QUANTUM PARTICLE MOTIONS IN POTENTIALS
POSITIVE POWERS OF THE DIRAC $\delta$ DISTRIBUTION

by

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ABSTRACT

One and three dimensional stationary motions are considered. The one dimensional wave function $\psi$ is given by $\psi''(x) + (k-\alpha(\delta(x))^m)\psi(x) = 0$, $x \in \mathbb{R}^1$, with $k, \alpha \in \mathbb{R}^1$, $m \in (0,\infty)$. The radial wave function $R$ of the three dimensional motion, assumed spherically symmetric and with zero angular momentum, is given by $(r^2R'(r))' + r^2(k-\alpha(\delta(r-a))^m)R(r) = 0$, $r \in (0,\infty)$, with $k, \alpha \in \mathbb{R}^1$, $a, m \in (0,\infty)$. The wave function solutions obtained are $C^\infty$ except eventually the support of the potential.

1. INTRODUCTION

Potentials with strong local singularities have recently been studied in scattering theory [1,3,4,8,9,16]. In [3], potentials given by a measure which need not be absolutely continuous with respect to the Lebesque measure were considered. The potentials in this paper, given by arbitrary positive powers of the Dirac $\delta$ distribution present the strongest local singularities considered yet. The positive powers of $\delta$ are defined within the associative and commutative algebras with unit element and containing $D'(\mathbb{R}^n)$, constructed in [10,11,12,15]. The present paper extends the approach suggested in [14], since the $\delta$ distribution need not be given by a non-symmetric representation. Moreover, all the positive powers of $\delta$ are defined within the algebras containing $D'$.

In §2, a short account of the construction of the algebras containing $D'$ is presented. The main result concerning the wave function solutions is given in §3.

2. ALGEBRAS CONTAINING $D'$

The set $\mathcal{W} = N \rightarrow C^\infty(\mathbb{R}^n)$ of all the sequences of complex valued smooth functions on $\mathbb{R}^n$ will be in the sequel the general background. If $s \in \mathcal{W}$, $v \in N$, $x \in \mathbb{R}^n$, then $s(v) \in C^\infty(\mathbb{R}^n)$, $s(v)(x) \in C^1$. For $\psi \in C^\infty(\mathbb{R}^n)$ denote $u(\psi) \in \mathcal{W}$, where $u(\psi)(v) = \psi$, $\forall v \in N$. $\mathcal{W}$ is an associative and commutative algebra over the complex numbers $\mathbb{C}^1$, considered with the term by term addition and multiplication of the sequences of smooth functions. The unit element is $u(1)$ and the null space is $O = \{u(0)\}$. Denote by $S_\mathcal{O}$ the set of all $s \in \mathcal{W}$ weakly convergent in $D'(\mathbb{R}^n)$ and by $V_\mathcal{O}$ the kernel of
the linear surjection $S \ni s \rightarrow \langle s, \cdot \rangle \in D'_{\mathbb{R}^n}$ where 
\[
\langle s, \psi \rangle = \lim_{\nu \to \infty} \int_{\mathbb{R}^n} s(\nu)(x)\psi(x)\,dx, \forall \psi \in D(\mathbb{R}^n).
\]
Then

(1) \[ S_0 / V_0 \ni (s + V_0) \rightarrow \langle s, \cdot \rangle \in D'_{\mathbb{R}^n} \]

is a vector space isomorphism. In [10, 11, 12, 15] it was shown that the following inclusion diagrams can be constructed:

\[
\begin{array}{ccc}
I & \rightarrow & A \\
\uparrow & & \uparrow \\
V & \rightarrow & S \leftarrow U \\
\uparrow & & \uparrow \\
V_0 & \rightarrow & S_0
\end{array}
\]

with $A$ subalgebra in $\hat{W}$, $I$ ideal in $A$ and $V, S$ vector subspaces in $S_0$, satisfying the condition:

(3) \[ I \cap S = V, V_0 \cap S = V, V_0 + S = S_0. \]

where $U = \{u(\psi) \mid \psi \in \mathcal{D}'(\mathbb{R}^n)\}$. A diagram (2) generates the linear embedding of $D'_{\mathbb{R}^n}$ into an associative and comutative algebra with unit element given in:

\[
\begin{array}{cccc}
D'_{\mathbb{R}^n} & \rightarrow & S_0 / V_0 & \rightarrow & S / V \\
\psi & \rightarrow & \psi & \rightarrow & \psi \\
\langle s, \cdot \rangle & \rightarrow & \langle \omega \rangle & \rightarrow & \langle s + V, \cdot \rangle \\
\text{Isom} & \rightarrow & \text{Isom} & \rightarrow & \text{Lin, Lin}
\end{array}
\]

Diagrams (2) are constructed as follows. A subalgebra $A$ in $\hat{W}$ is called admissible, only if it satisfies the following two conditions:

(5) \[ \forall w \in A, p \in \mathbb{N}^n : D^p w \in A \]

(6) \[ \forall w \in A \cap T_+, m \in (0, \infty) : w^m \in A \]
where \( D^P : W \rightarrow \tilde{W} \) with 
\[
(D^P w)(v) = D^P(w(v)), \quad \forall w \in W, \quad v \in N, \quad W_+ = \{w \in W \mid \\
\forall v \in N : w(v) \in C_+^\infty(R^n)\}, \quad w^m(v)(x) = (w(v)(x))^m, \quad \forall w \in W, \quad \forall v \in N, \quad x \in R^n
\]

and finally, 
\[
C_+^\infty(R^n) = \{\psi \in C_+^\infty(R^n) \mid \forall x \in R^n : \psi(x) \geq 0 \text{ and} \quad \forall m \in (0,\infty) : \psi^m \in C_+^\infty(R^n)\}. \quad \text{Obviously, } \tilde{W} \text{ is admissible and any intersection of admissible subalgebras in } \tilde{W} \text{ is an admissible subalgebra.}

Suppose \( V \) and \( S' \) are vector subspaces respectively in \( V_0 \) and \( S_0 \) such that

\[
(7) \quad S_0 = V_0 \oplus S', \quad U \subset S'
\]

For \( p \in \tilde{N}^n \) (\( \tilde{N} = N \cup \{\infty\} \)) denote \( V(p) = \{v \in V \mid \forall q \in N^n, q \leq p : D^q v \in V\}, \) \( A(V,S',p) \) the admissible subalgebra generated in \( \tilde{W} \) by \( V(p) \oplus S' \) and finally, \( I(V,S',p) \) the ideal generated in \( A(V,S',p) \) by \( V(p) \). In case the condition is satisfied

\[
(8) \quad I(V,S',0) \cap (V \oplus S') = V_0
\]

all the resulting inclusion diagrams of type (2), where \( p \in \tilde{N}^n \):

\[
\begin{align*}
I(V,S',p) & \rightarrow A(V,S',p) \rightarrow W \\
\uparrow & \quad \quad \uparrow \\
V(p) & \rightarrow V(p) \oplus S' \rightarrow U \\
\uparrow & \quad \quad \uparrow \\
V_0 & \rightarrow S_0
\end{align*}
\]

will satisfy the corresponding condition (3) and, therefore, will generate the linear embeddings of \( D'(R^n) \) into associative and commutative algebras with unit element:

\[
(9) \quad D'(R^n) \xrightarrow{e_P} A(V,S',p) = A(V,S',p)/I(V,S',p)
\]

where \( e_P = \beta_p \circ \alpha_p^{-1} \circ \omega^{-1} \) and \( \alpha_p : V(p) \oplus S'/V(p) \rightarrow S_0/V_0 \),
\[ \beta_p : V(p) \oplus S'/V(p) \rightarrow A(V,S',p) \] result from (4). The multiplication in \( A(V,S',p) \) induces on \( C^\infty(R^n) \) the usual multiplication of functions and the function \( \psi(x) = 1, \forall x \in R^n \) is the unit element in \( A(V,S',p) \), due to the fact that \( U \subset S' \). Linear mappings \( D^p_{p+q} : A(V,S',p+q) \rightarrow A(V,S',q) \) with \( D^p_{p+q}(s + I(V,S',p+q)) = D^p s + I(V,S',q) \), for \( p \in N^n, q \in N^n \) can be defined and the restriction of \( D^p_{p+q} \) to \( C^\infty(R^n) \) will be the usual derivative \( D^p \) of functions. Moreover, the Leibnitz rule of product derivative holds: if \( S,T \in A(V,S',p+q) \), for given \( p \in N^n, q \in N^n \), then the relation

\[ D^p_{p+q}(S \cdot T) = \sum_{k \in N^n} \left( \begin{array}{c} p \\ k \end{array} \right) (D^k_{p+q} S)(D^{p-k}_{p+q} T) \]

is valid in \( A(V,S',p+q) \), where \( \gamma_{\lambda,h} : A(V,S',\lambda) \rightarrow A(V,S',h) \), with \( \lambda, h \in N^n, h \leq \lambda \), is defined by \( \gamma_{\lambda,h}(s + I(V,S',\lambda)) = s + I(V,S',h) \).

Arbitrary positive powers of certain elements in \( A(V,S',p) \) are defined as follows. Suppose \( T \) is a vector subspace in \( S' \) and \( U \cap W_+ \subset T \subset S' \). Denote \( D^p_{T,+,R^n} = \{ \langle t, \cdot \rangle \mid t \in T \cap W_+ \} \), then obviously \( C^\infty(R^n) \subset D^p_{T,+,R^n} \subset A(V,S',p), \forall p \in N^n \). Given now \( m \in (0,\infty) \), one can define

(10) \[ D^p_{T,+,R^n} \exists T \rightarrow \imath^m \in A(V,S',p) \]

by \( \imath^m = \imath^m + I(V,S',p) \), where \( T = \langle t, \cdot \rangle, t \in T \cap W_+ \). It results that \( T = T, \imath^m + m'' = \imath^{m'} \cdot \imath^{m''}, \forall m', m'' \in (0,\infty) \) and \( (\imath^m)^{m'} = \imath^{m' \cdot m''}, \forall m', m'' \in (0,\infty) \), \( m' \in N \setminus \{0\} \). Moreover, the application in (10) restricted to \( C^\infty(R^n) \) is the usual power of functions. Finally, there exist \( T \) and \( t \in T \cap W_+ \) such that \( \langle t, \cdot \rangle = \delta \) ([15], Prop. 7, Chap. 1). Therefore, arbitrary positive powers of \( \delta \) can be defined according to the above procedure.
3. WAVE FUNCTION SOLUTIONS

The one dimensional wave function $\psi$ is given by

$$\psi''(x) + (k - U(x))\psi(x) = 0, \; x \in \mathbb{R}^1 \quad (k \in \mathbb{R}^1)$$

with the potential

$$U(x) = a(\delta(x))^m, \; x \in \mathbb{R}^1 \quad (a \in \mathbb{R}^1, m \in (0,\infty)).$$

The solution of (11), (12) is expected to be of the form

$$\psi(x) = \begin{cases} 
\psi_-(x) & \text{if } x < 0 \\
\psi_+(x) & \text{if } x > 0
\end{cases}$$

where $\psi_-, \psi_+ \in C^\infty(\mathbb{R}^1)$ are solutions of

$$\psi''(x) + k\psi(x) = 0, \; x \in \mathbb{R}^1$$

satisfying certain initial conditions

$$\psi_-(x_0) = y_0, \; \psi_-'(x_0) = y_1$$

$$\psi_+(x_1) = z_0, \; \psi_+'(x_1) = z_1$$

where $-\infty < x_0 \leq 0 \leq x_1 \leq \infty, y_0, y_1, z_0, z_1 \in \mathbb{C}^1$ are given and the vectors $\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$ might be dependent.

As known, [5], that happens in the case of $m = 1$ when for a potential $U(x) = a\delta(x), \; x \in \mathbb{R}^1 \quad (a \in \mathbb{R}^1)$, the connection in $x = 0$ between $\psi_-$ and $\psi_+$ is given by

$$\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
In the case of an arbitrary positive power $m \in (0, \infty)$, the following three problems arise:

1) to give a meaning to the power $(\delta(x)1)^m$, $x \in \mathbb{R}$;

2) to prove that the hypothesis (13) is correct, and

3) to obtain a relation generalizing (16).

The first problem is solved according to §2. The solution of the second problem results from [15], Theorem 8, Chap. 6 and is based on the smooth representation of $\delta$ constructed in §5. The third problem is solved in §4, using a standard "weak solution" approach. It is shown that the connection in $x = 0$ between $\psi_+$ and $\psi_-$ is given by

\[
(\psi_+(0)) = Z(m; \alpha) (\psi_-(0))
\]

where

\[
Z(m, \alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m \in (0, 1), \alpha \in \mathbb{R},
\]

\[
Z(1, \alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R} \quad \text{(see (16))},
\]

\[
Z(2, -\nu \pi)^2 = \begin{pmatrix} (-1)^\nu & 0 \\ 0 & (-1)^\nu \end{pmatrix}, \quad \nu = 0, 1, 2, \ldots
\]

\[
Z(m, \alpha) = \begin{pmatrix} \sigma & 0 \\ K & \sigma \end{pmatrix}, \quad m \in (2, \infty), \quad \alpha \in (-\infty, 0)
\]

with $\sigma = \pm 1$ and $-\infty \leq K \leq +\infty$ arbitrary.
For the values of \((m, a) \in (0, \omega) \times \mathbb{R}\) not mentioned in (17.1) - (17.4) the method employed in the present paper did not lead to solutions.

The interpretation of (17) in the case of the one-dimensional stationary motion of a quantum particle in the potential (12) results as follows:

1) If \(m \in (0,1)\), the potential (12), either well or wall, has no influence on the motion.

2) For \(m = 1\), the known, [5], motion is obtained.

3) If \(m = 2\), only for the discrete levels of the potential well

\[
U(x) = -\left(\frac{\alpha}{\pi}\right)^2(\delta(x))^2, \quad x \in \mathbb{R}, \quad \nu = 0, 1, 2, ...
\]

there is motion through \(x = 0\); moreover, for \(\nu\) even, the potential (18) has no influence on the motion, whereas for \(\nu\) odd, the potential (18) causes a sign change of the wave function \(\psi_+(x) = -\psi_-(x), x \in \mathbb{R}\).

4) If \(m \in (2, \omega)\), there is motion through \(x = 0\) in the case of a potential well only and the connection in \(x = 0\) between \(\psi_-\) and \(\psi_+\) is not uniquely determined due to the arbitrary values of \(\sigma\) and \(K\) in (17.4).

The three dimensional stationary motion, assumed spherically symmetric and with angular momentum zero, has the radial wave function \(R\) given by

\[
\left(r^2 R'(r)\right)' + r^2 \left(k - U(r)\right) R(r) = 0, \quad r \in (0, \omega) \quad (k \in \mathbb{R})
\]

with the potential concentrated on the sphere of radius \(a\)

\[
U(r) = \alpha(\delta(r-a))^m, \quad r \in (0, \omega) \quad (\alpha \in \mathbb{R}, m, a \in (0, \omega)).
\]
The solution of (19), (20) can be reduced, [5], to the solution of (11), (12). Therefore, the above interpretation for the one-dimensional motion will lead to a corresponding interpretation for the three-dimensional motion.

4. THE WEAK SOLUTION

The solution (13), (17) of (11), (12) is obtained in two steps. First, a particular nonsmooth representation of $\delta$ will give in Theorem 1 a weak solution of (11), (12). The second step, in §5, constructs a smooth representation of $\delta$ needed in the algebras containing $D^t(R^1)$. That representation gives the same weak solution, which proves to be a valid solution of (11), (12) within the mentioned algebras and therefore, independent of the representations used for $\delta$.

The nonsmooth representation of $\delta$, employed for the sake of simpler computation is given in

$$\delta(x) = \lim_{v \to \infty} V(\omega_v, 1/\omega_v, x), \ x \in R^1,$$

where

$$\lim_{v \to \infty} \omega_v = 0, \ \omega_v > 0, \ \forall \nu \in N$$

and

$$V(\omega, K, x) = \begin{cases} K & \text{if } 0 < x < \omega \\ 0 & \text{if } x \leq 0 \text{ or } x \geq \omega \end{cases}$$

for $\omega > 0$ and $K \in R^1$. 
Given $m \in (0, \infty)$, $\alpha \in \mathbb{R}^1$, $x_0 < 0$, $y_0$, $y_1 \in C^1$ and $v \in \mathbb{N}$, denote by $\psi_v \in C^\infty(\mathbb{R}^1 \setminus \{0, \omega_v\}) \cap C^1(\mathbb{R}^1)$ the unique solution of
\begin{equation}
(22) \quad \psi''(x) + (k - V(\omega_v, \alpha/(\omega_v)^m, x))\psi(x) = 0, \quad x \in \mathbb{R}^1,
\end{equation}
with the initial conditions
\begin{equation}
(22.1) \quad \psi(x_0) = y_0, \quad \psi'(x_0) = y_1.
\end{equation}

Given $k \in \mathbb{R}^1$ and $x_0 < 0$, denote by $M(k, x_0)$ the set of all $(m, \alpha) \in (0, \infty) \times \mathbb{R}^1$ for which there exists $(\omega_v \mid v \in \mathbb{N})$ satisfying (21.1), such that
\begin{equation}
(23) \quad \lim_{v \to \infty} \begin{pmatrix} \psi_v(\omega_v) \\ \psi_v'(\omega_v) \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}
\end{equation}
exists and finite, $\forall y_0$, $y_1 \in C^1$.

Now, for $(m, \alpha) \in M(k, x_0)$ and $y_0$, $y_1 \in C^1$ given, one can define $\psi_-, \psi_+ \in C^\infty(\mathbb{R}^1)$ as the unique solutions of (14), satisfying respectively the initial conditions
\begin{equation}
(24) \quad \begin{pmatrix} \psi_-(x_0) \\ \psi_-(x_0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} \psi_+(0) \\ \psi_+(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.
\end{equation}

**Theorem 1**

With $\psi_v$ given by (13) and $\psi_+, \psi_-, \psi_v$ defined above, the limit $\lim_{v \to \infty} \psi_v = \psi$ holds in the following sense:

1) $\psi = \psi_v$ on $(-\infty, 0)$, $\forall v \in \mathbb{N}$

2) for $p \in \mathbb{N}$, $\lim_{v \to \infty} D^p \psi_v = D^p \psi$ uniformly on every compact in $(0, \infty)$. 
Proof. For $h \in \mathbb{R}^1$ consider the differential equation

$$ (D^2 + h)\psi(x) = 0, \quad x \in \mathbb{R}^1, $$

and for $x \in \mathbb{R}^1$ define the $2 \times 2$ matrix $W(h,x) = \exp(xA_h)$, where

$$ A_h = \begin{pmatrix} 0 & 1 \\ -h & 0 \end{pmatrix}. $$

If $u \in C^\infty(\mathbb{R}^1)$ is the unique solution of (25) with the initial conditions $u(a) = b$, $Du(a) = c$, where $a \in \mathbb{R}^1$, $b$, $c \in \mathbb{C}^1$, then

$$ \begin{pmatrix} D^2Pu(x) \\ D^2P^{+1}(x) \end{pmatrix} = (-h)PW(h,x)W(h,a)^{-1} \begin{pmatrix} b \\ c \end{pmatrix}, \quad \forall \ p \in \mathbb{N}, \ x \in \mathbb{R}^1. $$

Thus, for given $\nu \in \mathbb{N}$ and $x \in \mathbb{N}$ one obtains

$$ \begin{pmatrix} D^2P\psi_\nu(x) \\ D^2P^{+1}\psi_\nu(x) \end{pmatrix} = (-k)PW(k,x)W(k,\omega_\nu)^{-1} \begin{pmatrix} \psi_\nu(\omega_\nu) \\ \psi_\nu^+(\omega_\nu) \end{pmatrix}, \quad \forall \ \nu \in \mathbb{N}. $$

But, obviously $\lim_{\nu \to \infty} W(k,\omega_\nu)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the proof is completed.

Theorem 2. The set $M(k,x_0)$ does not depend on $k \in \mathbb{R}^1$ and $x_0 < 0$, and

$$ M = ((0,1] \times \mathbb{R}^1) \cup \{(2) \times (-\nu \pi)^2 \mid \nu \in \mathbb{N}\} \cup ((2,\infty) \times (-\infty,0)) \cup ((0,\infty) \times \{0\}) $$

Proof. Assume $(m,\alpha) \in (0,\infty) \times \mathbb{R}^1$ and $p = 0$. Applying (26) for $x = x_0$ and then for $x = \omega_\nu$, one obtains

$$ \begin{pmatrix} \psi_\nu(\omega_\nu) \\ \psi_\nu^+(\omega_\nu) \end{pmatrix} = W(k - \alpha/(\omega_\nu)^m,\omega_\nu)W(k,x_0)^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \quad \forall \ \nu \in \mathbb{N}. $$
Therefore, \((m,a) \in M(k,x)\) only if

\[
\lim_{\nu \to \infty} W(k - a/(\omega_{\nu})^{m}, \omega_{\nu}) = Z(m,a) \text{ exists and it is finite.}
\]

Now, a direct computation will end the proof.

**Theorem 3.** Suppose given \((m,a) \in M\) and \(\gamma_{0}, \gamma_{1} \in C^{1}\). Then

\[
\begin{pmatrix}
\psi_{+}(0) \\
\psi'_{+}(0)
\end{pmatrix} = Z(m,a) \begin{pmatrix}
\psi_{-}(0) \\
\psi'_{-}(0)
\end{pmatrix},
\]

where \(Z(m,a)\) is given in (17.1)-(17.4). In addition, \(Z(m,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), \(\forall m \in (0,\infty)\).

**Proof.** It results easily from (27), (28) and (24).

5. A SMOOTH REPRESENTATION OF \(\delta\)

The smooth representation of \(\delta\) is obtained from (21) by "rounding off the corners". Suppose \(\beta, \gamma \in C^{\infty}(R)\) satisfy the conditions (see Lemma 1 below).

*1) \(\beta = 0\) on \((-\infty,1]\) and \(\beta = 1\) on \([1,\infty)\)

(29) *2) \(0 < \beta < 1\) on \((-1,1)\), where \(M \in [1,\infty)\)

*3) \(D^{p}\beta(0) \neq 0, \forall p \in N\)

respectively

*1) \(\gamma = 1\) on \((-\infty,1]\) and \(\gamma = 0\) on \([1,\infty)\)

(30) *2) \(0 < \gamma < 1\) on \((-1,1)\).
Suppose \(a, b: (0, \infty) \to (0, \infty)\), such that
\[
a(\omega) + b(\omega) < \omega, \quad \forall \omega \in (0, \infty).
\]
Define for \(\omega \in (0, \infty)\) the function \(V_*(\omega, \cdot) \in C^\infty(\mathbb{R}^1)\)
\[
(31) \quad V_*(\omega, x) = \beta(x/a(\omega))\gamma((x-\omega)/b(\omega))/\omega, \quad \forall x \in \mathbb{R}^1.
\]
Obviously,
\[
|1 - \int_{\mathbb{R}^1} V_*(\omega, x) \, dx| \leq ((2M-1)a(\omega) + b(\omega))/\omega, \quad \forall \omega \in (0, \infty)
\]
therefore, one can assume that \(a\) and \(b\) also satisfy the condition
\[
\lim_{\omega \to 0} \int_{\mathbb{R}^1} V_*(\omega, x) \, dx = 1.
\]
In that case
\[
(32) \quad \delta(x) = \lim_{\nu \to \infty} V_*(\omega_\nu, x), \quad x \in \mathbb{R}^1
\]
gives a weakly convergent (in \(D'(\mathbb{R}^1)\)) smooth representation of \(\delta\), for any \((\omega_\nu | \nu \in \mathbb{N})\) satisfying (21.1).

Moreover, the relations (29), (30) and (31) will imply
\[
(33) \quad (V_*(\omega_\nu, \cdot) | \nu \in \mathbb{N}) \in \mathcal{S}_0^+ \cap \mathcal{W}_+
\]
therefore, arbitrary positive powers of \(\delta\) can be defined by (32), according to the procedure in §2.

We prove now that the smooth representation of \(\delta\) given in (32) generates the same weak solution of (11) and (12) as the one obtained in Theorem 1. Indeed, given \((m, a) \in M, x_0 < 0, y_0, y_1 \in C^1\) and \(\nu \in \mathbb{N},\)
denote by \( \psi_{x,v} \in C^\infty(\mathbb{R}^1) \) the unique solution of
\[
\psi''(x) + (k - \alpha(V_x(\omega_y,x))^m)\psi(x) = 0, \ x \in \mathbb{R}^1
\]
with the initial conditions \( \psi(x_0) = y_0, \ \psi'(x_0) = y_1 \).

**Theorem 4** With \( \psi \) in Theorem 1 and \( \psi_{x,v} \) defined above, the limit
\[
\lim_{v \to 0} \psi_{x,v} = \psi
\]
holds in the following sense:

1) \( \psi = \psi_{x,v} \) on \( (-\infty, \max\{x_0,-\omega_y\}] \), \( \forall \ v \in \mathbb{N} \)

2) \( \lim_{v \to 0} \begin{pmatrix} \psi_{x,v} \\ \psi'_{x,v} \end{pmatrix} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \)
uniformly on every compact in \( (0,\infty) \).

**Proof.** It results by a standard argument based on Gronwall's inequality.

**Lemma 1.** There exists \( \beta \in C^\infty(\mathbb{R}^1) \) satisfying (29).

**Proof.** Define \( \beta_1 \in C^\infty(\mathbb{R}^1) \) with \( \beta_1(x) = 0 \) for \( x < 0 \) and
\( \beta_1(x) = \exp(-1/x) \) for \( x > 0 \). Assume \( 0 < c, \ d \leq 1 \) and define
\( \beta_2, \beta_3 \in C^\infty(\mathbb{R}^1) \), with \( \beta_2(x) = \beta_1(x + 1)/[\beta_1(x+1) + \beta_1(-x-c)] \) and
\( \beta_3(x) = \beta_1(1-x)/[\beta_1(1-x) + \beta_1(x-d)] \). Define now \( \beta \in C^\infty(\mathbb{R}^1) \) with
\( \beta(x) = (\beta_2(x)\exp(x) - 1)\beta_3(x) + 1 \). Then \( \beta \) will satisfy (29) with \( M = e \).

In order to complete the answer to the second problem formulated in
§3, we recall under a suitable form Theorem 8 in 15, Chap.6:

**Theorem 5.** Suppose given \( k \in \mathbb{R}^1, \ (m,\alpha) \in \mathcal{M} \) and a weak solution \( \psi \) of
(11), (12) as obtained in Theorem 1. Then there exist \( V, S^1 \) satisfying
(7), (8) and
(34) \( \{ \omega_v, \cdot \} | \nu \in N, \in S' \cap \mathbb{N}_+ \)

for a certain \( (\omega_v | \nu \in N) \) as in (21.1).

Further, there exists \( s \in S' \) such that

(35) \[ \psi = s + I(V, S', p) \in A(V, S', p), \forall p \in \mathbb{N} \]

and finally

(36) \[ \frac{d^2}{dp^2} \psi + (k - a(\delta)^m) \psi = 0 \in A(V, S', p), \forall p \in \mathbb{N}. \]
REFERENCES


