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Technical Report #117
January 1978
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ABSTRACT

A Baire category argument proves that a large class of vector subspaces in countable cartesian products of vector spaces on \( \mathbb{R}^1 \) are necessarily related to the ultraproduct construction, a result generalizing a known property of proper ideals in powers of \( \mathbb{R}^1 \) or \( \mathbb{C}^1 \).

AMS(MOS) subject classification (1970). Primary 02H15; secondary 54C50, 54J05, 02H13, 02H20, 26A98.

Key words and phrases. Ultraproducts, countable cartesian products of vector spaces, vector subspaces.
§1. ULTRAPRODUCTS OF VECTOR SPACES

Given a nonvoid family \((X_i \mid i \in I)\) of nontrivial vector spaces on \(\mathbb{R}^1\) and a set \(B\) of subsets in the index set \(I\), denote by \(Y_B\) the set of all elements \(x = (x_i \mid i \in I) \in \prod_{i \in I} X_i\) satisfying the condition:

\[\exists B \in B: B \subseteq B(x)\]

where \(B(x) = \{i \in I \mid x_i \neq 0\}\). Then \(Y_B\) is a proper vector subspace in \(X\) only if \(B\) is a filter base on \(I\).

The ultraproduct of the family \((X_i \mid i \in I)\) of vector spaces on \(\mathbb{R}^1\), according to the filter base \(B\), is the quotient vector space

\[\prod_{B} X_i = X/Y_B.

We call a proper vector subspace \(Y\) in \(X\) ultrasubspace, only if \(Y \subseteq Y_B\) for a certain filter base \(B\) on \(I\). One obtains easily:

Lemma 1. A proper vector subspace \(Y\) in \(X\) is an ultrasubspace, only if

\[G_Y = \{B(x) \mid x \in Y\}\]

is a filter generator on \(I\).
§2. A QUESTION

As known, the proper ideals in arbitrary powers of $\mathbb{R}^1$ or $\mathbb{C}^1$ are ultrasubspaces. That fact explains for instance, the role the ultrapower constructions plays in Nonstandard Analysis.

The question arises whether any proper vector-subspace in a cartesian product of vector spaces on $\mathbb{R}^1$ is an ultrasubspace?

An affirmative answer will be obtained under rather general conditions in the case of countable cartesian products of vector spaces on $\mathbb{R}^1$.

The countability condition proves to be essential and not due only to the Baire category argument employed, as can be seen in a simple counter example.

§3. REGULAR VECTOR SUBSPACES

A vector subspace $Y$ in $X$ is called regular, only if

\begin{equation}
B(x) \neq \emptyset, \forall x \in Y,
\end{equation}

which implies that $Y$ is a proper vector subspace in $X$.

Obviously, an ultrasubspace is a regular vector subspace. The converse of that property is established in:

**Theorem 1:** Any regular vector subspace in a countable cartesian product of vector spaces on $\mathbb{R}^1$ is an ultrasubspace.

**Proof.** Assume $Y$ is a regular vector subspace in $X$. According to Lemma 1, it suffices to show that $G_Y$ is a filter generator on $I$. That property
follows from Lemma 3 in §4, by taking \( \tau_i \) the projection mapping \( pr: X \to X_i \) and \( I = \{1\} \).

Remark: In case of an uncountable cartesian product of vector spaces on \( \mathbb{R}^1 \), the result in Theorem 1 does not necessarily hold. Indeed, suppose \( I = \mathbb{R}^1 \) and \( X_i = \mathbb{R}^1 \), with \( i \in I \). Define \( x = (x_i \mid i \in I) \), \( y = (y_i \mid i \in I) \in X = \prod X_i \), by

\[
x_1 = 1, \ y_1 = 1 + i(\pi/2 - \arctg l), \ \forall i \in I.
\]

The obviously,

\[
(4) \quad B(x) \cap B(y) = \emptyset.
\]

Denote by \( Y \) the vector subspace generated by \( \{x, y\} \) in \( X \). It can easily be seen that \( Y \) is a regular vector subspace in \( X \). However, due to (4), \( Y \) cannot be an ultrasubspace.

An additional characterization is given in:

**Lemma 2:** Suppose \( Y \) is a proper vector subspace in \( X \). Then \( Y = \bigcap_B \) for a certain filter base \( B \) on \( I \), only if:

\[
\forall x \in X: \ (\exists y \in Y: B(y) \subseteq B(x)) \Rightarrow x \in Y.
\]

**Proof:** Assume \( Y \) satisfies (5), then obviously \( Y = \bigcap_B \) with the notations in Lemma 1. Therefore, \( G_Y \) is a filter base on \( I \), since \( Y \) is a proper subspace in \( X \). The converse is obvious.
§4. A LEMA

The proof of Theorem 1 was obtained from the following lemma of a rather general interest (see [3], Chap.8, §3, for a particular version).

Suppose given a family of linear mappings

$$\tau_i : X \rightarrow X_i, \ i \in I,$$

between vector spaces on $R^1$. Define then

$$X \ni x \mapsto B(x) = \{ i \in I \mid \tau_i x = 0 \} \subset I.$$ 

Suppose $I$ is a set of nonvoid and countable subsets in $I$, and denote

$$X_I = \{ x \in X \mid B(x) \cap J \neq \emptyset, \ \forall J \in I \}.$$

Lemma 3. Suppose $Y$ is a vector subspace in $X$ and $Y \subset X_I$, then, for each $J \in I$, the set

$$G_{Y,J} = \{ B(x) \cap J \mid x \in Y \}$$

is a filter generator on $J$.

Proof: Assume, it is false and $J \in I$, $x_1, \ldots, x_h \in Y$ such that

$$B(x_1) \cap \ldots \cap B(x_h) \cap J = \emptyset.$$ 

Define

$$R^h \ni \lambda = (\lambda_1, \ldots, \lambda_h) \mapsto y_\lambda = \lambda_1 x_1 + \ldots + \lambda_h x_h \in Y$$

and

$$J \ni i \mapsto \Lambda_i = \{ \lambda \in R^h \mid \tau_i y_\lambda = 0 \}. $$
Obviously, $\Lambda_i$, with $i \in J$, are vector subspaces in $R^h$. Moreover,

\[(7) \quad \Lambda_i \neq R^h, \quad \forall i \in J.\]

Indeed, (6) implies that

\[\forall i \in J:
\exists k_i \in \{1, \ldots, h\}:
\tau_i x_{k_i} \neq 0\]

therefore, taking $\lambda_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^h$, with the coordinate 1 in the $k_i$-th position, it follows that $\lambda_i \notin \Lambda_i, \quad \forall i \in J$.

Now, (7) and the Baire category argument applied to $R^h$, will give

\[\bigcup_{i \in J} \Lambda_i \neq R^h\]

since $J$ is countable. Assume then

\[\lambda \in R^h \setminus \bigcup_{i \in J} \Lambda_i\]

it follows obviously that

\[\tau_i y_\lambda \neq 0, \quad \forall i \in J\]

hence

\[(8) \quad \mathcal{B}(y_\lambda) \cap J = \emptyset.\]

But $y_\lambda \in Y \subset X_I$, therefore (8) contradicts the definition of $X_I$.\]
REFERENCES


