ON FINDING AN OPTIMAL COMMUNICATION SPANNING TREE

by

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ABSTRACT

An optimum communication spanning tree of an undirected graph is a spanning tree which minimizes the sum of the lengths of the tree paths between all pairs of vertices. It is shown that finding an optimal communication spanning tree is NP-complete. The maximum communication cost of a spanning tree of an n-vertex graph is $O(n^3)$. The average cost over all trees is $O(n^{5/2})$. The maximum cost of a breadth first search tree may be worse than the cost of an optimum tree by a factor of at most $n^{2/3}$.

For $n$ sufficiently large, $1+\log/\log n$ bounds the diameter of most $n$-vertex graphs, in which the appearance of an edge is an independent random variable with probability $P \geq 16 \log n/n$. Therefore, the communication cost of a breadth-first-search tree on such graphs is bounded by $O(n^2 \log n/\log \log n)$.
1. INTRODUCTION

Suppose we are given an undirected graph $G = (V, E)$. The vertices represent cities which need to communicate with one another. The problem is to construct a spanning tree which minimizes communication cost. The cost of communication between two vertices is the length of the unique tree-path which connects the vertices, and the cost of the tree is the sum of communication costs between all pairs of vertices.

This problem, unit distance communication spanning tree (UDCST) is a special case of the distance communication spanning tree proposed by Professor F. Maffioli to T.C. Hu [H]. Hu assumed a complete graph with distances (real numbers) associated to the edges and gave sufficient conditions that the optimum (distance communication spanning) tree be of a special simple structure; thereby, finding the tree in those cases. In Section 2 we show that in general the problem is computationally hard, and that even our restricted case, UDCST, is NP-complete [K]. Hence, quite probably, there exists no polynomial-time-bounded algorithm to solve this and the more general communication spanning tree problems defined by T.C. Hu [H].

In Section 3 we investigate the cost of an arbitrary breadth-first search (BFS) tree. It is shown that the cost of this tree is at most $n^{2/3}$ times greater than that of an optimal tree ($n$ is the number of vertices).

In Section 4 we show that the average communication cost of labelled trees is $O(n^{2.5})$.

If the tree consists of a single path (i.e. the graph...
has a Hamiltonian path) then the cost is as high as $O(n^3)$. Posa proved that most graphs with more than $cn\log n$ edges have such a path $P$, and therefore have a high cost communication spanning tree. In Section 5 we prove that if the existence of an edge is an independent random variable with probability $K\log n/n$ ($K \geq 16$) then the diameter of almost all random graphs is at most $\log n/\log \log n$. Therefore, the communication cost of a breadth first search tree is bounded by $n^2\log n/\log \log n + O(n^2)$ for almost all random graphs. i.e. it is greater than the minimum by a factor of at most $\log n/\log \log n$.

(Here and in the sequel, unless otherwise stated, all logarithms are natural logarithms, to the base $e$.)
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If the tree consists of a single path (i.e. the graph
The bound on the communication cost is \( W = (N-S)n + (N-1)^2 \).

Claim: \( G \) has a spanning tree of cost \( W \) if and only if the given 3DM problem has a solution.

(i) Let \( M \) be a matching. Then \( T = (V,E) \) where

\[
E = \{(R,y_i) : i = 1, \ldots, k\} \cup \{(R,S_j) : j = 1, \ldots, m\} \\
\cup \{(S_j, x_i) : x_i \in S_j \text{ and } S_j \in M\}
\]

is a spanning tree of cost \( W \). In Figure 1 \( M = \{2,3\} \) and the corresponding tree consists of all the edges of \( G \) except \((S_1, x_1), (S_1, x_2), (S_1, x_3)\). Its cost is \( 23 \cdot 4 + 27^2 = 821 \).

(ii) Assume that 1DCST has a solution. First we prove the following lemma concerning the structure of any optimum tree for \( G \).
Lemma 2. Every optimum tree contains all the edges incident with R.

Proof. Since there are many Y-vertices it is worthwhile to keep the distance to them as small as possible. Formally: Let T be any spanning tree which contains all the edges incident with R, and T' a spanning tree in which at least one edge (R,S_j) is missing (T' must contain all the edges (R,Y_i)). It suffices to prove that the cost of T is less than that of T'. The cost of T:

\[ C(T) \leq k + m + 2n \]
- the cost from R
+ \( 2 \binom{k}{2} + 2km + 3kn \) - the cost from the Y-vertices
+ \( 2 \binom{m}{2} + 3mn \) - the cost from the S-vertices
+ \( 4 \binom{n}{2} \) - the cost between the X-vertices.

The cost of T':

\[ C(T') \geq k + (m+2) + 2n \]
- the cost from R
+ \( 2 \binom{k}{2} + 2k(m-1) + 4k + 3kn \) - the cost from the Y-vertices
+ \( 2 \binom{m}{2} + mn \) - the cost from the S-vertices
+ \( 2 \binom{n}{2} \) - the cost between the X-vertices.

\[ C(T') - C(T) \geq 2 + 2k - 2mn - 2 \binom{n}{2} > 0. \]

To complete the proof of the theorem, let T be an optimal tree, then by Lemma 2 all the X-vertices are leaves. For \( i = 0, 1, 2, 3 \) let \( a_i \) be the number of S-vertices connected in T to i X-vertices. Then:

1. \( a_0 + a_1 + a_2 + a_3 = m \)
2. \( a_1 + 2a_2 + 3a_3 = n \).

The tree contains \( a_0 + k + n = N(a_1 + a_2 + a_3 + 1) \) leaves and its cost is
Consider $C(T)$ as a function of $a_1, a_2$ and $a_3$. We wish to minimize this function under the restriction (2) and $a_1 \geq 0$. This is a linear programming problem and the minimum is obtained at one of the vertices of the polytop, i.e. at one of the points $(n,0,0)$, $(0,n/2,0)$ or $(0,0,n/3)$. Evaluating $C(T)$ at these points shows that a strict minimum is obtained at $a_1 = a_2 = 0$, $a_3 = n/3$. The cost of the optimal tree is

$$C(T) = (N-5)n + (N-1)^2 = W.$$ 

Therefore, if the cost of the optimal tree is $W$, then in $T$ $n/3$ of the $S$-vertices are each connected to three leaves. The three dimensional matching is:

$$M = \{ j \mid \text{ in } T, S_j \text{ is connected to 3 leaves} \}.$$ 

It is easy to verify that $M$ is indeed a matching.
3. BREADTH FIRST SEARCH SPANNING TREES

Since finding the optimum tree is NP-complete, we wish to find a tree whose cost is not too bad. Note that the complete graph $K_n$ has a Hamiltonian path which constitutes a spanning tree whose communication cost is $O(n^3)$. However, $K_n$ also contains a star-tree—a tree consisting of a vertex adjacent to all other vertices. The communication cost of the star-tree is $O(n^2)$. Therefore, a factor of at most $n$ can be gained by choosing a good tree.

A breadth first search tree is similar to the star-like tree. Let $B$ be any breadth first search tree with root $v$. Let us see how $B$ approximates an optimum tree whose cost is $C^*$.

**Theorem 2:** $C(B)/C^* \leq 2(6n)^{2/3}$.

**Proof.** The theorem is trivially true for $K_n$. Let $\ell(x,y)$ be the length of the shortest path in $G \neq K_n$ between $x$ and $y$. Let $r = \max_{x,y \in V} \ell(x,y)$ be the diameter of the graph. For $G \neq K_n$, the diameter is greater than 1. The length of the path from the root $v$ to a vertex $u$ in any breadth first search tree is bounded by the diameter and therefore the length of the unique tree-path between any two vertices is bounded by $2r$. Consequently, $C(B) \leq (\frac{n}{2}) \cdot 2r$. On the other hand, in the graph there are vertices which are $r$ edges apart. Hence, every optimal tree contains a path of length $r$ or greater. The vertices along this path contribute $(r^3-r)/6$ to the cost of the tree. Hence, $C^* \geq (r^3-r)/6$. Also, $C^* \geq (\frac{n}{2})$. Therefore,
\[ C(B)/C^* \leq \min\left\{ \frac{\binom{n}{2} \cdot 2r}{(r^3-r)/6}, \frac{\binom{n}{2} \cdot 2r}{\binom{n}{2}} \right\} \]

\[ \leq \min\left\{ \frac{6n(n-1)}{r^2-1}, 2r \right\} \]

\[ \leq \min\left\{ \frac{12n^2}{r^2}, 2r \right\}. \]

Consequently, \[ C(B)/C^* \leq 2 \cdot (6n)^{2/3}. \]
4. AVERAGE COST OF COMMUNICATION SPANNING TREES

Let \( T_n \) be the set of labelled trees with \( n \) vertices. Let \( \overline{c}(T_n) \) be the average communication cost of the trees in \( T_n \) that is,

\[
\overline{c}(T_n) = \frac{\sum_{T \in T_n} c(T)}{|T_n|}
\]

Theorem 3: There exist two constants \( K_1 \) and \( K_2 \) such that for every \( n \), \( K_1 n^{2.5} \leq \overline{c}(T_n) \leq K_2 n^{2.5} \).

Proof. We may assume that the vertices of every tree \( T \) in \( T_n \) are labelled by the numbers \( 1, 2, \ldots, n \). Let \( 1 \leq i < j \leq n \) be integers.

Let \( T \) be a tree containing the edge \((i, j)\). Deleting \((i, j)\) from \( T \) separates the tree into two subtrees \( T_i \) and \( T_j \). Let \( V_i \) and \( V_j \) be the set of labels of the vertices of \( T_i \) and \( T_j \) respectively and let \( m = |V_i| \). Besides \( i \), \( V_i \) contains \( m-1 \) elements of the set \( \{1, \ldots, n\} \{-i, j\} \).

Thus, there are \( \binom{n-2}{m-1} \) distinct \( V_i \)'s. Given \( V_i \) the set \( V_j \) is uniquely determined. The number of labelled trees with \( m \) vertices is \( m^{m-2} \) \([c]\). The number of labelled trees with \( n-m \) vertices is similarly \( (n-m)^{n-m-2} \). The number of paths in \( T \) which pass through the edge \((i, j)\) is \( m \cdot (n-m) \) independently of the shape of \( T_i \) and \( T_j \). The sum over all possible values of \( m \) of the contribution of \((i, j)\) to the communication cost of the trees to which it belongs is therefore

\[
\sum_{m=1}^{n-1} \binom{n-2}{m-1} m^{m-2} \cdot (n-m)^{n-m-2} \cdot m \cdot (n-m).
\]
Computing this value for every edge, and dividing by the number of labeled trees with $n$ vertices yields

$$\tilde{c}(T_n) = \frac{1}{n^{n-2}} \binom{n}{2} \sum_{m=1}^{n-1} (n-2 \choose m-1)^{m-2} \cdot (n-m)^{n-m-2} \cdot (n-m)^{n-m}$$

$$= \frac{n(n-1)}{2n^{n-2}} \sum_{m=1}^{n-1} (n-2 \choose m-1)^{m-1} \cdot (n-m)^{n-m-1}$$

(1) $$\tilde{c}(T_n) = \frac{1}{2n^{n-2}} \sum_{m=1}^{n-1} (n-2 \choose m-1)^{m-1} \cdot (n-m)^{n-m} .$$

Let $f(n) = 1 + 1/(12n-1)$, Stirling's formula yields

$$\binom{n}{m} \leq \frac{\sqrt{n^m n!}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{m!(n-m)!}} \cdot \frac{1}{(n-m)! m!} .$$

Substituting (2) in (1) yields

$$\tilde{c}(T_n) \leq \frac{f(n)}{\sqrt{8\pi}} \cdot n^{2.5} \cdot S(n)$$

where

$$S(n) = \sum_{m=1}^{n-1} \frac{1}{\sqrt{1/(n-m)! \cdot m!}} .$$

$S(n)$ is bounded from above by some constant $K$. Therefore,

$$\tilde{c}(T_n) \leq \frac{f(n)}{\sqrt{8\pi}} \cdot K \cdot n^{2.5} .$$

However, $f(n)$ is also bounded and therefore there exists a constant $K_2$ such that

$$\tilde{c}(T_n) \leq K_2 \cdot n^{2.5} .$$
On the other hand, let us look once again at (1). Let

\[ L_m = \binom{n}{m} m^m (n-m)^{n-m}. \]

For \( m < \frac{n}{2} \), \( L_m > L_{m+1} \) since \( L_{m+1}/L_m = \frac{(1 + 1/m)^m}{(1 + 1/(n-m-1))^{n-m-1}} \) and \( (1 + 1/x)^x \) is a monotonically increasing function of \( x \) (and \( m < n-m-1 \)). Therefore,

\[ L_1 > L_{\lfloor n/2 \rfloor} \]

and

\[ \bar{C}(T_n) > \frac{1}{2^{n-2}} \cdot (n-1)L_{\lfloor n/2 \rfloor}. \]

Using Stirling's formula again yields

\[ \bar{C}(T_n) \geq \frac{n^{2.5}}{\sqrt{2\pi} \cdot f(\frac{n}{2}) \cdot f(\lfloor \frac{n}{2} \rfloor)} \]

Since \( f(n) \) is bounded we conclude that there exists a constant \( K_1 \) such that \( \bar{C}(T_n) \geq K_1 \cdot n^{2.5}. \)
5. BREADTH FIRST SEARCH IN RANDOM GRAPHS

How well does a breadth first search tree approximate the optimum communication tree? It is shown that the cost of any breadth first search tree on 'most' of the random graphs is at most $n^2(2 + \log n / \log \log n)$. Since any communication tree costs at least $(n-1)^2$, this is not a bad approximation.

Let $Q$ be a property on graphs, $X_n$ a random variable whose values are graphs, then $Q$ is true with probability (for $X_n$) if there exist constants $a_Q, n_Q$ such that for $n > n_Q$, $\Pr(Q(X_n)) > 1 - n^{-a_Q}$. This definition differs from the usual one by imposing the convergence to be faster than $1 - n^{-a_Q}$ for some $a_Q$ independent of $n$.

To prove that a property $R$ is true with probability it suffices to show that property $Q$ is true with probability and if $Q$ occurs then $R$ is also true with probability. To simplify proofs, we assume that if a property is true with probability then it is actually true. This deduction rule holds as long as it is done a constant number of times.

**Lemma 3.** There is a probability of at least $1 - \exp(mp/8)$ to obtain $mp/2$ successes in $m$ Bernoulli trials each with probability $p$.

**Proof.** Fact 4(i) of [AV] states that for all $m, p, \beta$ with $0 \leq p \leq 1$ and $0 \leq \beta < 1$

\[
(1-\beta)^{mp} \sum_{k=0}^{m} k^p (1-p)^{m-k} > \exp(-\beta^2 mp/2).
\]

Applying the inequality to the case $\beta = 1/2$ yields:

\[
\Pr(\text{less than } mp/2 \text{ successes}) = \sum_{k=0}^{mp/2} \binom{m}{k} p^k (1-p)^{m-k} < \exp(-mp/8).
\]
Let $G_{n,p}$ be a random variable whose values are undirected graphs with $n$ vertices no parallel edges and no self loops, such that for each edge $e$ $Pr(e \in G_{n,p}) = p$. We start with $G_{n,p}$ for which $p = \frac{\log n}{n}$.

In undirected graphs the event $[(u,v) \in G_{n,p}]$ depends on the event $[(v,u) \in G_{n,p}]$. This causes difficulties in the probabilistic analysis of algorithms. To overcome this problem, we convert the undirected graphs to directed ones. In the probability space of directed graphs which we use, the existence of the edge $(u,v)$ is independent of the existence of the edge $(v,u)$.

Let $D_{n,p}$ be a random variable whose values are directed graphs and in which $Pr((u,v) \in D_{n,p}) = p$.

Let DIRECT be the following randomized algorithm for constructing a directed graph $D$ from an undirected graph $G$.

If $(u,v)$ is an edge of $G$ then with probability:

- $1/2 - p/4$ set $(u,v) \in D$ and $(v,u) \in D$;
- $1/2 - p/4$ set $(u,v) \in D$ and $(v,u) \in D$;
- $p/4$ set $(u,v) \in D$ and $(v,u) \in D$;
- $p/4$ set $(u,v) \in D$ and $(v,u) \in D$.

If $(u,v) \in G$ then set $(u,v) \in D$ and $(v,u) \in D$.

Lemma 4. [AV] $DIRECT(G_{n,p})$ is a random variable with the same distribution as $D_{n,p/2}$.

Note that no self loops or parallel edges are introduced by DIRECT.

Let $D$ be a directed graph and $v$ a vertex of $D$, then the outdegree of $v$, $\text{out}(v)$, is the number of edges emanating from $v$. 
The outdegree of \( D \), \( \text{out}(D) \), is the minimum outdegree of all the vertices of \( D \).

**Lemma 5** For \( K > 4 \) and \( p \geq 2K \log n/n \), \( \text{out}(D_{n,p}) \geq K \log n \) with probability.

**Proof.** Assume that all the edges emanating from some vertex \( v \) are scanned. For each vertex \( u \), the existence of the directed edge \((v,u)\) from \( v \) to \( u \) is an independent random variable with probability \( p \). For all vertices \( u \) of the graph we check whether \((v,u)\) actually belongs to the graph. By Lemma 3 the probability to get \((n-1)p/2\) successes is greater than \(1-\exp(-(n-1)p/8)\). Since \( p \geq 2K \log n/n \), the probability is greater than \(1-\exp(-2\gamma \log n/8) = 1-n^{-\gamma/4}\), for some \( 4 < \gamma < K \) independent of \( n \).

However, we wish that the degree of all the vertices be greater than \( K \log n \). Since the existence of edges emanating from a vertex \( v \) is independent of the part of the graph explored so far, we have

\[
P\left( \text{for all } u : \text{out}(v) \geq K \log n \right) = (Pr(\text{out}(v) \geq K \log n))^n \geq (1-n^{-\gamma/4})^n \geq 1-n^{-\gamma/4} = 1-n^{-\gamma/4-1}.
\]

However, \( 4 < \gamma < K \) and therefore set \( \gamma = \gamma /4-1 > 0 \) and the outdegree of \( D_{n,p} \) is greater than \( K \log n \) with probability.

Let \( BD_{n,d} \) be a random variable whose values are directed graphs with \( n \) vertices and outdegree greater than \( d \), such that for all three distinct vertices \( u,v,w \):

\[
Pr((u,v) \in BD_{n,d}) = Pr((u,w) \in BD_{n,d}).
\]
From the previous lemma the graph \( D_{n,2K\log n/n} \) has degree at least \( K\log n \) with probability and therefore we obtain a random variable which satisfies with probability the requirements for \( BD_{n,K\log n} \).

Let \( \text{ROOT} \) be a vertex of a directed graph \( D \). We shall perform a breadth first search from \( \text{ROOT} \) on \( D \) until reaching half of the vertices ([n/2] vertices). As a result of the search a partial breadth first search tree, \( \text{PBFS}(D) \), is obtained. The depth of a directed tree is the length of the longest directed path.

**Lemma 6.** For \( K > 4 \): depth \( (\text{PBFS}(BD_{n,K\log n})) \leq 1 + \log n / \log \log n \) with probability.

**Proof.** Let \( B = \text{PBFS}(BD_{n,K\log n}), d = \text{depth}(B) \), and \( L_i \) be the number of vertices in the \( i \)-th level of \( B \). Let us look at the breadth first search algorithm when it is constructing the \( i+1 \)-st level.

Consider the construction of a level which is not the last one, i.e. \( i + 1 < d \). There are at least \( n/2 \) vertices not scanned. With probability at least half, an edge emanating from a vertex in the \( i \)-th level reaches a new vertex. By Lemma 3 the probability that at least \( L_i \cdot K\log n/2 \) new vertices are encountered is greater than or equal to
\[
1 - \exp(-L_i K\log n/8) > 1 - \frac{L_i}{2}.
\]

Since \( K > 4 \), \( K\log n/2 > \log n \), \( \Pr(L_{i+1}/L_i > \log n) > 1 - \frac{L_i}{2} \).

\( L_1 \), the number of vertices of the first level is equal to \( \text{out}(\text{ROOT}) \geq K\log n \). Since \( K > 4 \), \( L_1 \geq 5 \). However, if \( L_2/L_1 \geq \log n \) then \( L_2 \geq 5 \) and \( \Pr(L_2/L_3 > \log n) > 1 - n^{-5/2} \). Similarly, for \( i \leq d \)
\[
\Pr(L_i/L_{i-1} > \log n) > 1 - n^{-5/2}.
\]
Therefore,

\[ \Pr(\text{for all } i < d-1: \frac{L_i}{L_{i-1}} > \log n) > (1-n^{-5/2})^n > 1-n^{-3/2}. \]

If for \( i < d \) \( \frac{L_i}{L_{i-1}} > \log n \) then \( (\log n)^{d-1} \leq n/2 \), and

\[ d \leq 1 + \log_{\log n} n/2 < 1 + \log n / \log \log n. \]

Therefore, \( d < 1 + \log n / \log \log n. \)

Let \( H \) be a subgraph of a directed graph \( D \), a vertex \( v \in H \) is linked to \( H \) in \( D \) if there exists a vertex \( w \in H \) and a directed edge in \( D \) from \( v \) to \( w \).

**Lemma 7.** Let \( B \) be a partial breadth first search tree of \( BD_n, K \log n \), with probability all the vertices not in \( B \) are linked to \( B \) in \( BD_n, K \log n \).

**Proof.** We first calculate the probability that a vertex \( v \in B \) is linked to \( B \).

The outdegree of \( v \) is at least \( K \log n \) and \( B \) contains \( \lfloor n/2 \rfloor \) vertices. Any edge emanating from \( v \) has an equal chance to reach any other vertex. Therefore, it has probability \( \lfloor n/2 \rfloor / (n-1) \geq \frac{1}{2} \) to reach a vertex in \( B \). If the first edge did not reach \( B \) then the subsequent edges have even a greater probability to reach \( B \).

\[
\Pr(v \text{ is not linked to } B) \leq (\frac{1}{2})^{\text{out}(D)} \leq (\frac{1}{2})^{K \log n} = e^{-(\log 2)K \log n} = e^{\log n \cdot (-K \log 2)} = n^{-K \log 2}.
\]

\[
\Pr(v \text{ is linked to } B) \geq 1 - n^{-K \log 2}.
\]
For two vertices \( v, w \in B \) the edges emanating from \( v \) are statistically independent from the edges emanating from \( w \). Therefore,

\[
Pr(\forall v \in B \; v \text{ is linked to } B) = Pr(\forall v \in B \; v \text{ is linked to } B) [n/2] > (1-n^{-K\log n/2})^{n/2} > 1 - \frac{1}{2} n^{-K\log n} > 1 - (K\log n - 1).
\]

Since \( k > 4 \), \( K\log n - 1 > 1 \). Therefore, for all \( v \in B \) \( v \) is linked to \( B \) with probability.

\[\square\]

We wish to estimate the depth of a breadth first search conducted on \( G_{n,K\log n} \) for \( n, K \geq 4 \). To this end, a directed graph with the same distribution as \( D_{n,2K\log n} \) was produced and its underlying undirected graph is contained in the original one. Then a partial breadth first search tree \( B \) was constructed. With probability, all the vertices either belong to \( B \) or are linked to \( B \). Combining this result with Lemma 6 yields:

**Theorem 4:** The depth of a breadth first search tree on \( G_{n,p} \) for \( p \geq 16 \log n/n \) is bounded by \( 2 + \log n/\log \log n \) with probability.

i.e. the diameter of \( G_{n,p} \) is bounded by \( 2 + \log n/\log \log n \) with probability.

\[\square\]

**Corollary.** With probability, \( O(n^2 \log n/\log \log n) \) bounds the communication cost of a breadth first search tree on \( G_{n,p} \) \((p \geq 16 \log n/n)\).
REFERENCES


