The Complexity of Finding Maximum Disjoint Paths with Length Constraints

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ABSTRACT

The following problem is considered: Given an integer K, a graph G with two distinct vertices s and t, find the maximum number of disjoint paths of length K from s to t. The problem has several variants: the paths may be vertex-disjoint or edge-disjoint, the lengths of the paths may be equal to K or bounded by K, the graph may be undirected or directed.

It is shown that except for small values of K all the problems are NP-complete. For each problem, the largest value of K for which the problem is not NP-complete is found. Whenever a polynomial algorithm exists, an efficient algorithm is described.
1. INTRODUCTION

In the reliability analysis of a communication network it is often required to find a maximum number of disjoint paths between two given vertices of the network. This (polynomially solvable) problem becomes more complicated when we insist that the length of each path does not exceed a given value. (We may wish to restrict the length so as to keep the noise interference under control.)

This research is strongly motivated by a practical analysis problem of a communication network. Problems concerning paths of definite length are also considered, because of their mathematical interest. A simpler problem of finding $K$ disjoint paths of minimum total cost is solved efficiently by Suurballe, [S].

Following [E] let $G = (V,E)$ be an undirected graph without multiple edges and self loops, and let $s$ and $t$ be two distinct vertices of $G$. Henceforth, path stands for a simple (not crossing itself) path between $s$ and $t$. The length of a path is the number of its edges.

Four different problems are considered in this paper. They can be stated jointly as follows: Find a maximum number of vertex (edge) disjoint paths of length (bounded by) $K$ in $G$.

In Section 3 we prove that these problems (stated as decision problems), are NP-complete both for vertices and edges, when $K \geq 4$ ($K \geq 5$ for the bounded case). Several other related problems are shown to be NP-complete.

In Section 2 we present polynomial algorithms for most of the cases of smaller lengths. We also solve the problem of finding maximum vertex (edge) disjoint shortest paths.
Each problem is represented by two letters and a number. The first letter is V or E depending on whether vertex or edge disjoint paths are sought. The second letter is B or D. B indicates that the length of the paths are bounded by K (the bounded case) while D indicates that the length of each path is equal to K (the definite case). The number is the value of K. For example, VB4 denotes the problem of finding vertex disjoint paths of length bounded by 4.

We also use the following notations: V-disjoint for vertex-disjoint, E-disjoint, for edge-disjoint. A \(K\)-path (K-bounded path) is a path of length (bounded by) \(K\). The sets, \(S = \{v \in V; (s,v) \in E\}\), \(T = \{v \in V; (v,t) \in E\}\). A path is sometimes denoted by \((v_1, \ldots, v_m)\) - the sequence of its vertices.
2. POLYNOMIAL ALGORITHMS

2.1 V-Disjoint Paths

V D1, V D2, VB1 and VB2 are trivial.

In the following problems we may assume that G does not contain an edge from s to t.

V D3: Let A = S ∩ T. Let G' = (V', E') be defined as follows:
V' = S ∩ T, E' = {(u, v) ∈ E: u ∈ S and v ∈ T} (see Fig. 2.1).

Figure 2.1

There exists a 3-path (s, u, v, t) in G, if and only if (u, v) ∈ E'.

Moreover, two 3-paths (s, u, v, t) and (s, x, y, t) are V-disjoint if and only if the edges (u, v) and (x, y) are V-disjoint. Thus, a maximum matching in G' yields a solution for our problem. (Note that G' is not necessarily bipartite.) Gabow [G] provides an O(∥V∥^3) algorithm for solving the maximum matching problem. This problem has recently been solved by Even and Kariv [EK] in O(∥V∥^2.5) time. It is easy to construct
a reduction from the maximum matching problem to VD3. Thus the two problems are linearly equivalent.

Maximum V-Disjoint Paths of Shortest Length

We first perform a breadth first search in $G$ starting from $s$. Thus, obtaining an auxiliary directed subgraph $G' = (V, E')$ containing all the edges of $G$ which are directed by the breadth first search from one level to the next. There exists a one-to-one correspondence between the shortest paths in $G$ and the directed paths (from $s$ to $t$) in $G'$. (Actually the following algorithm may become more efficient if we apply a second breadth first search backwards from $t$ to obtain a subgraph of $G'$ containing exactly all the edges of all shortest paths in $G$.) Maximum V-disjoint paths in $G'$ are obtained by applying maximum flow techniques in a flow network in which the vertices and edges have unit capacity. Even and Tarjan [ET] show that Dinic's [D] algorithm requires $O(|V|^2 |E|)$ time for this case. Thus, our algorithm requires $O(|V|^2 |E|)$ time.

**VD3:** There exists a solution in which the vertices of $A = S \cap T$ do not participate in any 3-path, since each such 3-path can be shortened into a 2-path. Thus, we start by producing $|A|$ V-disjoint 2-paths through the vertices of $A$. We now remove these vertices and apply the algorithm for maximum V-disjoint shortest paths.

**VD4:** As in VD3, we first produce $|A|$ V-disjoint 2-paths and remove the vertices of $A$ from $G$. Henceforth, we assume that $A = \emptyset$. Let $B$ denote the set of all vertices of $V - (S \cup T)$ which are adjacent with at least one vertex of $S$ and one vertex of $T$. Any vertex which does not belong to
\{(s,t) U S U T U B\} does not participate in any 3 or 4-path. Moreover, any edge connecting two vertices of B does not participate in any 3 or 4-path. We may also assume that the edges connecting two vertices of S (or two vertices of T), do not participate in our solution, since the corresponding 4-paths can be shortened. Hence, we are left with the graph G', which appears in Figure 2.2.

Figure 2.2

Let us place a new vertex on each edge which connects a vertex of S with a vertex of T, (i.e. if \((u,v)\) is such an edge, it is replaced by the edges \((u,uv)\) and \((uv,v)\), where \(uv\) is a new vertex). Our problem is then reduced to that of finding maximum Y-disjoint shortest (4-)paths.
2.2 E-Disjoint Paths

ED1, ED2, EB1 and EB2 are trivial.

Maximum E-Disjoint Paths of Shortest Length

We construct the same auxiliary subgraph of the V-Disjoint case. Applying Dinic's [D] algorithm for this subgraph with unit edge capacities yields on \(O(|V|^{2/3}|E|)\) solution, (see [ET]).

**EB3:** Let \(G' = (V', E')\) be defined by

\[ V' = \{s, t\} \cup S \cup T \]

\[ E' = E_s \cup E_t \cup \{(u, v) \in E: u \in S \text{ and } v \in T\}, \]

where \(E_s(E_t)\) denotes the set of edges incident with \(s(t)\), (see Fig. 2.3a).

Every 3-bounded path in \(G\) is contained in \(G'\). Thus, it is sufficient to solve the problem for \(G'\).

Figure 2.3
The graph $G' = (V', E')$ is obtained from $G$ by replacing each vertex $a \in A = S \cap T$ by two vertices $a'$ and $a''$ which are connected by an edge $(a', a'')$. An edge $(v, a)$ is replaced by $(v, a')$ if $v \in \{s\} \cup (T - A)$ and by $(v, a'')$ if $v \in \{t\} \cup (S - A)$. Each edge of the form $(a_1, a_2)$ where $a_1, a_2 \in A$, is replaced by the two edges $(a'_1, a''_2)$ and $(a''_1, a'_2)$ (see Figure 2.3).

There is a one-to-one correspondence between the 3-bounded paths in $G'$ and the 3-paths in $G'$. Moreover, any two paths in $G'$ which correspond to two $E$-disjoint 3-bounded paths in $G'$, are also $E$-disjoint. On the other hand, there is just one case in which the 3-bounded paths $Q_1$ and $Q_2$ which correspond to two $E$-disjoint 3-paths $P_1$ and $P_2$ in $G'$, are not $E$-disjoint in $G'$. This happens when $P_1 = (s, a_1', a_2'', t)$ and $P_2 = (s, a_2', a_1'', t)$. However, in this case $Q_1$ and $Q_2$ may be replaced by the two disjoint paths $(s, a_1, t)$ and $(s, a_2, t)$.

Thus, our problem is reduced to finding a maximum number of disjoint shortest (3-) paths in $G'$.

3. NP-COMPLETENESS

In this section we refer to the problems as decision problems. For example, EB5 denotes the following problem:

Given $G = (V, E); s, t \in V$ and a positive integer $m$. Are there $m$ $E$-disjoint paths of length bounded by $5'$?

Let 3-SAT denote the satisfiability problem of conjunctive normal forms with three literals per clause $[[C_1, [K]], \{x_1, \ldots, x_n\}$ are the variables and $\{C_1, \ldots, C_p\}$ are the clauses.]

**Lemma 3.1** Let $\Phi$ be an instance of 3-SAT, there exists a Boolean formula $\psi$ such that:
(a) \( \psi \) is an instance of 3-SAT.
(b) \( \psi \) is satisfiable if and only if \( \psi \) is
(c) the number of occurrences of \( x_i \) in \( \psi \) is equal to that of \( \bar{x}_i \) \( (i = 1, \ldots, n) \).

Proof. Let \( m_i(m_i') \) denote the number of occurrences of \( x_i(x_i') \).
Assume that \( m_i - m_i' = \xi > 0 \). We may add the clauses \( C_{p+1} = C_{p+2} = \ldots = C_{p+\xi} = [x_i \lor x_{n+1} \lor x_{n+1}] \) where \( x_{n+1}(x_{n+1}) \) is a new variable.
This addition does not effect the satisfiability of the formula since
\( C_{p+1}, \ldots, C_{p+\xi} \) are satisfiable in any truth assignment. Moreover
\( m_i = m_i' \) and \( m_{n+1} = m_{n+1} ' \).

Q.E.D.

Theorem 3.2. VB5 is NP-complete.

Proof. Obviously, VB5 belongs to the class NP.

We show that 3-SAT is polynomially reducible to VB5. Let
\( \phi = C_1 \land \ldots \land C_p \) be an instance of 3-SAT such that \( m_i = m_i' \) for \( i = 1, \ldots, n \).
We construct a graph \( G \) with the property that \( \phi \) is satisfiable if and
only if \( G \) contains \( p + m (m = \sum_{i=1}^{n} m_i) \) \( \nu \)-disjoint \( s \)-bounded paths.

\( G \) contains \( n \) subgraphs \( G_1, \ldots, G_n \) which have only \( s \) and \( t \) in
common. The subgraph \( G_i \) is associated with the variable \( x_i \). The vertex
\( x_{ik} (\bar{x}_{ik}) \) of \( G_i \) corresponds to the \( k \)-th occurrence of \( x_i(x_i) \) in \( \phi \).
\( G_i \) is depicted in Figure 3.1. (Figures 3.1a and 3.1b should be superimposed
one on top of the other, such that the vertex \( x_{ik} \), for example, is connected
to the vertices \( v_{ik}, w_{ik} \) and \( y_{ik} \).)
Apart from $G_1, \ldots, G_n$, the graph $G$ contains $p$ vertices $c_1, \ldots, c_p$, the edges between each $c_i$ and $t$ ($i = 1, \ldots, p$), and if the $k$-th occurrence of $x_1$ ($x_i$) is in $C_j$ an edge between $x_{ik}$ ($x_{ij}$) and $c_j$.

Figure 3.2 depicts the graph associated with

$$\varphi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor \overline{x}_3) \land (x_1 \lor \overline{x}_1 \lor x_2) \land (x_1 \lor \overline{x}_1 \lor \overline{x}_2).$$
Assume that $\varphi$ is satisfiable. Let $t(x_i)$ denote the truth value of $x_i$. If $t(x_i) = \text{true}$ ($t(x_i) = \text{false}$) then every $G_i$ contributes $m_i$ disjoint $S$-paths which pass through the vertices $x_{ik}$ ($x_{ik}$).

(See Figure 3.3.)

Thus, we have $m = \sum m_i$ disjoint paths which are called $A$-paths. Since $\varphi$ is satisfiable each $G_j$ contains a literal $x_i$, for example, such that $t(x_i) = \text{true}$. Therefore, the corresponding vertex $x_{ik}$ does not belong to any of the $A$-paths. Thus, the path $(s,u_i,k,v_i,k,x_{lk},k,c_j,t)$ is disjoint to all the $A$-paths. Such a path is called a $B$-path. Each $G_j$ contributes one $B$-path. These $p$ $B$-paths are pairwise disjoint. Thus, if $\varphi$ is satisfiable, there exist $m+p$ disjoint, $S$-bounded paths in $G$.

Assume now that $G$ contains $m+p$ vertex-disjoint $S$-bounded paths.
The degree of \( t \) is exactly \( m + p \) and therefore each edge entering \( t \) participates in a path. An edge of the form \((z_{ik}, t)\) may participate only in one of the two 5-paths \((s, w_{ik}, s_{ik}, y_{ik}, z_{ik}, t)\) or \((s, w_{ik}, k + (\text{mod } m), s_{ik}, y_{ik}, z_{ik}, t)\). Both are \( A \)-paths and are contained in \( G_i \). Thus, \( G_i \) must contain \( m \) disjoint \( A \)-paths. \( G_i \) is designed so that either all these \( A \)-paths contain the vertices \( x_{ik} \) and none of the vertices \( y_{ik} \), or vice-versa. In the first case, we set \( t(x_j) = \text{false} \) otherwise \( t(x_j) = \text{true} \). Thus, \( t(x_j) = \text{true} \) if and only if, \( x_{i1}, \ldots, x_{im} \smallsetminus \{x_{ik}\} \) do not belong to any \( A \)-path.

The other \( p \) 5-bounded paths contain \( c_1, \ldots, c_p \) and therefore they must be \( B \)-paths. (The vertices which participate in the \( A \)-paths.) If a \( B \)-path contains both \( c_j \) and \( x_{ik} \) then \( x_i \in C_j \) and \( t(x_j) = \text{true} \). Thus \( \varphi \) is satisfiable.

Q.E.D.

**Theorem 3.3**

EB5,VD4 and ED4 are NP-complete.

It can be shown that 3-SAT is reducible to each of these three problems. The constructions and the proofs of each case are very similar to those of Theorem 3.2. For each case, the appropriate \( G_i \) is given in Figure 3.4, the \( B \)-path appears in solid lines. The details can be easily derived.
Figure 3.4
Theorem 3.4

VDK, EDK (VBK, EBK) are NP-complete for all $K \geq 4$ ($K \geq 5$).

Proof. We show, for example, that VD4 is polynomially reducible to VDK for $K > 4$, as follows: Let $G$ be a VD4 problem, place $K-4$ vertices on every edge incident with $s$. A solution to the VDK problem yields a solution to the VD4 problem. Q.E.D.

Consider the following decision problem. Given a graph $G$, two vertices $s,t \in G$ and a positive integer $K$. Do there exist two vertex (edge) disjoint $K$-paths ($K$-bounded) paths in $G$? The notations 2VB, 2EB, 2VD, 2ED indicate which of the four versions of this problem is discussed.

Theorem 3.5

2VD and 2ED are NP-complete.

Proof. Let $G = (V,E)$ and let $u,v \in V$. The existence of a $(u,v)$ Hamiltonian path is NP-complete. Let $G'$ denote the graph in Figure 3.5. It is easy to see that $G'$ contains a Hamiltonian $(u,v)$ path if and only if $G'$ contains two $V$-disjoint (E-disjoint) paths of length $|V| + 1$.

Q.E.D.

An interesting open question is whether 2VB and 2EB are NP-complete. Assume that the edges of $G$ have weights and the length of a path is the sum of the weights of its edges. We then have:
Theorem 3.6. The weighted 2VB and 2EB are NP-complete.

Proof. We show that there exists a polynomial reduction to the weighted 2VB and 2EB from the following (NP-complete) partition problem [K]:

Given the integers $c_1, \ldots, c_n$, does there exist a set $I \subseteq N = \{1, \ldots, n\}$ such that $\sum_{i \in I} c_i = \sum_{i \in N/I} c_i$?

Consider the graph $G$ in Figure 3.6. It is easy to see that $G$ contains two $V$-disjoint ($E$-disjoint) paths of length $\leq \frac{1}{2} \sum_{i=1}^{n} c_i$ if and only if the partition problem has a solution. Q.E.D.

Figure 3.5

Figure 3.6
4. CONCLUSIONS

4.1 Summary

The following table summarizes the main results of this paper. It suggests that the definite length problems are more difficult than the bounded length problems and the E-disjoint problems are more complex than the V-disjoint problems.

<table>
<thead>
<tr>
<th>K</th>
<th>3</th>
<th>4</th>
<th>≥5</th>
<th>Shortest</th>
</tr>
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<td>VDK</td>
<td>$</td>
<td>V</td>
<td>^2.5$</td>
<td>NPC</td>
</tr>
<tr>
<td>VBK</td>
<td>$</td>
<td>V</td>
<td>^{1.1}</td>
<td>E</td>
</tr>
<tr>
<td>EDK</td>
<td>?</td>
<td>NPC</td>
<td>NPC</td>
<td>$</td>
</tr>
<tr>
<td>EBK</td>
<td>$</td>
<td>V</td>
<td>^{2/3.1}</td>
<td>E</td>
</tr>
</tbody>
</table>

4.2 The Directed Problems

All the results concerning V-disjoint paths remain valid when G is a directed graph and the paths are directed paths. However, there are differences between the directed and undirected problems concerning E-disjoint paths. The directed problems which differ from their corresponding undirected problems are:

(1) ED3 is polynomially solvable, using a construction similar to that of EB3. (Only the edges of the form (a',a'') are omitted.)

(2) ED4: The complexity of this problem is unknown.
These differences suggest that the E-disjoint undirected problems are more complex than the corresponding directed problems.

4.3 The Non Existence of a "Menger" Theorem

Let \( G \) be the graph shown in Figure 4.1.

Two edges must be removed in order to disconnect all the 4-(4-bounded) paths in \( G \). Nevertheless, there are no two E-disjoint 4-(4-bounded) paths in \( G \). Thus, the edge-form of Menger's theorem \([E]\) is not true as far as \( K-(K\text{-bounded}) \) paths are concerned. The line-graph of \( G \) can be used in order to construct a counter-example to the vertex-form of Menger's theorem for \( K(K\text{-bounded}) \) paths.

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REFERENCES


