ANALYSIS OF CONVERGENCE FOR SOME NON-LINEAR CONVERGENCE ACCELERATION METHODS

by

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Technical Report #122
October 1978
ABSTRACT

Recently the present author has given some convergence theorems of general nature for Levin's non-linear sequence transformations. In this work these theorems are extended and sharpened to cover the case of power series, both inside and on their circle of convergence. It is shown that one of the two limiting processes considered in the previous work can be used for analytic continuation and a realistic estimate of its rate of convergence is given. Three illustrative examples are also appended.
1. INTRODUCTION AND REVIEW OF RECENT RESULTS

In a recent paper, Sidi (1978), (from here on denoted as (I)), a partial study of the convergence properties of the non-linear sequence transformations due to Levin (1973) has been given. The purpose of the present work is to extend the results given in (I) to cover the case of power series (and Fourier series), and also to improve upon them. Since we shall be using the notation of (I) and its results, we shall give here its notation and, when needed, those results that are relevant to the present work.

Let the sequence \( A_r \), \( r = 1,2, \ldots \), be a convergent infinite sequence whose limit we denote by \( A \). \( T_{k,n} \) (from here on \( T \) for short), the approximation to \( A \), and the constants \( \gamma_i \), \( i = 0,1, \ldots, k-1 \), are defined as the solution of the \( k+1 \) linear equations

\[
\begin{align*}
A_r &= T + R_r \sum_{i=0}^{k-1} \gamma_i / r^i, \quad r = n,n+1, \ldots, n+k,
\end{align*}
\]

where \( R_r \) are preassigned numbers related to the sequence in consideration, see Levin (1973). Equations (1.1) have a simple solution for \( T \) which is given by, see Levin (1973),

\[
\begin{align*}
T &= \left( \sum_{j=0}^{k} (-1)^j (\frac{k}{j})(n+j)^{k-1} A_{n+j}/R_{n+j} \right) / \left( \sum_{j=0}^{k} (-1)^j (\frac{k}{j})(n+j)^{k-1} / R_{n+j} \right)
\end{align*}
\]

(1.2) can also be written in a more compact and revealing form as, see (I),

\[
\begin{align*}
T &= \frac{\Delta_k(n^{k-1} A_n / R_n)}{\Delta_k(n^{k-1} / R_n)}
\end{align*}
\]

(1.3) where \( \Delta \) is the forward difference operator operating on \( n \). Once \( T \) has been computed, the \( \gamma_i \) can be computed recursively from, see Theorem 5.1 in (I),
\[ \sum_{j=0}^{i} \gamma_j \Delta^k(n^{k+i-j}) = \Delta^n[k^{k+i}(A_n - T)/R_n], \quad i = 0, 1, \ldots, k-1, \]

in this order.

We now define two limiting processes for \( T \):

1) \( k \) is held fixed, \( n \to \infty \) (Process I),

2) \( n \) is held fixed, \( k \to \infty \) (Process II).

In the analysis given in (1) it is assumed that the members of the sequence \( \{A_r\} \) satisfy

\[ A_r = A + R_r f(x), \quad r = 1, 2, \ldots, \]

where \( f(x) \), as a function of the continuous variable \( x \), is defined for all \( x > 1 \), including \( x = \infty \), and as \( x \to \infty \), has a Poincaré-type asymptotic expansion in inverse powers of \( x \), given by

\[ f(x) \sim \sum_{i=0}^{\infty} \beta_i / x^i, \quad \text{as } x \to \infty, \beta_0 \neq 0. \]

(For Process II it is also assumed that \( f(x) \) is an infinitely differentiable function of \( x \) for all \( x > 1 \) including \( x = \infty \).)

Remark 1. If the sequence \( A_r \) has the above property, then for Process I, which is the more effective of the two processes, \( T \) converges to \( A \) extremely quickly as various computations in the literature show.

If, on the other hand, the sequence does not possess the above property, then no meaningful results can be expected from \( T \) as computations have shown.

Therefore, the property above seems to be necessary for \( T \) to work at all.

Remark 2. As can be seen easily, if (1.5) and (1.6) are satisfied, then we can express (1.5) in the form:
\( \sum_{r=1}^{\infty} a_r x^r = a + \sum_{r=1}^{\infty} R_r \tilde{f}(r), \quad r = 1, 2, \ldots, \)

where \( R_r = R_r g(r) \), and \( g(x) \), as a function of the continuous variable \( x \), as \( x \to \infty \), has a Poincaré-type asymptotic expansion in inverse powers of \( x \) like that of \( f(x) \) with \( \lim_{x \to \infty} g(x) \neq 0 \). Therefore, \( \tilde{f}(x) = f(x)/g(x) \), has the same properties as \( f(x) \). Thus, by Remark 1, the \( R_r \) in (1.2) can be replaced by \( \tilde{R}_r \) without affecting \( T \) numerically very much.

The plan of this paper is as follows: In Section 2 it is shown that for some power series with finite radius of convergence (1.5) and (1.6) hold. Furthermore, the results of (I) are extended to cover the case of some divergent sequences. In Section 3 Process I is analyzed for the power series considered in Section 2 and convergence theorems for it are proved. In Section 4 a new approach to Process II is presented which makes the analysis of this process more amenable. Using this approach we prove some useful convergence theorems that show, to some extent, the mechanism by which Process II works in some cases including that of power series considered in Section 2, both inside and outside their circle of convergence.

The results of Sections 2, 3 and 4 are illustrated with three interesting examples in Section 5.

2. ASYMPTOTIC EXPANSIONS FOR REMAINDERS OF SOME POWER SERIES AND EXTENSION OF SOME PREVIOUS RESULTS

Our purpose here is to show, with the help of Theorem 6.1 in (I), under what conditions Levin's transformations can be applied to power series. We begin by recalling Theorem 6.1 of (I), which is a special case of a more general theorem given by Levin and Sidi (1975), for future reference.
Theorem 2.1: (see Theorem 6.1 of (I)). Let the sequence \( A_r = \sum_{m=1}^{r} a_m \), 
\( r = 1,2, \ldots \), be such that the terms \( a_r \) satisfy a linear first order homogeneous difference equation of the form:

\[
(2.1) \quad a_r = p(r) \Delta a_r, \quad r = 1,2, \ldots,
\]

where \( p(x) \), considered as a function of the continuous variable \( x \), as \( x \to \infty \), has a Poincaré-type asymptotic expansion in inverse powers of \( x \), of the form:

\[
(2.2) \quad p(x) \sim x^\tau (p_0 + p_1/x + p_2/x^2 + \cdots), \quad p_0 \neq 0,
\]

for \( \tau \) an integer \( \leq 1 \). Let \( \lim_{r \to \infty} A_r = A \), \( A \) finite. Assume

\[
(2.3) \quad \lim_{r \to \infty} p(r) a_r = 0,
\]

and

\[
(2.4) \quad \bar{\rho} \neq 1, \quad \ell = -1,1,2,3, \ldots,
\]

where \( \bar{\rho} = \lim_{x \to \infty} p(x)/x \). Then \( A - A_{r-1} \), as \( r \to \infty \), has an asymptotic expansion of the form:

\[
(2.5) \quad A - A_{r-1} = \sum_{m=r}^{\infty} a_m \sim \Delta a_r \tau (\beta_0^r / r + \beta_1^r / r^2 + \cdots).
\]

Furthermore, from the constructive proof of this theorem it follows that

\[
\beta_0^r = -p_0 / (\bar{\rho} + 1) \neq 0.
\]

If we now subtract \( a_r \) from both sides of (2.5) and rearrange, we obtain

\[
(2.6) \quad A_r \sim A + a_r \tau (\beta_0^r / r + \beta_1^r / r^2 + \cdots),
\]

where
(2.7) \[ \beta_i = \begin{cases} -\beta_i' & \text{if } i \neq \tau \\ -\beta_i' + 1 & \text{if } i = \tau \end{cases}, \quad i = 0, 1, 2, \ldots \]

Remark. It follows from (2.6), (1.5) and (1.6) that a very natural way to choose \( R_\tau \) is by letting \( R_\tau = a_\tau r^\tau \), see Levin (1973).

It turns out that there is a large class of infinite power series satisfying the conditions of Theorem 2.1 as the following theorem shows.

Theorem 2.2: Let \( A_\tau = \sum_{m=1}^{\infty} a_m \tau^m, \tau = 1, 2, \ldots \), and suppose \( a_\tau \) is of the form:

\[ a_\tau = z^{r-1} w(r) \]

where \( z \) is a complex parameter and \( w(x) \), as a function of the continuous variable \( x \), as \( x \to \infty \), has a Poincaré-type asymptotic expansion of the form:

\[ w(x) \sim x^\alpha (w_0 + w_1/x + w_2/x^2 + \ldots), \quad w_0 \neq 0, \]

such that 1) \( \alpha \) can be any real number when \( |z| < 1 \), 2) \(-1 \leq \alpha < 0\) when \( |z| \leq 1 \), \( z \neq 1 \), 3) \( \alpha < -1 \) when \( |z| \leq 1 \). Then all the conditions of Theorem 2.1 are satisfied simultaneously, hence an asymptotic expansion of the form (2.6) exists.

Proof. As can be seen easily from (2.8) and (2.9) the radius of convergence of the infinite series \( \sum_{r=1}^{\infty} a_\tau \) is 1 for all values of \( \alpha \), which explains 1). The convergence of the infinite series on the unit circle with the exception of \( z = 1 \) explains 2). The convergence of the infinite series everywhere on the unit circle explains 3). Hence 1), 2), 3) guarantee the convergence of \( A_\tau \) for the specified ranges of \( z \).
Now the terms satisfy a difference equation of the form (2.1), where \( p(r) \) is simply

\[
(2.10) \quad p(r) = \left( \frac{a_{r+1}}{a_r} - 1 \right)^{-1},
\]

therefore, \( 1/p(x) = zw(x+1)/w(x)-1 \). Now \( 1/p(x) \), as \( x \to \infty \), has a Poincaré-type asymptotic expansion which can be shown to be

\[
(2.11) \quad \frac{1}{p(x)} = (z-1) + \frac{az}{x} + o(x^{-2}).
\]

Hence \( p(x) \), as \( x \to \infty \), has a Poincaré-type asymptotic expansion which is given by

\[
(2.12) \quad p(x) = \begin{cases} 
\frac{1}{(z-1)} - \frac{az}{(z-1)^2}/x + o(x^{-2}), & z \neq 1 \\
\frac{x}{\alpha} + o(1), & z = 1.
\end{cases}
\]

From (2.12) it follows that \( \tau = 0 \), hence \( \bar{p} = 0 \), whenever \( z \neq 1 \) and \( \tau = 1 \) with \( \bar{p} = 1/\alpha \) for \( z = 1 \), where \( \bar{p} \) and \( \tau \) have been defined in Theorem 2.1. Using these last results it is easy to verify that (2.3) and (2.4) are satisfied. This completes the proof of the theorem.

If we now choose \( R_x \) as explained in the remark following Theorem 2.1, then we have

\[
(2.13) \quad R_x = z^{r-1} g(x),
\]

where \( g(x) = w(r)x^r \). Considered as a function of the continuous variable \( x \), as \( x \to \infty \), \( g(x) \) has a Poincaré-type asymptotic expansion of the form

\[
(2.14) \quad g(x) \sim x^{\alpha + \tau} (w_0 + w_1/x + w_2/x^2 + \ldots).
\]

We note that for this case \( R_x \) can also be taken as

\[
(2.15) \quad R_x = z^{r-1} x^{\alpha + \tau}.
\]
which is the dominant term of $A_r F(z)$ as $r$ becomes large, in accordance with Remark 2 in Section 1.

**Remark.** The infinite series $\sum_{r=1}^{\infty} a_r z^{r-1}$ represents an analytic function $F(z)$ in the open disc $|z| < 1$. The point $z = 1$ is usually a point of singularity of this function, a branch point or a pole, while other points on the unit circle are regular points. It turns out usually that the relation

$$ (2.16) \quad A_r = F(z) + R_r f(r) $$

together with (1.6), whose existence, whenever $\lim_{r \to \infty} A_r$ exists, has been proved above, can be continued analytically to the unit circle and its exterior. (Examples of this will be given later.) That is, in some cases (2.16) is valid even when $\lim_{r \to \infty} A_r$ does not exist. (We recall that in Theorem 2.1 we assumed the existence of $\lim_{r \to \infty} A_r$.)

In view of the remark above we now extend Theorems 3.1 and 3.2 in (I) to cover also the case of some non-convergent sequences as follows:

**Theorem 2.3.** Let the sequence $A_r, r = 1, 2, \ldots$ (convergent or not) satisfy (1.5), where $f(x)$ is as explained in Section 1 and satisfies (1.6). If in addition, $R_n = O(n^\alpha)$ for some $\alpha$ as $n \to \infty$, and, for $k > \alpha$, fixed, $\sup_n |\delta^k((-1)^n R_{n-1}/R_n)|/\delta^{k-1}(n^{k-1}/R_n) < \infty$, then $T_A$ as $n \to \infty$; actually $T_A = O(n^{-k+\alpha})$.

This theorem extends Corollary 2 of Theorem 3.1 in (I) and its proof is similar to that given in (I).

**Theorem 2.4.** Let the sequence $A_r, r = 1, 2, \ldots$ (convergent or not), $f(x)$, and $R_n$ be as in Theorem 2.3 and assume that $f(x)$ is infinitely
differentiable for all \( x \geq n \), including \( x = \infty \). If, for \( n \) fixed, 
\[
\sup_k |\delta^k((-1)^n n^{-1}/|R_n|)| / \delta^k(n^{-1}/R_n) | < \infty ,
\]
then \( T \rightarrow A \) as \( k \rightarrow \infty \); 
actually \( T-A = o(k^{-\lambda}) \) for any \( \lambda > 0 \).

This theorem extends Corollary 2 of Theorem 3.2 in (I) and its proof is similar to that given in (I).

As in (I) these last theorems can be applied immediately to oscillatory sequences for which \( R R+1 < 0 , \ r = 1,2, \ldots \), since for these sequences 
\[
|\delta^k((-1)^n n^{-1}/|R_n|)| / \delta^k(n^{-1}/R_n) | = 1 , \text{ see (I),}
\]
giving us an extension of Theorem 4.1 in (I).

**Theorem 2.5** Let the sequence \( A_r , \ r = 1,2, \ldots \), (convergent or not), \( f(x) \), and \( R_n \) be as described in Theorem 2.3 and assume \( R R+1 < 0 , \ r = 1,2, \ldots \). 
Then when \( k > a , k \) fixed, \( T-A = O(n^{-k+\alpha}) \) as \( n \rightarrow \infty \). If in addition \( f(x) \) is infinitely differentiable as described in Theorem 2.4; then, for \( n \) fixed, 
\( T-A = o(k^{-\lambda}) \) as \( k \rightarrow \infty \) for any \( \lambda > 0 \).

The above theorems can now be applied to the power series that have been considered in Theorem 2.2 and the remark following it, inside and on the circle of convergence. Especially when \( z = -1 \), Theorem 2.5 can be applied to the partial sums of the infinite series 
\[
\sum_{m=1}^{\infty} (-1)^{m-1} w(m) , \text{ where } w(m) > 0 \text{ for all } m \text{ and } w(x) \text{ is as in (2.9).}
\]

Although the results of Theorems 2.3-5 are stronger than their predecessors given in (I), they are still not the best, due to their general nature. In the next sections we shall improve on them by making certain (realistic) assumptions about the sequences to which Levin's transformations are applied.
3. APPLICATION OF PROCESS I TO POWER SERIES AND FOURIER SERIES.

The purpose of this section is to extend Theorems 4.2 and 5.2 of (I) which were stated and proved for some monotone sequences, to cover the case of infinite power series such as those that we have considered in the previous section, inside and on the unit circle, taking into account the remark following the proof of Theorem 2.2. Our new results will be stated in slightly more general terms. They seem to be the best that one can obtain under the given conditions.

Theorem 3.1: Let the sequence $A_r, r = 1, 2, \ldots$, (convergent or divergent) depending on the complex parameter $z$, satisfy

$$A_r = F(z) + R_rf(r),$$

where $F(z)$ is a function depending on $z$ such that $\lim_{r \to \infty} A_r = F(z)$ whenever this limit exists, and

$$R_r = z^{r-1}g(r),$$

where $g(x)$, as a function of the continuous variable $x$, when $z \neq 1$ has a Poincaré-type asymptotic expansion of the form

$$g(x) \sim \sum_{i=0}^{\infty} \rho_i x^{i+\sigma}, \quad \rho_0 \neq 0$$

and $f(x)$, considered as a function of the continuous variable $x$, has a Poincaré-type asymptotic expansion of the form $(1.6)$ with the same notation. Let $T$ be as given in $(1.3)$. Then, when $z \neq 1,$

$$T-F(z) = z^{n-1/2} \frac{2k+\sigma}{n} \left[ \mathcal{O}(n^{-1}) \right], \quad n \to \infty.$$
where
\[ p = \rho_0 \beta_k \cdot k! \cdot (1-1/z)^{-k}. \]

**Proof.** Equation (3.6) in (1) reads
\[ T^{-1}(z) = \frac{\Delta_k[n^{-1} w_k(n)]}{\Delta_k(n^{-1}/R_n)} \]
where
\[ w_k(x) = f(x) - \sum_{i=0}^{k-1} \beta_i x^i. \]

Now \[ w_k(x) = \beta_k x^k + O(x^{-k+1}) \] as \( x \to \infty \), therefore \[ x^{-k} w_k(x) = \beta_k / x + O(x^{-2}) \] as \( x \to \infty \), consequently \[ \Delta_k[n^{-1} w_k(n)] = O(n^{-k-1}) \] as \( n \to \infty \). Using the fact that
\[ \Delta_{k-1} = (-1)^{k-1} k! / [n(n+1) \ldots (n+k)], \]
which can easily be proved by induction, we can actually write for the numerator of (3.6)
\[ \Delta_k[n^{-1} w_k(n)] = (-1)^k \cdot k! / [n(n+1) \ldots (n+k)], \]

As for the denominator of (3.6) we proceed as follows: since \( g(x) \) has a Poincaré-type asymptotic expansion, so does \( 1/g(x) \) and its asymptotic expansion is given by
\[ 1/g(x) \sim \sum_{i=0}^{\infty} \varepsilon_i x^{-i}, \]

where \( \varepsilon_0 = 1/\rho_0 \). We now need the asymptotic behavior of \( \Delta_k(z^{-n} a) \) as \( n \to \infty \). First of all we have
\[ \Delta_k(z^{-n} a) = \sum_{j=0}^{k} (-1)^{k-j} j! (n+j)! a^{-n-j}, \]
which, as \( n \to \infty \), can be shown to behave like

\[
\Delta^k(z^{-n} a) = \left[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} z^{-n(j)} \right] z^{-n a[1 + O(n^{-1})]}
\]

\[
= (-1)^k(1-1/z)^k z^{-n a[1 + O(n^{-1})]}.
\]

Combining (3.10) and (3.12) we obtain for the denominator of (3.6)

\[
\Delta^k(n^{-1}/R_n) = \Delta^k[n^{-1} z^{-n+1} \epsilon_0 \sigma(1 + O(n^{-1}))]
\]

\[
= (-1)^k(1-1/z)^k \epsilon_0 z^{-n+1} k+\sigma-1[1 + O(n^{-1})], \text{ as } n \to \infty.
\]

Substituting now (3.9) and (3.13) in (3.6) and using the fact that \( \epsilon_0 = 1/p_0 \), we obtain (3.4) together with (3.5), thus proving the theorem.

**Corollary.** If \( |z| \leq 1, z \neq 1 \), then \( T \to F(z) \), as \( n \to \infty \), provided \( k \) is chosen so that \( 2k + \sigma > 0 \). For \( |z| > 1 \), however, \( T \) diverges as \( n \to \infty \), i.e., Process I cannot be used for analytic continuation beyond the circle of convergence of the infinite series considered in Section 2.

**Proof.** The proof follows by observing that the right hand side of (3.4) tends to zero as \( n \to \infty \) for \( |z| < 1, z \neq 1 \), only if \( 2k + \sigma > 0 \). For \( |z| > 1 \), however \( T-F(z) = O(z^n) \) as \( n \to \infty \), thus completing the proof.

**Remark 1.** (3.1), (3.2) and (3.3) imply that \( \lim_{\mathcal{R} \to \infty} A_\mathcal{R} = F(z) \) for 1) \( |z| < 1 \) for all \( \sigma \), and 2) \( |z| \leq 1 \) for \( \sigma > 0 \). The corollary above tells us that \( T \to F(z) \) as \( n \to \infty \) for all \( |z| \leq 1, z \neq 1 \), no matter what \( \sigma \) is, i.e., whether \( \lim_{\mathcal{R} \to \infty} A_\mathcal{R} \) exists or not, provided \( k \) is chosen large enough so that \( 2k + \sigma > 0 \).
Remark 2. Equation (3.4) tells us that for \( z \neq 1 \) whenever \( A_n \) converges to \( F(z) \) as \( n \to \infty \), \( T \) converges to \( F(z) \) more quickly, in fact

\[
\frac{T - F(z)}{A_n - F(z)} = O(n^{-2k}) \quad \text{as} \quad n \to \infty.
\]

Remark 3. From the expression for \( D \) given in (3.5) we can see that problems will arise as we approach \( z = 1 \). Indeed, there is a drastic fall in the rate of convergence of \( T \) to \( F(z) \) as numerical experiments show.

Also Theorem 4.2 in (I) shows that if \( \lim_{r \to \infty} A_r = F(1) \) exists, we have

\[
\frac{T - F(1)}{A_n - F(1)} = O(n^{-k}) \quad \text{as} \quad n \to \infty,
\]

as opposed to (3.14).

Going back to \( A_r = \sum_{m=1}^{r} w(m)z^{m-1} \), where \( w(x) \) is as described in the previous section, we can see that, on \( |z| = 1 \), \( A_r \) is a partial sum of the complex Fourier series \( \sum_{m=1}^{\infty} w(m)e^{i(m-1)\theta} \), where we have put \( z = e^{i\theta} \).

Hence Theorems 2.2 and 3.1 cover the case of complex Fourier series whose coefficients \( w(m) \) are as described in Section 2.

Theorem 3.2: Let the sequence \( A_r, r = 1,2, \ldots \), satisfy all the conditions of Theorem 3.1 with the notation therein and let \( \gamma_i, i = 0,1, \ldots, k-1 \), be as in (1.1). Then for \( z \neq 1 \), we have

\[
\gamma_1 - \beta_1 = O(n^{-k+1}) \quad \text{as} \quad n \to \infty.
\]

Proof. The proof of (3.16) proceeds along the same lines as that of Theorem 5.2 in (I). Equation (5.6) in (I) reads

\[
[F(z)-T]A^{k}[\Delta^{k+1}f(n)] = \sum_{j=0}^{k} \gamma_j \Delta^j (A^{k+1-1}), \quad i = 0,1, \ldots, k-1.
\]
Now \( \Delta^n (n^{k+1}/R_n) = z^{-n+1} 0(n^{k+1}+\sigma) \) as \( n \to \infty \), which can be proved in a way similar to that in Theorem 3.1. Also \( F(z) - T = z^{-n-1} 0(n^{-2k-\sigma}) \) as \( n \to \infty \) which follows from (3.4). Therefore the first term on the right hand side of (3.17) is just

\[
(F(z) - T) \Delta^n (n^{k+1}/R_n) = 0(n^{-k+1}) \text{ as } n \to \infty.
\]

Once this has been established the rest of the proof is exactly the same as that of Theorem 5.2 in (I), therefore we shall omit it.

We note that Theorem 5.2 in (I) covers the case \( z = 1 \) and for this case too \( \gamma_1 - \beta_1 = 0(n^{-k+1}) \), \( i = 0, \ldots, k-1 \).

4. ANOTHER APPROACH TO THE ANALYSIS OF LEVIN'S TRANSFORMATIONS

In Theorem 2.4 it was assumed that the sequence \( A_r, r = 1, 2, \ldots \), (convergent of not) satisfies (1.5) where \( f(x) \), as a function of the continuous variable \( x \), is defined and is infinitely differentiable for all \( x \geq 1 \), including \( x = \infty \), and has a Poincaré-type asymptotic expansion of the form (1.6). We shall now assume further that \( f(x)/x = \tilde{f}(x) \) is the Laplace transform of a function \( \psi(t) \) which is an infinitely differentiable function of \( t \) for \( 0 \leq t < \infty \), i.e.,

\[
\tilde{f}(x) = L[\psi(t); x] = \int_0^\infty e^{-xt} \psi(t) dt.
\]

Then, using Watson's lemma, see, for example, Oliver (1974, p. 71), we have

\[
\tilde{f}(x) \sim \sum_{i=0}^{\infty} \phi^{(i)}(0) x^{i+1}, \text{ as } x \to \infty.
\]
where we immediately identify $\phi^{(1)}(0)$ as $\beta_1$. (Examples of this will be given in Section 5.)

Equation (3.7) in (I) reads:

$$
(4.3) \quad T - A = \frac{\Delta^k[n - f(n)]}{\Delta^k(n - R_n)}
$$

which, in view of the assumptions above, can be expressed as

$$
(4.4) \quad T - A = \frac{\Delta^k[n^k f(n)]}{\Delta^k(n^k - R_n)}
$$

Now, from the theory of the Laplace transform we have, see Sneddon (1972, p.147),

$$
(4.5) \quad \mathcal{L}[\phi^{(m)}(t);x] = x^m \mathcal{L}[\phi(t);x] - \sum_{i=0}^{m-1} \phi^{(m-i-1)}(0)x^i.
$$

Letting $x = n$, $m = k$, and applying the operator $\Delta^k$ to both sides of (4.5) and using the fact that $\Delta^k p(n) = 0$ when $p(n)$ is a polynomial in $n$ of degree at most $k-1$, we obtain

$$
(4.6) \quad \Delta^k[n^k \tilde{r}(n)] = \Delta^k[\mathcal{L}[\phi^{(k)}(t);n]] = \Delta^k[\int_0^\infty e^{-nt}\phi^{(k)}(t)dt].
$$

Since the operator $\Delta^k$ operates only on $n$ and since

$$
(4.7) \quad \Delta^k(e^{-nt}) = e^{-nt}(e^{-t} - 1)^k,
$$

we can express (4.6) in the form

$$
(4.8) \quad \Delta^k[n^k \tilde{r}(n)] = \int_0^\infty e^{-nt}(e^{-t} - 1)^k\phi^{(k)}(t)dt.
$$
We have therefore proved the following:

Theorem 4.1: Let the sequence $A_r, r = 1, 2, \ldots, \infty$ (convergent or not) be as described in the first paragraph of this section. Then

$$T - A = \frac{\mathcal{F}[(e^{-t-1})k\phi(k)]}{\Delta^k(n^{k-1}/R_n)}$$

(4.9)

If Eq. (4.9) is used in the analysis of Process I ($k$ fixed, $n \to \infty$) it seems that one can obtain only those results that were given previously, so that there is not much to be gained from (4.9) as far as Process I is concerned.

As for Process II ($n$ fixed, $k \to \infty$), which is the more effective of the two processes, yet the more difficult to analyze, Theorem (4.1) does seem to represent a breakthrough. Of course, eventually one has to analyze the asymptotic behavior of $\mathcal{F}[(e^{-t-1})k\phi(k)]$ and of $\Delta^k(n^{k-1}/R_n)$ as $k \to \infty$, which is not an easy task in general. The following results and the examples in the next section do, however, give an indication about the mechanism by which Process II works and why it works so efficiently.

Lemma 4.1: Let $\phi(\xi)$ be analytic and uniformly bounded in the half strip $S(u) = \{\xi \mid \text{Re}\xi \geq -u, \ |\text{Im}\xi| \leq u\}$, for some $u > 0$. Then,

$$\mathcal{F}[(e^{-t-1})k\phi(m)] \leq M\ m! \ k!/ [u^{m+1}(n+1) \ldots (n+k)],$$

(4.10)

where $M$ is the uniform bound of $\phi(\xi)$ in $S(u)$; i.e., $|\phi(\xi)| \leq M$ for $\xi \in S(u)$. 

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Proof. Since \( \phi(\xi) \) is analytic in \( S(u) \) we can write, using Cauchy's formulas,

\[
\phi^{(m)}(t) = \frac{m!}{2\pi i} \oint_{|\xi-t|=u} \frac{\phi(\xi)}{(\xi-t)^{m+1}} \, d\xi ,
\]

where \( t \in [0, \rho) \). Taking the modulus of both sides of (4.11) and using the assumption of uniform boundedness we obtain

\[
|\phi^{(m)}(t)| \leq M \frac{m!}{|u|^{m+1}} .
\]

Making use of (4.12), we therefore have

\[
|\mathcal{L}[e^{-t-1} \phi^{(m)}(t);n]| \leq M \frac{m!}{|u|^{m+1}} \int_0^\infty e^{-nt}(1-e^{-t})^k \, dt .
\]

But

\[
\int_0^\infty e^{-nt}(1-e^{-t})^k \, dt = (-1)^k \Delta^k \left( \int_0^\infty e^{-nt} \, dt \right) = (-1)^k \Delta^k (n^{-1})
\]

by (3.8). Substituting (4.14) in (4.13), (4.10) now follows.

**Corollary.** If \( m = k+p \), where \( p \) is fixed, then

\[
|\mathcal{L}[e^{-t-1} \phi^{(k+p)}(t);n]| \leq \bar{M} \frac{k!}{(n+1) \ldots (n+k)},
\]

for some constant \( \bar{M} > 0 \) which is independent of \( k \).

The proof of (4.15) follows easily from (4.10).
We shall now apply Theorem 4.1 and the corollary of Lemma 4.1 to the power series considered in Sections 2 and 3.

**Theorem 4.2:** Let the sequence \( A_r \), \( r = 1, 2, \ldots \), be as in Theorems 3.1 and 4.1 and Lemma 4.1 with the notation therein. Then for \( z \) real and negative and \(|z| < (ue)^2\),

\[
(4.16) \quad T \cdot F(z) = 0(q^{-k}) \quad \text{as } k \to \infty ,
\]

at least, where \( q = ue|z|^{-\frac{1}{2}} > 1 \).

**Proof.** From the conditions above it is clear that (4.15) holds; therefore, the numerator of the expression on the right hand side of (4.9) is at least \( 0(k! u^{-k}) \) as \( k \to \infty \). As for the denominator of this expression we proceed as follows: Since \( g(m) \) satisfies (3.3), \( g(m) \sim p_0 m^{-\sigma} \) as \( m \to \infty \) and has a fixed sign (that of \( q_{-\frac{1}{2}} \)) for \( m \geq m_0 \), for some positive integer \( m_0 \).

Denoting

\[
(4.17) \quad b_j = (-1)^j \left( \frac{k}{j} \right) (n+j)^{-k-1}/R_{n+j}
\]

we can write for the denominator

\[
(4.18) \quad \left| \sum_{j=0}^{m_0} b_j \right| > \left| \sum_{j=m_0}^{k} b_j \right| \quad \left| \sum_{j=0}^{k} b_j \right| - \left| \sum_{j=m_0}^{k} b_j \right| .
\]

Now since \( b_j \) are all of the same sign for \( j \geq m_0 \), and \( m_0 \) is fixed, we can write

\[
(4.19) \quad \left| \sum_{j=m_0}^{k} b_j \right| > \left| b_{[k/2]} \right| = \left( \frac{k}{[k/2]} \right) (n+[\frac{k}{2}])^{-k-1} |z|^{-n-[k/2]+1}/g(n+[\frac{k}{2}])
\]

\[ = 0 \left[ k! (e|z|^{-\frac{1}{2}})^k \right] \quad \text{as } k \to \infty ,
\]
which can be provided by using Stirling's formula, \( k! \sim k^{k - \frac{1}{2}} e^{-k} \sqrt{2\pi k} \) as \( k \to \infty \). Essentially this is a \( O(k^k) \)-like behavior. The sum \( \sum_{j=0}^{m_k} b_j \), on the other hand, can grow at most like some fixed power of \( k \) as can readily be verified. Therefore,

\[
(4.20) \quad \left| \sum_{j=0}^{k} b_j \right| \sim \left| \sum_{j=m_k}^{k} b_j \right| \quad \text{as} \quad k \to \infty.
\]

Combining these results for the numerator and denominator in (4.9), the result follows.

Remark. By replacing (4.19) by:

\[
(4.19') \quad \left| \sum_{j=0}^{k} b_j \right| > \left| b_{\lfloor ak \rfloor} \right|, \quad 0 < a < 1,
\]

we can show, by using the method above, that (4.16) holds with \( q = [a/(1-a)]^{1-a} u \epsilon |z|^{-a} \) provided \( z \) is chosen such that \( q > 1 \). Now one can choose \( a \) so as to make \( q \) as large as possible.

We now give a result that will be useful in dealing with monotonic sequences.

Theorem 4.3: Let the sequence \( A_r, r = 1, 2, \ldots \), be as in Theorem 4.1 and Lemma 4.1 with \( u > 1 \) and the notation therein, and with \( R_r = r^{-\alpha} \), for some \( \alpha \). Then

\[
(4.21) \quad T_r - A = O(u^{-k}) \quad \text{as} \quad k \to \infty.
\]
Proof. As in Theorem 4.2 the numerator of the expression on the right hand side of (4.9) is at least $0(kl^{-k})$ as $k \to \infty$. Now the denominator of this expression becomes $\Delta_h^{k}(n^{k+\alpha-1})$. From the calculus of finite differences we know that, see Isaacson and Keller (1966, p.262),

\[ \Delta_h^k(x) = h^{(k)}(y) \quad \text{for some } y \in (x,x+k). \]

Therefore

\[ \Delta_h^k(x) = \prod_{j=0}^{k-1} (\sigma-j)^{y_{\sigma-k}} \quad \text{for some } y \in (n,n+k). \]

Hence the denominator becomes

\[ \Delta_h^k(n^{k+\alpha-1}) = \frac{(k+\alpha-1)!}{(\alpha-1)!} \quad \text{for some } m \in (n,n+k). \]

Using Stirling's approximation we have $\Delta_h^k(n^{k+\alpha-1}) = O(k!)$ as $k \to \infty$.

Combining these results in (4.9), (4.21) follows.

We are now going to consider the $\gamma_{\tau}$ in (1.1).

**Theorem 4.4:** If the sequence $A_{r}$, $r = 1,2,\ldots$, is as in Theorem 4.1, then

\[ \Delta_{h}^{k} \left[ n^{k+i} f(n) \right] = L[(e^{-t_{-1}})^{k+i+1} \phi(t);n] + \sum_{j=0}^{i} \beta_{j} \Delta_{h}^{k} [n^{k+i-j}], \]

\[ i = 0,1,\ldots,k-1. \]

**Proof.** Using (3.17) we just have to prove that

\[ \Delta_{h}^{k} \left[ n^{k+i} f(n) \right] = L[(e^{-t_{-1}})^{k+i+1} \phi(t);n] + \sum_{j=0}^{i} \beta_{j} \Delta_{h}^{k} [n^{k+i-j}], \]

\[ i = 0,1,\ldots,k-1. \]
This can be proved easily by using (4.5) with \( x = n \) and \( m = k+1 \) and applying \( \Delta^k \) to both sides, keeping in mind that \( \Delta^k p(n) = 0 \) when \( p(n) \) is a polynomial of degree at most \( k-1 \) and that \( \beta_j = \phi(j)(0), \ j = 0, 1, \ldots \).

**Theorem 4.5:** Let the sequence \( A_r, r = 1, 2, \ldots \), be as in Theorem 4.3 and consider the case \( z = 1 \) with \( R_r = r^{1+\alpha} \). Then, for fixed \( i \),

\[
(4.27) \quad \gamma_i - \beta_i = O(u^{-k}) \quad \text{as} \quad k \to \infty.
\]

**Proof:** We shall prove (4.27) by induction on \( i \). For \( i = 0 \), (4.25) becomes

\[
(4.28) \quad (\gamma_0 - \beta_0) = \left[ F(z) - T \right] \frac{\Delta^k(n^k/R_n)}{k!} + \left[ (e^{-t} - 1)^k \phi^{(k+1)}(t) \right] \frac{1}{k!}.
\]

Now \( F(z) - T = O(u^{-k}) \) as \( k \to \infty \). From (4.23) \( \Delta^k(n^k/R_n) = O(k!) \) as \( k \to \infty \).

Therefore, the first term on the right hand side of (4.28) is \( O(u^{-k}) \) as \( k \to \infty \).

Using (4.15) in the corollary of Lemma 4.1, we can see that the second term is also \( O(u^{-k}) \) as \( k \to \infty \). Hence we have shown that (4.27) holds for \( i = 0 \).

Let us now assume that (4.27) is true for \( i \leq m-1 \). For \( i = m \) we have from (4.25),

\[
(4.29) \quad \gamma_m - \beta_m = \left[ F(z) - T \right] \frac{\Delta^k(n^{k+m}/R_n)}{k!} + \left[ (e^{-t} - 1)^k \phi^{(k+m+1)}(t) \right] \frac{1}{k!} + \sum_{j=0}^{m-1} (\gamma_j - \beta_j) \frac{\Delta^k(n^{k+i-j})}{k!}.
\]

Using in (4.29) the same technique that was used in (4.28), we again have \( \gamma_m - \beta_m = O(u^{-k}) \), thus proving the theorem.

It seems, in general, that \( \tilde{T}(x) \) is a Laplace transform as in (4.1) and that \( \phi(x) \) satisfies the conditions of Lemma 4.1 so that (4.10) and
hence (4.15) hold. These points will be illustrated with three typical examples in the next section.

5. EXAMPLES

In this section we shall show through three typical examples that the assumptions made in the previous sections are realistic and we shall especially be concerned with the application of Process II to these examples, keeping in mind the results of Section 4.

Example 1. $A_r = \sum_{m=1}^{r} \frac{z^{m-1}}{m}, \ r = 1, 2, ...$

This sequence satisfies the conditions of Theorem 2.2 with $a = -1$ in (2.7), therefore Theorem 2.2 applies to it. Now $\lim_{r \to \infty} A_r = -(1/z) \log(1-z) = F(z)$, provided $|z| < 1$, $z \neq 1$. $z = 1$ is a branch point of $F(z)$ and we put the branch cut along the real interval $[1, \infty)$. This being the case, Theorem 3.1 applies and $T-F(z) = 0(n^{-2k-1})z^n$ as $n \to \infty$.

Taking $z \in [1, \infty)$ and integrating both sides of the equality

\[(5.1) \quad \frac{1}{1-s} = \sum_{m=0}^{r-1} s^m \frac{s^r}{1-s} \]

from $s = 0$ to $s = z$ along a straight line in the $s$-plane, and dividing by $z$, we obtain

\[(5.2) \quad F(z) = A_r + \frac{1}{z} \int_0^z \frac{s^r}{1-s} ds \]

Letting $s = ze^{-\xi}$ in the integral on the right hand side of (5.1), the contour in the $s$-plane is mapped to the positive real line in the $\xi$-plane,
and (5.2) becomes

\[ F(z) = A_r + z^r \int_0^\infty e^{-rt} (e^t - z)^{-1} \, dt. \]

Defining \( R = z^{r-1} / r \), the \( r \)-th term of the infinite series \( \sum_{m=1}^{\infty} z^{m-1} / m \), as in the \( t \)-transformation of Levin, we can express (5.3) in the form (1.5) with \( f(x) = x \tilde{f}(x) \), where \( \tilde{f}(x) = \mathcal{L}[\phi(t); x] \) and \( \phi(t) = z(z-e^{-t})^{-1} \).

Since \( \phi(t) \) is analytic at \( t = 0 \) and for any \( t > 0 \) provided \( z \in [1, \infty) \), applying (4.2) we therefore obtain:

\[ f(x) \sim \frac{x}{z-1} + \frac{z}{(z-1)^2} \frac{1}{x} + \cdots, \]

with \( \beta_0 = z/(z-1) \) as predicted by Theorem 2.1, and this expansion is valid both for \( |z| < 1 \), \( z \neq 1 \), and for \( |z| > 1 \), \( z \in [1, \infty) \), see the remark following the proof of Theorem 2.2.

Since \( \tilde{f}(p) \) is a Laplace transform it is analytic for \( \text{Re} \, p > -1 \); therefore so is \( f(p) \). However, since \( \phi(\xi) \) is not an entire function, (5.4) diverges for all \( x \), hence \( f(x) \) is not analytic at infinity. On the other hand it is easy to show that \( f(x) \) is infinitely differentiable at \( x = \infty \). This is an important property that \( f(x) \) was required to have in Process II in (I).

Now the function \( \phi(\xi) \) is meromorphic and its only poles are \( \xi = \log z + i 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots \), i.e., all the singularities are on the straight line \( \text{Re} \xi = \log |z| \). Furthermore, \( \phi(\xi) \) is uniformly bounded as \( \text{Re} \xi \to \infty \), in fact \( |\phi(\xi)| \ll |z| |(e^\xi - z)|^{-1} = O(e^{-t}) \) as \( \text{Re} \xi = t \to \infty \).

Hence the strip \( S(u) \) in Lemma 4.1 exists and \( u \) is determined as follows: For \( |z| < 1 \), \( u = |\log |z| + i \arg z| - \delta^\circ \); for \( |z| > 1 \), \( \arg z \neq 0 \), \( u = \arg z| - \delta \) for \( \delta > 0 \) and as small as we wish. Therefore, Theorem 4.2 applies and consequently, (4.16) holds.
For example for $z = -1$, $T\text{-}F(-1) = O[((\pi - \delta) e)^{-k}]$ at least, as $k \to \infty$.

For this case the sequence $A_r, r = 1, 2, \ldots$, is a very slowly converging oscillatory sequence. For $z = -2$, $T\text{-}F(-2) = O[((\pi - \delta)e/\sqrt{x})^{-k}]$ at least, as $k \to \infty$, and for this case the sequence $A_r, r = 1, 2, \ldots$, is a strongly diverging oscillatory sequence.

**Example 2.** $A_r = \sum_{m=1}^{r} z^{-m/r^2}, r = 1, 2, \ldots$.

This sequence satisfies the conditions of Theorem 2.2 with $\alpha = -2$ in (2.7).

Now the $A_r$ of this example are the partial sums of the MacLaurin series of the function $F(z) = \frac{1}{z} \int_{0}^{z} \frac{\log(z/s)}{1-s} ds$, where the integral is taken along the straight line in the $s$-plane joining $s = 0$ to $s = z$. Then $F(z)$ has a branch point at $z = 1$ and a branch cut along the real interval $[1, \infty)$. By using the expansion in (5.1) we can express $F(z)$ as follows:

\[
(5.9) \quad F(z) = A_r + \frac{1}{z} \int_{0}^{z} \frac{s^r \log(z/s)}{1-s} ds.
\]

Making the change of variable $s = z e^{-\xi}$ in the integral on the right hand side of (5.9), exactly as in the previous example, we obtain

\[
(5.10) \quad F(z) = A_r + z^r \int_{0}^{\infty} e^{-rt} \frac{t}{e^t - z} dt,
\]

where $t = \Re \xi$. Defining $R_x = z^{-1/r^2}$ for $z \neq 1$, again as in the $t$-transformation of Levin, we obtain $f(x) = z^{x^2} \int_{0}^{\infty} e^{-xt} t/(z-e^t) dt$, which on using Watson's lemma for $x \to \infty$, becomes

\[
(5.11) \quad f(x) \sim \frac{e^x}{z^{x^2}} + \frac{2z}{x(z-1)^2} \frac{1}{x} + \ldots, \quad z \notin [1, \infty),
\]

as in the previous example. Hence also for this example we see that (1.5)
and (1.6) are valid beyond the circle of convergence of \( \sum_{m=1}^{\infty} \frac{z^{-m}}{m^2} \).

Using the fact that

\[ \mathcal{L}[g'(t);p] = p\mathcal{L}[g(t);p] - g(0), \]

we can express \( f(x) \) in the form \( f(x) = x\tilde{f}(x) \), where \( \tilde{f}(x) = \mathcal{L}[\phi(t);x] \) and \( \phi(t) = z[1/(z-t) + t(z-t)^2]. \)

Now this \( \phi(\xi) \) has the same properties as that \( \phi(\xi) \) of the previous example. Therefore, the conclusion of the previous example concerning Process II is valid also for the present example.

We now want to investigate Process II for \( z = 1 \) for which \( \sum_{m=0}^{\infty} \frac{1}{m^2} \) is a monotonic series. For this case

\[ \mathcal{L}(5.13) F(1) = A'r + \int_{0}^{\infty} e^{-rt} \frac{t}{(e^t-1)} \, dt. \]

Choosing \( R = 1/r \) as in the \( u \)-transformation of Levin, we have

\( f(x) = x\tilde{f}(x), \) where \( \tilde{f}(x) = \mathcal{L}[\phi(t);x], \) with \( \phi(t) = t/(1-e^t). \)

Again using Watson's lemma, we obtain:

\[ \mathcal{L}(5.14) f(x) \sim -\sum_{i=0}^{\infty} B_i/x^i, \text{ as } x \to \infty. \]

where \( B_i \) are the Bernoulli numbers. Again \( B_0 = -B_1 = -1 \) as predicted by Theorem 2.1. Now \( \phi(\xi) \) satisfies all the conditions of Lemma 4.1 with \( u = 2\pi - \delta \) and therefore the result of Theorem 4.3 holds and

\[ T-F(1) = O((2\pi - \delta)^{-k}) \text{ as } k \to \infty. \]
Example 3: \[ A_r = \sum_{m=1}^{r} \frac{1}{m} - \log r, \quad r = 1, 2, \ldots \]

It is known that \( \lim_{r \to \infty} A_r = C \), Euler's constant. Denoting \( a_1 = 1, \quad a_r = \frac{1}{r} + \log(1-1/r) \), for \( r = 2, 3, \ldots \), we can see that \( a_r \) is as in Theorem 2.2, with \( \alpha = -2 \) and \( z = 1 \). Therefore, (1.5) and (1.6) hold with \( R_r = 1/r \), in accordance with (2.15).

Now let us show that, also for this case \( f(x) = xf(x) \) where \( \overline{f}(x) = \mathcal{L}[\phi(t); x] \) with \( \phi(t) = t^{-1} (e^t - 1)^{-1} \).

Using the fact that \( \psi(r+1) = -C + \sum_{m=1}^{r} \frac{1}{m} \) and Gauss' formula for the Psi function, see Olver (1974, pp. 39-40), we have

(5.15) \[ \sum_{m=1}^{r} \frac{1}{m} = C + \int_{0}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-rt}}{t-1} \right) dt. \]

Now for the integral on the right hand side of (5.15) we can write

(5.16) \[ \psi(r+1) = \int_{0}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-rt}}{t-1} \right) dt = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-rt}}{t-1} \right) dt. \]

Making the change of variable \( t = rt' \) in the integral \( \int_{\epsilon}^{\infty} \frac{e^{-t}}{t} \) we can express (5.16) as

(5.17) \[ \psi(r+1) = \lim_{\epsilon \to 0^+} \left[ \int_{\epsilon}^{\infty} \frac{e^{-rt}}{t} dt + \int_{\epsilon}^{\infty} \frac{e^{-rt}}{t} dt \right]. \]

The second integral in (5.17) can easily be shown to be equal to \( \log r + O(\epsilon) \) as \( \epsilon \to 0^+ \). Letting now \( \epsilon \to 0^+ \), the desired result follows.
Now \( \phi(\xi) \) is a meromorphic function with poles at \( \xi = i2\pi n \),
\( n = \pm 1, \pm 2, \ldots \), and is uniformly bounded as \( \text{Re}\xi \to \infty \), actually
\( \phi(\xi) = O(\xi^{-1}) \) as \( \text{Re}\xi \to \infty \). Therefore all the conditions of Lemma 4.1
are satisfied with \( u = 2\pi - \delta \), for \( \delta > 0 \) but as small as we wish. Hence
(4.15) holds. Consequently, Theorem 4.3 holds and \( T-C = O((2\pi-\delta)^{-k}) \)
at least, as \( k \to \infty \).

Finally, we note that the well known convergence acceleration methods
due to Euler (see Bromwich (1942, p.62) and Shanks (1955)) or its equivalent
\( \varepsilon \)-algorithm of Wynn (1956) fail to accelerate the convergence of the
monotonic sequences considered in Examples 2 and 3.
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