)}_n - A^{(m,j)} \right| \leq \sum_{k=0}^{\mu} \gamma_{j,k} \epsilon_k(j) \left( n_{k+1} + 0(y_j) \right),$$

where $\epsilon_k(j) = \max_{j \leq i \leq j+\mu} |\phi_k(y_j)|$, and if $\lim_{y \to 0^+} A(y) = A$, then

$$\lim_{y \to 0^+} \epsilon_k(j) = 0,$$

and as $j \to \infty$.

$$\left| A^{(m,j)}_n - A^{(m,j)} \right| \leq \sum_{k=0}^{\mu} \gamma_{j,k} \alpha(y_j) \text{ as } j \to \infty,$$

where

$$\alpha = \min \left\{ \left( n_{k+1} + 0(y_j) \right) \right\}.$$  

**Proof.** (3.13) follows from (3.12) easily if we recall that

$$w_k(y) = \beta_k s + 0(y),$$

as $y \to 0^+$, since (1.2) is a Poincaré-type asymptotic expansion, and also that $y_j > y_{j+1} > \ldots$. From Definition 1.1, if $\lim_{y \to 0^+} A(y) = A$, then $\lim_{y \to 0^+} \phi_k(y) = 0$, in which case $\epsilon_k(j) = o(1)$ as $y \to 0^+$, $j \to \infty$. Using this together with (3.15), (3.14) now follows.
As an immediate consequence of Corollary 2, we obtain the following:

**Corollary 3:** If \( \lim_{y \to 0^+} A(y) = A \), and

\[
(3.16) \quad \sup_{j} \left( \sum_{l=0}^{N} |f_l(y)| \right) \leq L < \infty, \tag{3.16}
\]

then \( |A_n^{(m,j)} - A| \to 0 \) as \( j \to \infty \) and the rate of convergence is given by

\[
(3.17) \quad |A_n^{(m,j)} - A| = o(y^a), \tag{3.17}
\]

at least.

**Remark.** If \( \phi_k(y) = o(y^k) \) as \( y \to 0^+ \) for some constants \( r_k \) and if

\[
(3.16) \quad \tau_k = r_k + (n_k + 1)r, \quad 0 \leq k \leq \mu, \quad \text{as can be seen from (3.13). This means that}
\]

if \( n_k \) are sufficiently large such that \( r_k > 0 \) for all \( k \), then as \( j \to \infty \)

\( A_n^{(m,j)} \to 0 \) provided (3.16) is satisfied, whether \( \lim_{y \to 0^+} A(y) \) exists

or not. (For an example of this situation see Sidi (1978).)

b) Process II.

**Theorem 3.2** The approximation \( A_n^{(m,j)} \) satisfies

\[
(3.18) \quad A_n^{(m,j)} = \sum_{l=0}^{N} f_l(y) \sum_{k=0}^{\mu} \phi_k(y_j + \xi) u_k^{(y_j + \xi)} n_k^{(y_j + \xi)}, \tag{3.18}
\]

where

\[
(3.19) \quad u_k^{(y)} = \beta_k(y) - \pi_k^{(y)}, \quad k = 0, \ldots, \mu, \tag{3.19}
\]

such that
\begin{equation}
\kappa_s(y) = \sum_{i=0}^{s} a_{s,i} y^i
\end{equation}

is the best polynomial approximation of degree \( s \) to \( \beta_k(y) \) in
powers of \( y^k, 0 \leq k \leq \mu \), on the interval \([0,y_j]\).

**Proof.** Let us subtract from the first row of \( M_1 \) the sum of the
products of the rows \((k,i)\) by \( a_{n_k,i} \) \( i = 0, 1, \ldots, n_k \),
and leave 2nd, \ldots, \((N+1)th\) rows unchanged. Let us denote the new matrix
by \( M'' \). The first row of \( M'' \), by (3.19) and (3.20), is

\begin{equation}
\left( \sum_{k=0}^{\mu} \phi_k(y_j) y^k_n \right), \ldots, \left( \sum_{k=0}^{\mu} \phi_k(y_{j+N}) y^k_n \right),
\end{equation}

and furthermore, \( \det M'' = \det M_1 \). (3.18) is now obtained by expanding \( \det M'' \)
and \( \det K \) with respect to their first rows.

The following results can easily be derived from (3.18).

**Corollary 1.** The approximation \( A_n^{(m,j)} \) satisfies the inequality

\begin{equation}
|A - A_n^{(m,j)}| \leq \left( \sum_{k=0}^{N} |y_k^j| \right) \sum_{k=0}^{\mu} \left( \max_{k \leq j \leq j+N} |\phi_k(y_j)| \right) \left( \max_{j \leq i \leq \infty} |u^k_n(y_i)| \right).
\end{equation}

**Corollary 2.** If \( A(y) \in F_{n}^{(m)} \) (see Definition 1.1), then as \( n_k \to \infty \),

\begin{equation}
|A - A_n^{(m,j)}| \leq \left( \sum_{k=0}^{N} |y_k^j| \right) \sum_{k=0}^{\mu} n_k(j) \omega(n_k^{-\lambda_k}), \text{ any } \lambda_k > 0,
\end{equation}

where \( n_k(j) = \max_{j \leq i \leq j+N} |\phi_k(y_i)| \), \( 0 \leq k \leq \mu \), and if \( \lim_{y \to 0+} A(y) = A \), then

\begin{equation}
|A - A_n^{(m,j)}| \leq \left( \sum_{k=0}^{N} |y_k^j| \right) \omega(v^{-\lambda}) \text{ and } \lambda > 0, \text{ where}
\end{equation}

\[ v = \min\left(n_0, n_1, \ldots, n_{\mu}\right). \]
Proof. The proof of (3.23) follows from (3.22) and the fact that
\[ \max |u_s^k(y)| = o(s^{-\lambda}) \] as \( s \to \infty \), for any \( \lambda > 0 \), which is a standard result of approximation theory. The proof of (3.24) follows from (3.23) and the fact that \( \lim_{y \to 0^+} \phi_k(y) = 0 \), hence \( \phi_k(y) = o(1) \) for \( 0 \leq y \leq y_j \), \( 0 \leq k \leq \mu \).

Corollary 3: If \( A(y) \in F^m_{\infty} \) and \( \lim_{y \to 0^+} A(y) = A \), and
\[ (3.25) \sup_{n_0, n_1, \ldots, n_\mu} \left( \sum_{\ell=0}^{N} |y_n^{(\ell)}| \right) = L < \infty, \]
then, as \( n_k \to \infty, 0 \leq k \leq \mu \),
\[ (3.26) |A-A_n^{(m,j)}| = o(v^{-\lambda}), \] for any \( \lambda > 0 \).

Remark: If \( A(y) \in F^m_{\infty} \), \( \phi_k(y) = o(v^{y/\lambda}), y \to 0^+ \), \( n_k = o(v^{y/\lambda}) \) as \( n \to \infty \), \( 0 \leq k \leq \mu \), and if \( \lim_{y \to 0^+} A(y) \) does not exist, i.e., \( \lim_{y \to 0^+} \phi_k(y) = \infty \), for at least one \( k \), then (3.24) still holds provided \( y_1 \) are chosen such that \( y_1 = o(v^{y/\lambda}) \) for some \( p < 0 \). If furthermore (3.25) is satisfied, then (3.26) holds too.

There are two immediate practical conclusions that one can draw from Theorems 3.1 and 3.2 and their corollaries.

1. The smaller \( \sum_{\ell=0}^{N} |y_n^{(\ell)}| \) the smaller the error bounds are expected to be in (3.12) and (3.22). Now \( \sum_{\ell=0}^{N} |y_n^{(\ell)}| \geq 1 \), therefore one should adjust the \( y_1 \) in (1.3) such that \( \sum_{\ell=0}^{N} |y_n^{(\ell)}| \) will be small and as close to 1 as possible. The implications of this from the numerical point of view will be taken up in Section 6.

2. As can be seen from the corollaries to Theorems 3.1 and 3.2, the fact that \( u_s^k(y) \to 0 \) as \( s \to \infty \) much faster than \( w_s^k(y) \to 0 \) as \( y \to 0^+ \), suggests that Process II would have much better convergence properties than Process I.
Both of these conclusions seem to be correct as a large number of numerical examples of various kinds have shown. One such example will be given in Section 5. For a theoretical verification of the last conclusion for Levin's transformations see Sidi (1978, 1979).

Finally, we note that it is difficult to check rigorously under what circumstances conditions (3.16) and (3.25) hold. (As a matter of fact, in general, no simple expression for the $\gamma_{\ell}$ is available.) However, in some special cases the behavior of $\sum_{\ell=0}^{N} |\gamma_{\ell}|$ can be analyzed quite simply, see Sidi (1979). We shall say more on this point in the next section.

Before we close this section, we shall give another closed expression for the error $A-A_n^{(m,j)}$.

Theorem 3.3: Let the function $\beta_k(\xi)$ in Definition 1.2 be such that $\beta_k(\xi) \equiv \beta_k(\xi)\xi^{-r_k}$ are of the form

$$\beta_k(\xi) = L[w_k(t)\xi^{-r_k}] = \int_{0}^{\infty} e^{-t/\xi} \omega_k(t)dt, \quad k = 0, 1, \ldots, \mu,$$

(i.e., they are Laplace transforms), where the functions $\omega_k(t)$ are infinitely differentiable on $[0, \infty)$.

Then the error satisfies

$$A-A_n^{(m,j)} = \sum_{\ell=0}^{N} \gamma_{\ell} \sum_{k=0}^{\mu} \phi_k(y_{j+\ell})y_{j+\ell} L[w_k(t); y_{j+\ell}].$$

Proof. From Laplace transform theory we have

$$\beta_k(\xi) = \frac{\beta_k(\xi)}{\xi^{-r_k}} = \xi^{-r_k} L[w_k(t); \xi^{-r_k}] + \sum_{i=0}^{n_k} \omega_k(i)\xi^{-r_k},$$

$$k = 0, 1, \ldots, \mu.$$
Now, subtracting from the first row of the matrix $M_1$ the sum of the products of the rows $(k,i)$ by $\omega_k^{(i)}(0)$, $i = 0,1,\ldots,n_k$, $k = 0,1,\ldots,\mu$, and repeating the arguments which lead to (3.8) and (3.18) and using (3.29) the result follows.

**Remark.** Using Watson's lemma in (3.27), it is easy to identify the $\omega_k^{(i)}(0)$ as $\beta_{k,i}$, $i = 0,1,\ldots$, $k = 0,1,\ldots,\mu$.

We note that Theorem 3.3 is the generalization of Theorem 4.1 in Sidi (1978). The latter has enabled the author to prove convergence theorems on Process II for the $d_{n}^{(1,j)}$ approximations (or Levin's $T$-transformations), which are more powerful than Theorem 3.2. Applications of Theorem 3.3 in connection with the $D$-transformation for Fourier and Hankel transforms will be taken up in a future paper.
4. GREP AS A SUMMABILITY METHOD

Definition 4.1: The infinite matrix

\[
\Lambda = \begin{bmatrix}
\lambda_{00} & \lambda_{01} & \lambda_{02} & \cdots \\
\lambda_{10} & \lambda_{11} & \lambda_{12} & \cdots \\
\lambda_{20} & \lambda_{21} & \lambda_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\] (4.1)

is said to be regular if for every convergent sequence \( \{z_k\}_{k=0}^{\infty} \) of numbers, the sequence \( \{s_n\}_{n=0}^{\infty} \), where \( s_n = \sum_{k=0}^{\infty} \lambda_{nk} z_k \), converges and to the same limit.

Theorem 4.1 (Silverman-Toeplitz). The infinite matrix \( \Lambda \) in (3.1) is regular if and only if

1) \( \lim_{n \to \infty} \lambda_{nk} = 0 \) for all \( k \),

2) \( \lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{nk} = 1 \),

3) \( \sup_{n} \sum_{k=0}^{\infty} |\lambda_{nk}| \leq L < \infty \) for some \( L > 0 \).

The proof of this theorem can be found in Powell and Shah (1972, pp. 23-27).

It turns out that for Process I and Process II we can define infinite matrices \( B \) and \( C \) of the form (4.1). For simplicity we take \( n_0 = n_1 = \ldots = n_v = v \) and denote \( A^{(m,j)}_n \) by \( A^{(m,j)}_v \) and the corresponding \( \gamma_{B,j} \) by \( \gamma_{v,j} \).
For Process I we define the matrix \( B \) as follows:

\[
(4.2) \quad b_{j,k} = \begin{cases} 
0 & 0 \leq k < j \quad \text{if } j \neq 0 \\
\gamma_{v,k-j} & j \leq k \leq j+N \\
0 & k > j+N
\end{cases}
\]

As can be seen from (4.2) the matrix \( B \) is a band matrix since \( N \) is fixed and \( A_{v}^{(m,j)} = \sum_{k=0}^{\infty} b_{j,k} A(y_{k}), \quad j = 0,1,2, \ldots \).

For Process II the matrix \( C \) is defined as follows:

\[
(4.3) \quad c_{v,\ell} = \begin{cases} 
0 & 0 \leq \ell < j \quad \text{if } j \neq 0 \\
\gamma_{v,\ell-j} & j \leq \ell \leq j+N, \quad N = m(v+1) \\
0 & \ell > j+N
\end{cases}
\]

From (4.3) it follows that \( C \) is a "stair case" type matrix in that each row has \( m \) nonzero elements more than the previous row, since \( j \) is fixed, and \( N \) increases by \( m \) when \( v \) increases by 1. Also, for this case \( A_{v}^{(m,j)} = \sum_{\ell=0}^{\infty} c_{v,\ell} A(y_{\ell}), \quad v = 0,1,2, \ldots \).

Now the three conditions of Theorem 4.1 are sufficient for the matrix \( A \) to be regular. They become necessary if we require that the sequence \( \{a_{n}\}^{\infty}_{n=0} \) converge to the limit of \( \{z_{k}\}^{\infty}_{k=0} \) for any \( \{z_{k}\}^{\infty}_{k=0} \).

In our case the matrices \( B \) and \( C \) are applied only to very special sequences. This then raises the question whether the matrices \( B \) and \( C \) have to satisfy the conditions of Theorem 3.1 and under what circumstances they do. Now it can easily be verified that the second condition of Theorem 4.1 is automatically satisfied by both \( B \) and \( C \) since

\[
\sum_{k=0}^{\infty} b_{j,k} = \sum_{\ell=0}^{N} \gamma_{\ell} = 1 \quad \text{and} \quad \sum_{\ell=0}^{N} c_{v,\ell} = \sum_{\ell=0}^{N} \gamma_{\ell} = 1.
\]
automatically satisfied by matrix B since \( b_{j,k} = 0 \) for \( j \) large enough and \( k \) fixed. Numerous computations for convergent infinite integrals and series, by using the D and \( d^- \) transformations of Levin and Sidi (1975) have shown that this condition is satisfied for Process II too, although no proof of this is available yet. The same computations have shown that the third condition of Theorem 4.1, which is just (3.16) for Process I and (3.25) for Process II, is satisfied when the functions \( \phi_k(y) \) are all oscillatory as \( y \to 0^+ \) and it is not satisfied when some of the \( \phi_k(y) \) are monotonic as \( y \to 0^+ \) and \( \phi_k(y) \) vary slowly as \( \ell \) increases. No proof of this observation is available yet either. For some simple cases with \( m = 1 \), like the ordinary Romberg integration, see Bauer, Rutishauser, and Stiefel (1963), and the \( t^- \) and \( u^- \)transformations of Levin (1973) (see also Sidi (1978) as applied to oscillatory sequences, all three conditions of Theorem 4.1 can be shown to hold.

Finally, we note that the numerical experience gained by the use of the D and \( d^- \) transformations and some theoretical results in Sidi (1978, 1979) suggest that whether the third condition of Theorem 4.1 is satisfied or not, convergence takes place in both Process I and Process II, in some cases. The numerical rate of convergence, however, depends very strongly on the size of \( \sum_{\ell=0}^{N} |\gamma_{\ell}| \) and/or on the rate at which \( \gamma_{\ell} \to 0 \) as \( n_k \to \infty \), \( k = 1, \ldots, m \), for fixed \( \ell \).

Actually, the following have been observed to be satisfied simultaneously:

1) \( A_n^{(m,j)} \to A \) quickly (both Process I and Process II)

2) \( \gamma_{\ell} \to 0 \) quickly as \( n_k \to \infty \), \( k = 1, \ldots, m \), \( \ell \) fixed (Process II)

3) \( \sum_{\ell=0}^{N} |\gamma_{\ell}| \) is small, and if it increases its increase is slow (Process I and Process II).

The numerical example in the next section will clarify these points further.
5. A NUMERICAL EXAMPLE

We shall now apply the D-transformation of Levin and Sidi to the integral \( \int_{0}^{\infty} J_0(t) \, dt = 1 \), where \( J_0(t) \) is the Bessel function of the first kind of order zero. Now \( f(t) = J_0(t) \) satisfies all the conditions of Theorem 2.1 with \( m = 2, i_0 = -1, i_1 = 0 \) as can be seen from Bessel's equation of order zero, \( f = -(1/t)f' - f'' \). Therefore, a relation of the form (2.6) exists with \( \rho_0 < -1, \rho_1 < 0 \). Actually, see Longman (1959), as \( x \to \infty \),

\[
\int_{x}^{\infty} J_0(t) \, dt = J_0(x) \left( \frac{1}{x} - \frac{1^2 \cdot 3}{x^3} + \frac{1^2 \cdot 3^2 \cdot 5}{x^5} - \ldots \right) \\
+ \left[ J_0(x) \right]' \left( 1 - \frac{1}{x^2} + \frac{3^2}{x^4} - \ldots \right).
\]

Hence \( \rho_0 = -1, \rho_1 = 0 \) exactly, \( \sigma_0 \) and \( \sigma_1 \), by their definition, turn out to be equal to 0. Computing the finite integrals \( \int_{0}^{x} f(t) \, dt \) numerically (and accurately) and solving equations (2.11) we obtain the approximation \( D_n^{(2, j)} \) to \( \int_{0}^{\infty} J_0(t) \, dt = 1 \). We consider, as in Section 4, the approximations with \( n_0 = n_1 = v \) and use the notation therein. For further details the reader is referred to Levin and Sidi (1975).

a) Process I.

In Table 1 we exhibit some of the results obtained for \( D_n^{(2, j)} \) and \( \sum_{\ell=0}^{N} |\gamma_{\nu, \ell}| \) with \( \nu = 2, 4 \), using \( x_{\ell} = 3(\ell+1)/2, \ell = 0, 1, \ldots \).

b) Process II.

In Tables 2a, 2b, and 2c we exhibit some of the results obtained for \( \gamma_{\nu, 0}, \sum_{\ell=0}^{N} |\gamma_{\nu, \ell}| \), and \( D_n^{(2, 0)} \) using \( x_{\ell} = \ell+1, x_{\ell} = 3(\ell+1)/2, x_{\ell} = 2(\ell+1) \), respectively, \( \ell = 0, 1, 2, \ldots \).
\[ \sum_{\ell=0}^{N} |\gamma_{v,\ell}(j)| \]

\[ 1 - D^{(2,1)}_{v} \]

<table>
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<tr>
<th>(v = 2)</th>
<th>(v = 4)</th>
</tr>
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<tr>
<td>(j)</td>
<td>(\sum_{\ell=0}^{N}</td>
</tr>
<tr>
<td>1</td>
<td>2.662</td>
</tr>
<tr>
<td>3</td>
<td>2.006</td>
</tr>
<tr>
<td>5</td>
<td>1.714</td>
</tr>
<tr>
<td>7</td>
<td>1.547</td>
</tr>
<tr>
<td>9</td>
<td>1.443</td>
</tr>
</tbody>
</table>

**TABLE 1.**

\[ x_{\ell} = \ell+1, \ \ell = 0, 1, 2, \ldots \]

\[ \gamma_{v,0}^{(0)} \]

\[ \sum_{\ell=0}^{N} |\gamma_{v,\ell}^{(0)}| \]

\[ 1 - D^{(2,0)}_{v} \]

| \(v\) | \(\gamma_{v,0}^{(0)}\) | \(\sum_{\ell=0}^{N} |\gamma_{v,\ell}^{(0)}|\) | \(1-D^{(2,0)}_{v}\) |
|-------|-----------------|-----------------|-----------------|
| 1 | \(2.5 \times 10^{-1}\) | \(3.16 \times 10^{1}\) | \(2 \times 10^{-2}\) |
| 3 | \(-4.1 \times 10^{-4}\) | \(1.09 \times 10^{1}\) | \(2 \times 10^{-4}\) |
| 5 | \(7.2 \times 10^{-7}\) | \(4.11 \times 10^{2}\) | \(2 \times 10^{-6}\) |
| 7 | \(6.2 \times 10^{-7}\) | \(6.93 \times 10^{5}\) | \(3 \times 10^{-7}\) |
| 9 | \(-1.0 \times 10^{-12}\) | \(4.30 \times 10^{3}\) | \(1 \times 10^{-8}\) |

**TABLE 2a**
Table 3 exhibits part of the matrix C for Process II (see previous section) obtained by using \( x_\ell = 3(\ell + 1)/2, \ell = 0, 1, 2, \ldots \).
Let us now compare Process I and Process II with the help of Tables 1 and 2b, which have been computed by taking \( x_k = 3(k+1)/2, \ k = 0, 1, \ldots \), hence by using the same sequence of finite integrals \( \int_0^{x_k} J_0(t)dt \). As is seen the rate of convergence for Process II is much greater than that of Process I. Also if we compare two approximants, one from each table, whose computations are done by using about the same number of finite integrals, like \( D^{(2)}_{-2} \) in Table 1 (16 finite integrals) and \( D^{(2,0)}_{-2} \) (17 finite integrals), we see that Process II is superior to Process I in this kind of comparison too. This second kind of comparison becomes especially meaningful for the \( d \)-transformation for infinite series, since it implies that given a finite number of terms of the series, Process II gives much better accuracy than Process I. This observation is in agreement with one of the two conclusions of Section 3.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \gamma_{v,0}^{(0)} )</th>
<th>( \gamma_{v,1}^{(0)} )</th>
<th>( \gamma_{v,2}^{(0)} )</th>
<th>( \gamma_{v,3}^{(0)} )</th>
<th>( \gamma_{v,4}^{(0)} )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1.5 \times 10^{-2} )</td>
<td>( 4.3 \times 10^{-2} )</td>
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<td>( 8.2 \times 10^{-2} )</td>
<td>( 5.4 \times 10^{-1} )</td>
</tr>
<tr>
<td>3</td>
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<td>( 1.1 \times 10^{-2} )</td>
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</tr>
<tr>
<td>5</td>
<td>( 2.0 \times 10^{-8} )</td>
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<tr>
<td>7</td>
<td>( 7.4 \times 10^{-12} )</td>
<td>( -1.2 \times 10^{-10} )</td>
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<td>( -3.3 \times 10^{-6} )</td>
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</tr>
<tr>
<td>9</td>
<td>( 1.6 \times 10^{-15} )</td>
<td>( -3.2 \times 10^{-12} )</td>
<td>( 1.8 \times 10^{-9} )</td>
<td>( -4.4 \times 10^{-8} )</td>
<td>( 1.9 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

**TABLE 3**
6. REMARKS ON COMPUTATIONAL ASPECTS

It turns out that the matrix $Q$ of Equations (1.3), as the $n_k$ become larger, becomes very ill-conditioned. This causes the computed values of the $p_{k,l}$ to be very inaccurate. However, the accuracy of the computed value of $A_n^{(m,j)}$ seems to be unaffected by the ill-conditioning of $Q$. What does seem to have an effect on the accuracy of the computed value of $A_n^{(m,j)}$ is the size of $\sum_{l=0}^{N} |\gamma_{l}|$, the same quantity that affects the error $A-A_n^{(m,j)}$ in the true approximation. If we let $A_n^{(m,j)}$ be the computed value of $A_n^{(m,j)}$, then $|A-A_n^{(m,j)}|$ and $|A_n^{(m,j)}-\tilde{A}_n^{(m,j)}|$ increase (decrease) simultaneously as $\sum_{l=0}^{N} |\gamma_{l}|$ increases (decreases).

The effect of $\sum_{l=0}^{N} |\gamma_{l}|$ on $|A_n^{(m,j)}-\tilde{A}_n^{(m,j)}|$ can be explained to some extent as follows: Suppose that the $\gamma_{s}$ have been computed with an error of $\epsilon_{s}$, $s = 0,1,2,\ldots$. Then

$$\tilde{A}_n^{(m,j)} = \sum_{l=0}^{N} \gamma_{l} [A_n^{(m,j)+\epsilon_{l}}]$$

which together with (1.5) implies

$$|A_n^{(m,j)} - \tilde{A}_n^{(m,j)}| \leq \left( \sum_{l=0}^{N} |\gamma_{l}| \right) \max_{j \leq s \leq j+N} |\epsilon_{s}|.$$

Hence if $A(y_s)$ are of the same order of magnitude and have $r$ correct significant decimal digits, then $\epsilon_{s}/A(y_s) \sim 10^{-r+1}$, and if $A_n^{(m,j)}$ is of the same order of magnitude as the $A(y_s)$ and $\sum_{l=0}^{N} |\gamma_{l}| \sim 10^{q}$ for some integer $q \geq 0$ then $|A_n^{(m,j)}-\tilde{A}_n^{(m,j)}|/A_n^{(m,j)} \sim 10^{-r+1+q}$, i.e., $\tilde{A}_n^{(m,j)}$ has $r-q$ correct significant decimal digits. Now if $A_n^{(m,j)}$
agrees with \( A \) in the first \( q' \) significant decimal digits and \( q' \leq r-q \), then \( \tilde{A}_{n}^{(m,j)} \) can be taken as \( A_{n}^{(m,j)} \).

If \( r \) is large enough, that is the \( A(y_{s}) \) have been computed sufficiently accurately, then even if \( \sum_{k=0}^{N} |y_{k}| \) may be large, \( \tilde{A}_{n}^{(m,j)} \) will be sufficiently accurate to be taken as \( A_{n}^{(m,j)} \).

From what has been said above we can conclude that if the vector \( \gamma = (\gamma_{0}, \gamma_{1}, \ldots, \gamma_{N})^{T} \) is known, then we can simultaneously (1) compute \( \tilde{A}_{n}^{(m,j)} \), and (2) through \( \sum_{k=0}^{N} |y_{k}| \) obtain an estimate of the correct number of significant figures in \( \tilde{A}_{n}^{(m,j)} \), the computed value of \( A_{n}^{(m,j)} \).

The vector \( \gamma \) can be found by solving the set of linear equations

\[
Q^{T} \gamma = e_{1},
\]

where \( e_{1} = (1, 0, 0, \ldots, 0)^{T} \). Therefore, the amount of computing to be done for determining \( \tilde{A}_{n}^{(m,j)} \) and \( \gamma \) is the same as that for \( A_{n}^{(m,j)} \) and the \( \tilde{b}_{k,i} \).
7. FURTHER RESULTS ON GREP

So far we have been concerned with $A_n^{(m,j)}$ and have given bounds on $|A_n^{(m,j)}|$. It turns out that the $\bar{\beta}_{k,1}$ are approximations to the $\beta_{k,1}$ and tend to them in the limit for both Process I and Process II.

We start by solving equation (1.3) for $\bar{\beta}_{p,q}$ for $0 \leq p \leq m$ and $0 \leq q \leq n$. Using Cramer's rule again the result is $\bar{\beta}_{p,q} = \det \tilde{M} / \det K$, where $K$ is as described in Section 3 and $\tilde{M}$ is the matrix obtained from $K$ by replacing the $(p,q)$th row of $K$ by the row vector in (3.1). Let us denote the cofactor of $A(y_{j+\ell})$ in the $(p,q)$th row of $\tilde{M}$ by $\delta_{\ell}$, $\ell = 0,1,\ldots,N$. Then expanding $\det \tilde{M}$ and $\det K$ with respect to their $(p,q)$th rows we obtain

$$ -\bar{\beta}_{p,q} = \frac{1}{\det K} \sum_{\ell=0}^{N} \delta_{\ell} A(y_{j+\ell}) $$

from which we immediately identify

$$ -\bar{\gamma}_{\ell} = \frac{\delta_{\ell}}{\det K}, \quad \ell = 0,1,\ldots,N, $$

where $(\bar{\gamma}_0,\bar{\gamma}_1,\ldots,\bar{\gamma}_N)$ is that row of $Q^{-1}$ which corresponds to $\bar{\beta}_{p,q}$ in the vector $c$ in (1.4), i.e.,

$$ \bar{\beta}_{p,q} = \sum_{\ell=0}^{N} \bar{\gamma}_{\ell} A(y_{j+\ell}). $$

Lemma 7.1 $\bar{\beta}_{p,q}$ satisfies

$$ \beta_{p,q} - \bar{\beta}_{p,q} = \sum_{\ell=0}^{N} \bar{\gamma}_{\ell} \left[ \sum_{k=0}^{m} \phi_k(y_{j+\ell}) \beta_k(y_{j+\ell}) - \beta_{p,q} \phi_p(y_{j+\ell}) y_{j+\ell} \right]. $$
Proof. Substituting (1.1) in (7.1) we obtain

\[ \bar{\beta}_{p,q} = \frac{A \sum_{\ell=0}^{N} \delta_{\ell} \sum_{\ell=0}^{N} \delta_{\ell} \mu_{k} \phi_{k}(y_{j+\ell}) \beta_{k}(y_{j+\ell}) - \sum_{\ell=0}^{N} \delta_{\ell} \phi_{p}(y_{j+\ell}) y_{j+\ell}^{p}}{\sum_{\ell=0}^{N} q_{r}(y_{j+\ell}) y_{j+\ell}^{r}} \]

Now \( \sum_{\ell=0}^{N} \delta_{\ell} = 0 \) since it is just \( \det \bar{K} \), where \( \bar{K} \) is the matrix obtained from \( K \) by replacing its \( (p,q) \)th row by the vector \((1,1,...,1)\), thus giving two identical rows in \( \bar{K} \), namely the first and the \( (p,q) \)th.

(7.3) then follows by adding \( \beta_{p,q} \) to both sides of (7.4) and using (7.2).

Corollary.

(7.5) \( \beta_{p,q} - \bar{\beta}_{p,q} = \det \bar{M}_{1}/\det K \),

where \( \bar{M}_{1} \) is obtained from \( \bar{M} \) by replacing its \( (p,q) \)th row by the vector \((a_{0}, a_{1}, ..., a_{N})\), where

\[ a_{\ell} = \sum_{k=0}^{\mu} \phi_{k}(y_{j+\ell}) \beta_{k}(y_{j+\ell}) - \beta_{p,q} \phi_{p}(y_{j+\ell}) y_{j+\ell}^{p}, \quad \ell = 0,1,...,N. \]

Process I

Theorem 7.1 \( \bar{\beta}_{p,q} \) satisfies the equality

\[ \beta_{p,q} - \bar{\beta}_{p,q} = \sum_{\ell=0}^{N} \gamma_{\ell} \sum_{k=0}^{\mu} \phi_{k}(y_{j+\ell}) \omega_{k}^{k}(y_{j+\ell}), \]

where \( \omega_{k}^{k}(y) \) are as defined in (3.9).
Proof. Let us subtract from the \((p,q)\)th row of \(\tilde{M}_1\) the products of the rows \((k,i)\) by \(\beta_{k,i}\), \(0 \leq i \leq n_k\, 0 \leq k \leq v, \ (k,i) \neq (p,q)\), and leave the rest of the rows unchanged, and call the new matrix \(\tilde{M}'\).

The \((p,q)\)th row of \(\tilde{M}'\) is now given by (3.10). Expanding \(\det \tilde{M}'\) with respect to the \((p,q)\)th row and using (7.2), (7.7) follows.

Process II

Theorem 7.2 \(\bar{\beta}_{p,q} \) satisfies the equality

\[
\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{k=0}^{N} \sum_{\ell=0}^{v} \frac{w_{k}}{y_{j+\ell}} \phi_{k}^{(j+\ell)}(y_{j+\ell}) \phi_{k}^{(j+\ell)}(y_{j+\ell}) \frac{y_{j+\ell}}{y_{p}} u_{n_p,q}(y_{j+\ell}) \]

where \(u_{s}(y)\) are as defined in (3.19) and (3.20), and \(u_{n_p,q}(y)\) is the best polynomial approximation to \(w_{q}(y)/y^{p}\) in powers of \(y^{p}\) of degree \(n_p-q-1\), in the interval \([0,y_j]\).

Proof. Similar to those of Theorem 7.1 and Theorem 3.2.

Starting with (7.7) and (7.8) we can give upper bounds for \(|\beta_{p,q} - \bar{\beta}_{p,q}|\) and prove convergence theorems under some special circumstances as we did for \(|A-A_n^{(m,j)}|\) in Section 3.

We now give another result that corresponds to Theorem 3.3.

Theorem 7.3 Let \(A(y)\) be as in Theorem 3.3. Then

\[
\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{k=0}^{N} \sum_{\ell=0}^{v} \frac{w_{k}}{y_{j+\ell}} \phi_{k}^{(j+\ell)}(y_{j+\ell}) \phi_{k}^{(j+\ell)}(y_{j+\ell}) L[w_{k},(t);y_{j+\ell}] \]

Proof. Like that of Theorem 7.1 and Theorem 3.3.
Special cases of Theorems 7.1, 7.2, and 7.3 for the case of the T-transformations of Levin have been used by the present author (see Sidi (1978, 1979)), to prove convergence of $\bar{\beta}_{p,q}$ to $\beta_{p,q}$ for both Process I and Process II.

Note. Numerous computations with Process II have shown that $\bar{\beta}_{p,q} \rightarrow \beta_{p,q}$ as $n_k \rightarrow \infty$, $0 \leq k \leq n$, this convergence being very quick for $q = 0$, less quick for $q = 1$, etc. What happens is that as $q$ increases the $\bar{\gamma}_{k}$ become very large in absolute value and do not have a fixed sign. This, through (7.2a), introduces very severe round-off error propagation in the computation of $\bar{\beta}_{p,q}$. 
REFERENCES


8. I.M. Longman (1959), "A short table of \( \int_0^\infty J_0(t)t^{-n}dt \) and \( \int_0^\infty J_1(t)t^{-n}dt \)," MTAC, 13, pp. 306-311.


