NONLINEAR SHOCK WAVES AND DISTRIBUTION MULTIPLICATION

by

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ABSTRACT

The distribution space $D'(\mathbb{R}^n)$ is embedded into associative and commutative algebras with unit element. The shock wave solutions of the non-linear partial differential equation

$$u_t(x,t) + a(u(x,t)) u_x(x,t) = 0; \quad x \in \mathbb{R}^1, \ t > 0, \ u(x,0) = u_0(x), \ x \in \mathbb{R}^1,$$

where $a$ is a polynomial, will satisfy that equation in the usual algebraic sense, with the multiplication in the algebras containing $D'(\mathbb{R}^n)$.

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1. INTRODUCTION

The problem of defining a suitable multiplication for any pair of distributions in $D'(\mathbb{R}^n)$ appeared early in the development of the theory of distributions. The successful application of that theory to the study of linear partial differential equations pointed out the possible role a distribution multiplication could play in the case of nonlinear partial differential equations. Certain known basic facts came as a confirmation of that idea. For instance, the appearance of shock discontinuities in the solutions of nonlinear hyperbolic partial differential equations, [13] (see also [6], [9], [25]) corresponding even to analytic initial data indicated that in the nonlinear case the distribution multiplication could be needed starting with a rigorous and general definition of the notion of solution.

The problem of defining the convolution for arbitrary pairs of distributions originated from the study of linear partial differential equations. However, due to the Fourier transform, that problem is naturally related to the problem of distribution multiplication [10]. The well known examples of nonassociativity of the convolution suggested the possible difficulties within a distribution multiplication theory.

L. Schwartz's paper [26] in 1954, presented a first account of those difficulties. It showed that even in the case of $D'(\mathbb{R}^1)$, it was not possible to define an associative multiplication with unit element and satisfying a few rather natural embedding conditions.

A similar event happened in 1957, when H. Lewy gave a simple example of first order linear partial differential equation with three independent variables and the coefficients polynomials of degree at most one, without
solution in \( D' \). In 1960, L. Hormander obtained general sufficient conditions for a linear partial differential equation with variable coefficients in order to be unsolvable in \( D' \). Later, in 1963, L. Nirenberg and F. Treves gave necessary and sufficient conditions for solvability in \( D' \) in the case of first order linear partial differential equations with analytic coefficients.

The problem of defining a multiplication for all the distributions in \( D' \) was approached in numerous occasions. The first significant attempt was due to H. Konig [11] in 1955. Several of the more important further attempts were those in [1], [2], [4], [5], [7], [8], [12], [16], [17], [24], [27] and [28]. The multiplications suggested failed to be in the same time associative, commutative, with unit element, defined for each pair in \( D' \) and possessing certain natural embedding properties.

The associative and commutative multiplication with unit element constructed here for the distributions in \( D'(\mathbb{R}^n) \) is presented in Chapter 1 under a general form and illustrated in Chapter 2 in a particular case, useful in the study of nonlinear shock waves. That multiplication theory develops older ideas, first presented in [19], [20] and improved later in [21], [22]. The ideas and methods belong to the "sequential approach" of the distributions, suggested almost three decades ago by J. Mikusinski [15] and outlined perhaps in the best way what concerns the multiplication and other "irregular operations" in [16]. Within that approach the distributions are replaced by classes of weakly convergent sequences of smooth functions, in the spirit of various notions of 'weak solutions', used in the applications of partial differential equations. However, from the point of view of the distribution multiplication, the most important pattern of the sequential
The approach is that it offers a natural associative and commutative multiplication with unit element, namely the term by term multiplication of any two sequences of smooth functions. The distribution multiplication will be obtained by a special regularization of that multiplication of sequences, embedding $D'(\mathbb{R}^n)$ into associative and commutative algebras with unit element, the elements of the algebras being classes of sequences of smooth functions. In addition to being associative, commutative and with unit element, the multiplication will possess the strongest - in a certain sense - properties of embedding and of derivative, which are still weaker than those in [26] implying the impossibility of multiplication. Several other properties, not mentioned in [26], will be obtained, for instance, the existence of positive powers of certain distributions, among them the Dirac $\delta$ distribution (see [27]).

The concept of multiplication introduced, enables to establish - see Chapter 3 - that shock wave solutions of nonlinear partial differential equations [14],

$$u_t(x,t) + a(u(x,t)) \cdot u_x(x,t) = 0, \quad x \in \mathbb{R}^1, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^1$$

where $a$ is an arbitrary polynomial, satisfy these equations in the usual algebraic sense, with the multiplication in the algebras containing $D'$. 

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CHAPTER 1. ALGEBRAS CONTAINING $D'(\mathbb{R}^n)$

1.1 Algebras of sequences of smooth functions

The set

$\mathcal{H} = \mathbb{N} \rightarrow C^\infty(\mathbb{R}^n)$

of all sequences of complex valued smooth functions on $\mathbb{R}^n$ will be the general background in the sequel. If $s \in \mathcal{H}$, $v \in \mathbb{N}$, $x \in \mathbb{R}^n$, then $s(v) \in C^\infty(\mathbb{R}^n)$, $s(v)(x) \in C^1$. For $\psi \in C^\infty(\mathbb{R}^n)$ denote $u(\psi) \in \mathcal{H}$, where $u(\psi)(v) = \psi$, $\forall v \in \mathbb{N}$.

With the term by term addition respective multiplication of the sequences, $\mathcal{H}$ is an associative and commutative algebra with the unit element $u(1)$. The null subspace of $\mathcal{H}$ is $O = \{u(0)\}$; (The vector spaces and algebras are considered throughout over the field $\mathbb{C}^1$ of the complex numbers.)

Denote by $\mathcal{S}_o$ the set of all $s \in \mathcal{H}$ weakly convergent in $D'(\mathbb{R}^n)$ and by $\mathcal{V}_o$ the kernel of the linear surjection:

$\mathcal{S}_o \ni s \rightarrow <s, \cdot> \in D'(\mathbb{R}^n)$

where $<s, \psi> = \lim_{v \to \infty} \int_{\mathbb{R}^n} s(v)(x)\psi(x)dx$, $\forall \psi \in D(\mathbb{R}^n)$.

Then

$\mathcal{S}_o/\mathcal{V}_o \ni (s + \mathcal{V}_o) \xrightarrow{\omega} <s, \cdot> \in D'(\mathbb{R}^n)$

is a vector space isomorphism.

Since $\mathcal{H}$ is an algebra, one can ask whether it is possible to define a product of any two distributions $<s, \cdot>$, $<t, \cdot> \in D'(\mathbb{R}^n)$, by the product...
of the classes of sequences $s + V_0$ and $t + V_0$.

A simple way would be to construct diagrams of inclusions:

$$
\begin{array}{c}
I \longrightarrow A \longrightarrow \mathcal{W} \\
\begin{array}{c}
\uparrow \\
V_0 \longrightarrow S_0
\end{array}
\end{array}
$$

with $A$ subalgebra in $\mathcal{W}$, $I$ ideal in $A$ and

(4.1) $I \cap S_0 = V_0$

which would generate the following linear embedding of $D^1(R^n)$ into associative and commutative algebras with unit element:

$$
\begin{array}{cccc}
D^1(R^n) & S_0/V_0 & A/I \\
\psi & \psi & \psi
\end{array}
$$

However, diagrams of type (4) cannot be constructed since

(5) $(V_0 \cdot V_0) \cap S_0 \neq V_0$

for instance, if $n = 1$, take $v(v)(x) = \cos(v+1)x$, $v \in \mathbb{N}$, $x \in \mathbb{R}$ then $v \in V_0$, $v^2 \in S_0$ and $v^2 \notin V_0$ since $\langle v^2, \cdot \rangle = 1/2$.

An other way could be given by diagrams:

$$
\begin{array}{c}
I \longrightarrow A \longrightarrow \mathcal{W} \\
\begin{array}{c}
\downarrow \\
V_0 \longrightarrow S_0
\end{array}
\end{array}
$$

with $A$ subalgebra in $\mathcal{W}$, $I$ ideal in $A$ and
(6.1) \( V_0 \cap A = I \)

(6.2) \( V_0 + A = S'_0 \)

which would generate the following linear injection of associative and commutative algebras onto \( D'(\mathbb{R}^n) \):

\[
\begin{array}{ccc}
D'(\mathbb{R}^n) & \xrightarrow{\psi} & S'_0 / V_0 \xrightarrow{\phi} A / I \\
\langle s, \cdot \rangle & \xrightarrow{\text{isom}} & s + V_0 & \xrightarrow{\text{lin sur}} & s + I
\end{array}
\]

Here, the problem arises connected with (6.2). It is not possible to choose \( A \) containing even some of the frequently used types of "\( \delta \) sequences", \([17]\), as results from (see the proof in Appendix, §1.8):

**Lemma 1** Suppose \( s \in \mathcal{H} \) is a sequence of real valued smooth functions, such that \( \text{supp } s(v) \subseteq B(0, a_v) \), \( \forall v \in \mathbb{N} \), where \( a_v > 0 \) and \( \lim_{v \to \infty} a_v = 0 \).

(for \( x \in \mathbb{R}^n \), \( a > 0 \) we denote \( B(x, a) = \{ y \in \mathbb{R}^n | d(x, y) < a \} \), where \( d( , , ) \) is the Euclidian distance).

Then

1) \( s \in S'_{0} \) and \( \langle s, \cdot \rangle = \delta \) (the Dirac distribution), only if \( \lim_{v \to \infty} \int_{\mathbb{R}^n} s(v)(x) dx = 1 \);  

2) if \( s \in S'_{0} \) and \( \langle s, \cdot \rangle = \delta \), then \( s^2 \in S'_{0} \).

The importance of the "\( \delta \) sequences" mentioned above is well known due to the special smooth approximations which they generate by the convolutions \( f_v = f \ast s(v) \), for functions \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \).
In [19], [20], [21], [22] it was shown (see also Theorem 1 in §1.4) that the following, slightly more complicated inclusion diagrams can be constructed:

\[
\begin{align*}
& I \to A \to W \\
& V \to S \\
& V_o \to S_o
\end{align*}
\]

with \( A \) subalgebra in \( W \), \( I \) ideal in \( A \) and \( V, S \) vector subspaces in \( S_o \), satisfying the conditions

(7.1) \( I \cap S = V \)

(7.2) \( V_o \cap S = V \)

(7.3) \( V_o + S = S_o \)

and thus generating the following linear embedding of \( D'(R^n) \) into associative and commutative algebras with unit element:

\[
\begin{align*}
& D'(R^n) \xrightarrow{\psi} S_o/V_o \xrightarrow{\psi} S/V \xrightarrow{\psi} A/I \\
& <s, \star> \xrightarrow{\text{isom}} s + V_o \xrightarrow{\text{isom}} s + V \xrightarrow{\text{lin, inj}} s + I
\end{align*}
\]

The intermediate quotient space \( S/V \) has the role of a regularization of the representation of the distributions in \( D'(R^n) \) by classes of sequences in \( S_o/V_o \), given in (3).
1.2 Simpler inclusion diagrams

In constructing inclusion diagrams (7), the main problem is the choice of the regularizing quotient space $S/V$. One can think of reducing that problem to the choice of $S$ only, since $V$ will be given by (7.2). However, it will be convenient - see §1.4 - to consider

\[ S = V \oplus S' \]

with $S'$ vector subspace in $S$. Assuming (9), the conditions (7.2) and (7.3) become

\[ S_o = V_o \oplus S' \]

and the choice of $S/V$ is replaced by the choice of the pair $(V, S')$. The difficult task is to fulfil (7.1). In that respect, taking in (7) the smallest possible $I$, will obviously be convenient. That can easily be done assuming (see (20.2) in §1.4).

\[ u(1) \in S, \]

in which case the ideal $I(V, A)$ generated by $V$ in $A$ will be the smallest $I$. Moreover, $I(V, A)$ being the vector subspace generated in $A$ by $V \cdot A$, a useful insight into the structure of the algebra $A/I(V, A)$ containing $D'(R^d)$ will be obtained. Therefore, the diagrams (7) will be considered under the particular form:

\[ \begin{array}{cccc}
I(V, A) & \longrightarrow & A & \longrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
V & \longrightarrow & V \oplus S' \ni u(1) & \downarrow & \\
\downarrow & & \downarrow & & \downarrow \\
V_o & \longrightarrow & S_o = V_o \oplus S' & & 
\end{array} \]
Correspondingly, the embedding \((8)\) will obtain the particular form:

\[
\begin{array}{cccc}
D'(R^n) & S_V/V & V \oplus S'/V & A/1(V,A) \\
\phi & \phi & \phi & \phi \\
<s,\cdot> & \text{isom} & s + V & \text{isom} & s + V \\
& & \text{lin, inj} & s + I(V,A) \\
\end{array}
\]

1.3 Admissible Properties on \(W\)

In order to obtain \((12)\) for a given pair \((V,S^4)\), it suffices to choose \(A\). Several properties of the algebras \(A/1(V,A)\), as the existence of derivative operators on the algebras, the existence of positive powers for certain elements of the algebras, etc., will depend on corresponding properties of \(A\). A uniform approach of these properties can be obtained by the following definition. A property \(P\), valid for certain subalgebras in \(W\), is called **admissible** only if

\[
\begin{align}
(14.1) & \quad W \text{ has the property } P; \\
(14.2) & \quad \text{an intersection of subalgebras in } W, \text{ each having the property } P, \text{ will also have that property.}
\end{align}
\]

Suppose \(P\) and \(Q\) are admissible properties, then \(P\) is called **stronger** than \(Q\), denoted \(P \succ Q\), only if each subalgebra in \(W\) satisfying \(P\) will also satisfy \(Q\).

Obviously, if \(P_1, \ldots, P_m\) are admissible properties, then their conjunction \(P = (P_1 \text{ and } \ldots P_m)\) is also an admissible property and
$P = \max\{P_j, \ldots, P_m\}$ with the above defined partial order $\leq$.

Three of the more important admissible properties of subalgebras $A$ in $\mathcal{W}$ are the following ones:

(15) \hspace{1em} $A$ is \underline{derivative invariant}:

\begin{equation}
D^p A \subseteq A, \quad \forall p \in \mathbb{N}^n
\end{equation}

where $D^p: \mathcal{W} \rightarrow \mathcal{W}$ with $(D^p(s))(v)(x) = (D^p(s(v)))(x)$, $\forall s \in \mathcal{W}, \; v \in \mathbb{N}, \; x \in \mathbb{R}^n$;

(16) \hspace{1em} $A$ is \underline{positive power invariant}:

\begin{equation}
\begin{cases}
(16.1.1) & s(v)(x) \geq 0, \quad \forall v \in \mathbb{N}, \; x \in \mathbb{R}^n \\
(16.1.2) & s^\alpha \in \mathcal{W}, \quad \forall \alpha \in (0, \infty)
\end{cases}
\end{equation}

where $s^\alpha(v)(x) = (s(v)(x))^\alpha$, $\forall v \in \mathbb{N}, \; x \in \mathbb{R}^n$;

(17) \hspace{1em} $A$ is \underline{cofinal invariant}:

\begin{equation}
\begin{cases}
(17.1) & \exists s \in A, \; \mu \in \mathbb{N}:
\end{cases}
\end{equation}

where $\forall v \in \mathbb{N}, \; v \geq \mu: t(v) = s(v)$

The construction of the algebras containing $D^1(\mathbb{R}^n)$ is carried out assuming that a certain admissible property $P$ was specified in advance.

Suppose now, given an admissible property $Q$, such that $Q \leq P$.

Given $(\mathcal{V}, S')$, the choice of $A$ needed in order to obtain (12) will be:
(18) $A^Q(V,S')$ the smallest subalgebra in $W$ with the property $Q$ and containing $V \oplus S'$.

In that case, the notation

(19) $I^Q(V,S') = I(V,A^Q(V,S'))$

will be useful.

1.4 Constructing the Algebras Containing $D'(R^n)$

Suppose given an admissible property $P$. Denote by $R(P)$ the set of all pairs $(V,S')$, with $V$ and $S'$ vector subspaces respectively in $V_0$ and $S_0$, and satisfying

(20.1) $S_0 = V_0 \oplus S'$

(20.2) $U = V(p) \oplus S'$, \forall p \in \mathbb{N} \cup \{\infty\}$

(20.3) $I^P(V,S') \cap (V \oplus S') \subset V_0$

where

(21) $U = \{u(\psi) \mid \psi \in \mathcal{C}(R^n)\}$

and, for $p \in \mathbb{N}^n$.

(22) $V(p) = \{v \in V \mid \forall r \in \mathbb{N}^n, r \preceq p: D^rv \in V\}$.

Theorem 1 $R(P)$ is not void. Suppose given $(V,S') \in R(P)$ and an admissible property $Q$, with $Q \preceq P$.

Then:
1) for each \( p \in \mathbb{N}^n \), the inclusion diagram holds:

\[
\begin{align*}
\text{Diagram}
\end{align*}
\]

\((23)\)

\[
\begin{align*}
\text{and}
\end{align*}
\]

\((23.1)\)

\[
\begin{align*}
\text{and}
\end{align*}
\]

2) for each \( p, q \in \mathbb{N}^n \), \( p \leq q \), the inclusion diagram holds:

\[
\begin{align*}
\text{Diagram}
\end{align*}
\]

\((24)\)

Proof. \( R(P) \) is not void (see Remark 3, §2.5, Chap.2). The relations

\((23)\) and \((24)\) results easily. We prove now \((23.1)\). Obviously, \( V(p) \subseteq V \), hence \( A^Q(V(p), S') \subseteq A^Q(V, S') \). Noticing that \( A^Q(V, S') \subseteq A^P(V, S') \) one obtains \( I^Q(V(p), S') \subseteq I^P(V(p), S') \). Therefore, \( I^Q(V(p), S') \cap (V(p) \oplus S') \subseteq I^P(V, S') \cap (V(p) \oplus S') = V(p) \), the last inclusion resulting from \((20.3)\).

Now, obviously \( I^Q(V(p), S') \cap (V(p) \oplus S') \subseteq V(p) \cap (V(p) \oplus S') = V(p) \) and the inclusion \( \subseteq \) in \((23.1)\) is proved. The converse inclusion results from \((23)\).

And now, the definition of the family of associative and commutative algebras with unit element and containing the distributions in \( D'(\mathbb{R}^n) \).

For given \((V, S') \in R(P)\) and admissible property \( Q \), with \( Q \subseteq P \), denote

\[
(25)\quad A^Q(V, S', p) = A^Q(V(p), S') / I^Q(V(p), S') \quad \text{with} \quad p \in \mathbb{N}^n.
\]
The algebras $A^Q(V,S',p)$ will be called derivative algebras, positive power algebras or cofinal algebras if $Q$ is stronger than respectively the admissible properties (15), (16) and (17).

1.5 Properties of the Algebras Containing $D'(R^n)$

The next three theorems present the main properties of the embeddings $D'(R^n) \subset A^Q(V,S',p)$.

Theorem 2  Suppose given $(V,S') \in R(P)$, an admissible property $Q$, with $Q \leq P$ and $p,q,r \in \mathbb{N}$, $p \leq q \leq r$. Then

1) $A^Q(V,S',p)$ is an associative, commutative algebra with the unit element $u(1) + 1^Q(V(p),S')$;

2) the following linear applications exist:

$$S \cdot V < \frac{a}{b} V(p) \oplus S'/V(p) \rightarrow \frac{b}{a} A^Q(V,S',p)$$

with $\alpha_p(s + V(p)) = s + V(p)$

$$\beta_p(s + V(p)) = s + 1^Q(V(p),S')$$

3) the linear injective application (embedding) exists:

(26) $D'(R^n) \xrightarrow{\epsilon_p} A^Q(V,S',p)$

with $\epsilon_p = \beta_p \circ \alpha_p^{-1} \circ \omega^{-1}$

4) the multiplication in $A^Q(V,S',p)$ induces on $C^\infty(R^n)$ the usual multiplication of functions;

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5) the diagram of algebra homomorphisms is commutative:

\[
\begin{array}{c}
A^Q(V, S', r) \xrightarrow{\gamma_{r,p}} A^Q(V, S', q) \xrightarrow{\gamma_{q,p}} A^Q(V, S', p) \\
\end{array}
\]

with \( \gamma_{q,p} (s + l^Q(V(q), S')) = s + l^Q(V(p), S') \);

6) the diagram is commutative:

\[
\begin{array}{c}
A^Q(V, S', q) \xrightarrow{\gamma_{q,p}} A^Q(V, S', p) \\
\end{array}
\]

\[
\begin{array}{c}
V(q) \oplus S'/V(q) \xrightarrow{\eta_{q,p}} V(p) \oplus S'/V(p) \\
\end{array}
\]

\[
\begin{array}{c}
D'(R^n) \leftarrow \xrightarrow{\omega \alpha_q} D'(R^n) \\
\end{array}
\]

with \( \eta_{q,p} (s + V(q)) = s + V(p) \);

7) the diagram is commutative:

\[
\begin{array}{c}
A^Q(V, S', q) \xrightarrow{\gamma_{q,p}} A^Q(V, S', p) \\
\end{array}
\]

\[
\begin{array}{c}
D'(R^n) \leftarrow \xrightarrow{\epsilon_q} D'(R^n) \\
\end{array}
\]

therefore, \( \gamma_{q,p} \) restricted to \( \epsilon_q(D'(R^n)) \) is injective.

Proof. 4) It results from (20.2)

The rest follows from Theorem 1.
Theorem 3: In the case of derivative algebras (see §1.4), suppose given \((V,S') \in R(P)\), an admissible property \(Q\), with \(Q \leq P\) and \(p \in \mathbb{N}^n\), \(q,r \in \mathbb{N}^n\). Then

1) there exists the linear mapping:

\[
\begin{align*}
D^P_{q+p}: A^Q(V,S',q+p) & \longrightarrow A^Q(V,S',q) \\
\end{align*}
\]

defined by

\[
(27) \quad D^P_{q+p}(s + l^Q(V(q+p),S')) = D^P_s + l^Q(V(q),S')
\]

2) the restriction of \(D^P_{q+p}\) to \(C^\infty(\mathbb{R}^n)\) is the usual derivative \(D^P\) of functions;

3) the diagram is commutative:

\[
\begin{array}{ccc}
A^Q(V,S',r+p) & \xrightarrow{D^P_{r+p}} & A^Q(V,S',r) \\
\downarrow \gamma_{r+p,q+p} & & \downarrow \gamma_{r,q} \\
A^Q(V,S',q+p) & \xrightarrow{D^P_{q+p}} & A^Q(V,S',q) \\
\end{array}
\]

4) the mapping \(D^P_{q+p}\) satisfies the "Leibnitz rule of product derivative":

\[
(28) \quad D^P_{q+p}(S,T) = \sum_{k \in N^n} c_k^p(\gamma_{q+p-k,q+p}^kS^k) \cdot \gamma_{q+k,q+p}^kT
\]

for any \(S, T \in A^Q(V,S',q+p)\); in particular, if \(p = (p_1, \ldots, p_n)\), with \(p_1 + \ldots + p_n = 1\), the usual rule of product derivative results:

\[
(28.1) \quad D^P_{q+p}(S \cdot T) = (D^P_{q+p}S) \cdot \gamma_{q+p,q+p}^T + (\gamma_{q+p,q+p}^{q+p}S) \cdot D^P_{q+p}T
\]
Proof. 1) Obviously

\[(29) \quad \forall p \in \mathbb{N}^n, q \in \mathbb{N}^n: D^p \varphi_{q+p} \subseteq \varphi_q\]

But

\[(30) \quad \forall p \in \mathbb{N}^n, q \in \mathbb{N}^n: D^p A^0(\varphi_{q+p}, \varphi') \subseteq A^0(\varphi_q, \varphi')\]

due to the presence of derivative algebras. Further, (24) implies:

\[(31) \quad \forall p \in \mathbb{N}^n, q \in \mathbb{N}^n: A^0(\varphi_{q+p}, \varphi') \subseteq A^0(\varphi_q, \varphi')\]

Comparing (29), (30), (31) and (24), one obtains:

\[(32) \quad \forall p \in \mathbb{N}^n, q \in \mathbb{N}^n: D^p A^0(\varphi_{q+p}, \varphi') \subseteq A^0(\varphi_q, \varphi').\]

Now, (31) and (32) will imply (27).

2) It results from (20.2).

3) and 4) result from Theorems 1 and 2 and 1).

Remark 1. In §1.7 it is shown that under rather minimal requirements on the distribution multiplication, it is not possible to embed $D'(\mathbb{R}^n)$ into a unique algebra $A$ possessing derivative operators $D^p: A \to A$, with $p \in \mathbb{N}^n$.

Now, positive powers will be defined for certain elements of the algebras $A^0(\varphi, \varphi', p)$. Denote

\[C^\infty_+(\mathbb{R}^n) = \{ \psi \in C^\infty(\mathbb{R}^n) \mid \begin{array}{l}
* \quad \psi(x) \geq 0, \forall x \in \mathbb{R}^n \\
** \quad \psi^\alpha \in C^\infty(\mathbb{R}^n), \forall \alpha \in (0, \infty)\end{array} \} \]
Obviously, if \( \psi \in C^\infty(R^n) \) and \( \psi(x) > 0 \), \( \forall x \in R^n \), then \( \psi \in C^\infty_+(R^n) \).

But, there exist \( \psi \in C^\infty(R^n) \), with \( \psi(x) \gg 0 \), \( \forall x \in R^n \), such that \( \psi \in C^\infty_+(R^n) \), for instance, \( \psi(x) = x_1^2 \cdots x_n^2 \), \( \forall x = (x_1, \ldots, x_n) \in R^n \).

Nevertheless, defining

\[
\psi(x) = \begin{cases} 
\exp(-(1/x_1 + \cdots + 1/x_n)) & \text{if } \forall i \in \{1, \ldots, n\}: x_i > 0, \\
0 & \text{if } \exists i \in \{1, \ldots, n\}: x_i \leq 0.
\end{cases}
\]

it results \( \psi \in C^\infty_+(R^n) \).

Denote further

\[
W_+ = \{ s \in W | s(v) \in C_+(R^n), \forall v \in N \} .
\]

It follows that, for \( s \in W_+ \), one can define any positive power \( s^\alpha \), by

\[
(33) \quad s^\alpha(v)(x) = (s(v)(x))^\alpha, \forall \alpha \in (0, \infty), \forall v \in N, x \in R^n
\]

and one obtains \( s^\alpha \in W_+ \).

The condition in (16), given for a subalgebra \( A \) in \( W \) in order to be positive power invariant, can be reformulated as:

\[
(34) \quad \forall s \in A \cap W_+, \alpha \in (0, \infty): s^\alpha \in A.
\]

Suppose \( T \) is vector subspace in \( S_o \) and denote

\[
D^\prime_{T, +}(R^n) = \{ s \cdot t : t \in T \cap W_+ \}.
\]

The distributions in \( D^\prime_{T, +}(R^n) \) will be called \( T \) non negative.
Theorem 4  In the case of positive power algebras, suppose given 
\((V, S') \in R(P),\) an admissible property \(Q,\) with \(Q \leq P\) and \(p \in \mathbb{N}^n.\)
Assume \(T\) is a vector subspace in \(S'\) such that

\[ U \cap \hat{W}^+ \subset T \subset V(q) \oplus S', \quad \forall q \in \mathbb{N}^n. \]  

Then

1) \(C^\infty_+(\mathbb{R}^n) \subset D^1_{T,+}(\mathbb{R}^n) ;\)

2) for \(\alpha \in (0, \infty),\) one can define
\(D_{T,+}(\mathbb{R}^n) \ni T \longrightarrow T^\alpha \in A^Q(V, S', p)\)
by: \(T = \langle t, o \rangle \longrightarrow T^\alpha = t^\alpha + 1^Q(V(p), S')\) where \(t \in T \cap \hat{W}^+\)

3) for \(T \in D_{T,+}(\mathbb{R}^n),\) the relations result:

\[(36.1) \quad T^1 = T \]
\[(36.2) \quad t^{\alpha + \beta} = t^{\alpha} \cdot t^{\beta}, \forall \alpha, \beta \in (0, \infty) \]
\[(36.3) \quad (t^\alpha)^m = t^{\alpha \cdot m}, \quad \forall \alpha \in (0, \infty), \quad m \in \mathbb{N}\setminus\{0\} \]

4) the mapping in 1) restricted to \(C^\infty_+(\mathbb{R}^n)\) is the usual power of
functions ;

5) suppose in addition the case of derivative algebras and
\(q = (q_1, \ldots, q_n) \in \mathbb{N}^n,\) with \(q_1 + \ldots + q_n = 1,\) then the relation holds
in \(A^Q(V, S', p) :\)

\[(37) \quad d_q^{p+q} t^\alpha = \alpha \cdot t^{\alpha - 1} d_q^p t, \quad \forall T \in D_{T,+}^1(\mathbb{R}^n), \alpha \in (1, \infty). \]

Proof: 1) It follows from (35); 2) it follows from (35) and 2) in Théorème 2;
3) it results easily; 4) it follows from 1) and 2); 5) it follows from (28.1).
Remark 2
1) The condition (35) can be easily fulfilled due to (20.2).

2) As known, [22], there exist sequences \( s \in S^+ P \) such that \( \langle s, s \rangle = \delta \), the Dirac distribution. Choosing \( T \in s \), one obtains \( \delta \in D_{T,+}(R^n) \) therefore, arbitrary positive powers of \( \delta \) can be defined, according to 2) in Theorem 4. (See also [27].)

1.6 Maximaliy

Remark 3
There exists an interest in constructing the algebras \( A_+(V, S', p), p \in \mathbb{N}^n \), based on \( (V, S') \in R(P) \) with \( V \) maximal. Indeed, taking into account (25), the larger the ideals \( I_+(V(p), S'), p \in \mathbb{N}^n \), are, the more and in an easier way equalities can be obtained in \( A_+(V, S', p), p \in \mathbb{N}^n \). Due to (20.1), the first chance in enlarging \( I_+(V(p), S'), p \in \mathbb{N}^n \), seems to depend on enlarging \( V \) only.

Define on \( R(P) \) a partial order \( \leq \) by

\[
(V_{11}, S_{11}) \leq (V_{22}, S_{22}) \iff V_{11} \subseteq V_{22}, S_{11} = S_{22}.
\]

The admissible property \( P \) is called regular, only if \( (R(P), \leq) \) is inductive.

Denote by \( \bar{P} \) the admissible property satisfied by \( W \) only, that is, the property of subalgebras \( A \) in \( W \) that \( A = W \). Obviously, \( \bar{P} \) is the strongest admissible property.

Theorem 5
\( \bar{P} \) is a regular admissible property.
Proof. Suppose \( \{ (V_{\lambda}, S') \mid \lambda \in \Lambda \} \) is a totally ordered family of elements in \( (R(\vec{P}), \triangleleft) \). Denote \( V = \bigcup_{\lambda \in \Lambda} V_{\lambda} \). We shall prove that \( (V, S') \in R(\vec{P}) \).

Obviously, \( V \) is a vector subspace in \( V_{o} \) and (20.1), (20.2) hold. It remains to prove (20.3). First, we notice that \( \vec{P}(V, S') = W \), since \( \vec{P} \) holds only for \( W \). Therefore, \( \vec{P}(V, S') = \vec{P}(V, W) \) and the condition (20.3) becomes:

\[
(38) \quad I(V, W) \cap (V \oplus S') \subset V_{o}.
\]

Suppose now, \( s \in I(V, W) \cap (V \oplus S') \). Then, \( s \in I(V, W) \) results in:

\[
(39) \quad s = \sum_{1 \leq i \leq k} v_{i} \cdot w_{i}, \text{ with } v_{i} \in V, w_{i} \in W.
\]

Suppose \( \lambda' \in \Lambda \) such that \( V_{i} \in V_{\lambda'} \) for \( 1 \leq i \leq k \). Then (39) results in:

\[
(40) \quad s \in I(V_{\lambda'}, W).
\]

On the other side, \( s \in V \oplus S' \) results in \( s = v + s' \), with \( v \in V \), \( s' \in S' \), therefore:

\[
(41) \quad s = v_{\lambda''} + s', \text{ with } v_{\lambda''} \in V_{\lambda''}, s' \in S'.
\]

Now, choose \( \lambda \in \{ \lambda', \lambda'' \} \) such that \( V_{\lambda}, V_{\lambda''} \subset V_{\lambda'} \). Then, (40) and (41) imply \( s \in I(V_{\lambda}, W) \cap (V_{\lambda} \oplus S') \) hence \( s \in V_{o} \), since \( (V_{\lambda}, S') \in R(\vec{P}) \) satisfies (20.3), and the proof is completed.

Remark 4. 1) Suppose \( (V, S') \in R(\vec{P}) \) is maximal, then \( V \nsubseteq V_{o} \). Indeed, if \( V = V_{o} \), then (20.1) and (20.3) result in \( \vec{P}(V_{o}, S') \cap S_{o} \subset V_{o} \) which contradicts (5).
2) In the case of \((V, S') \in \mathcal{R}(\tilde{P})\), the following theorem gives an upper bound for \(V\), offering a relevant information about the necessary structure of a wide class of distribution multiplications encountered in applications (see Chapters 2, 3 and [21], [22], [23]). It results that, in case \(V \bigoplus S'\) contains "\(\delta\) sequences" (Lemma 1, §1.1 and [17]), the sequences in \(V\) have to vanish infinitely many times on every neighbourhood of each point in \(\mathbb{R}^n\).

In Chapter 2 (see Remark 3, §2.5) better upper bounds, under the form of stronger local vanishing properties will be presented in the case of specific constructions of algebras containing \(D^1(\mathbb{R}^n)\).

For \(p \in \mathbb{N}^n\) denote

\[
\mu^p = \{w \in W \mid \forall G \subseteq \mathbb{R}^n, G \neq \emptyset, G \text{ open, } q \in \mathbb{N}^n, q \leq p, \mu \in \mathbb{N} : \\
\exists x \in G, v \in \mathbb{N}, v \geq \mu : \\
D^q w(v)(x) = 0\}
\]

Call a subset \(F\) of \(W\) cofinal invariant, only if

\[
\forall t \in W : \left( \forall v \in \mathbb{N}, v \geq \mu : \\
\left( t(v) = s(v) \right) \Rightarrow t \in F \right).
\]

Theorem 6 Suppose \((V, S') \in \mathcal{R}(\tilde{P})\) such that

1) \(V \bigoplus S'\) is cofinal invariant;

2) \(\forall x \in \mathbb{R}^n : \exists s_x \in V \bigoplus S' : \\
2.1) \langle s_x, \cdot \rangle = \delta_x \text{ (the Dirac } \delta \text{ distribution in } x \in \mathbb{R}^n\rangle \\
2.2) \text{ supp} s_x(v) \subseteq B(x, a_v), \forall v \in \mathbb{N}, \text{ where } a_v > 0 \text{ and } \lim_{v \to \infty} a_v = 0.

Then:

\[V(p) \subseteq \mu^p, \forall p \in \mathbb{N}^n.\]
Proof. It suffices to prove the inclusion $V \subseteq \mathcal{W}'$ only. Suppose $v \in V \setminus \mathcal{W}'$, then

$$\exists G \subseteq \mathbb{R}^n, \ G \neq \emptyset, \ G \text{ open, } \mu' \in N:\ 
\forall x \in G, \ v \in N, \ v \geq \mu' : 
\nu(x) \neq 0.$$ 

Suppose $x_0 \in G, \ a > 0$ such that $B(x_0, a) \subseteq G$. Then, $\exists \mu'' \in N:$

$$\forall v \in N, \ v \geq \mu'' : 0 < a_v < a.$$ 

Define $w \in \mathcal{W}'$ by

$$(44) \quad w(v)(x) = \begin{cases} 
0 & \text{if } v < \mu = \max\{\mu', \mu''\} \\
\frac{s_{x_0}(v)(x)}{v(x)} & \text{if } v \geq \mu 
\end{cases}$$

then

$$(45) \quad \forall v \in N, \ v \geq \mu : \ v(v) \cdot w(v) = s_{x_0}(v)$$

therefore, $v \cdot w \in V \oplus S'$ due to the conditions 1) and 2). But $v \cdot w \in \bar{I}(V, S') = \bar{I}(V, S')$, since $v \in V$. We can conclude that $v \cdot w \in \bar{I}(V, S') \cap (V \oplus S')$ which implies :

$$(46) \quad v \cdot w \in V_o$$

taking into account (20.3), which holds for $(V, S') \in \mathcal{R}(\mathcal{P})$. The relations (45) and (46) contradict 2.1) and the proof is completed.
1.7 **Stronger Conditions for Derivatives**

It will be shown that in the case of \( n = 1 \), the stronger condition on derivatives mentioned in Remark 1, §1.5, leads to a trivial distribution multiplication.

Suppose \( A \) is an associative and commutative algebra such that:

\[
\mathcal{P}(R^1) \oplus D^*_\delta(R^1) = A
\]

(here \( \mathcal{P}(R^1) \) is the set of complex valued polynomials on \( R^1 \) and \( D^*_\delta(R^1) \) is the set of distributions in \( D'(R^1) \) with finite support).

\[
\text{the multiplication in } A \text{ induces on } \mathcal{P}(R^1) \text{ the usual multiplication of polynomials and the polynomial } \psi(x) = 1, x \in R^1 \text{ is the unit element of the algebra } A;
\]

\[
\text{there exists a linear mapping } D: A \rightarrow A, \text{ such that:}
\]

\[
D \text{ is identical on } \mathcal{P}(R^1) \oplus D^*_\delta(R^1) \text{ with the usual derivative;}
\]

\[
D \text{ satisfies on } A \text{ the "rule of product derivative":}
\]

\[
D(ab) = (Da)b + a(Db), \forall a, b \in A;
\]

\[
(x - x_0) \ast \delta_{x_0} = 0 \in A, \forall x_0 \in R^1.
\]

**Proposition 2** Within the algebra \( A \), the relations hold:

\[
(x - x_0)^p \cdot D^q \delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p, q \in N, p > q;
\]

\[
(p + 1) \cdot D^p \delta_{x_0} + (x - x_0) \cdot D^{p+1} \delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p \in N;
\]

\[
(x - x_0)^p \cdot (D^q \delta_{x_0})^q = 0 \in A, \forall x_0 \in R^1, p, q \in N, q \geq 2;
\]

\[
(\delta_{x_0})^2 = \delta_{x_0} \cdot D \delta_{x_0} = 0 \in A, \forall x_0 \in R^1.
\]
Proof. Applying $D$ to (50) and taking into account (49.1) and (49.2), one obtains:

$$\delta_{x_0} + (x - x_0) \cdot D\delta_{x_0} = 0 \in A, \forall x_0 \in R^1$$

which multiplied by $(x - x_0)$ gives according to (50), $(x-x_0)^2 \cdot D\delta_{x_0} = 0 \in A, \forall x_0 \in R^1$. Applying $D$ to that relation and then, multiplying by $(x-x_0)$, one obtains $(x-x_0)^3 \cdot D^2\delta_{x_0} = 0 \in A, \forall x_0 \in R^1$. Repeating the procedure, one obtains (51).

The relation (52) results applying repeatedly $D$ to (55).

Multiplying (52) by $(x-x_0)^p$, one obtains:

$$(p+1)(x-x_0)^p \cdot D^p\delta_{x_0} + (x-x_0)^p \cdot D^{p+1}\delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p \in N.$$ 

Multiplying that relation by $(D^p\delta_{x_0})^{q-1}$ and taking into account (51), one obtains (53).

Taking $p = 0, q = 2$ in (53) one obtains $(\delta_{x_0})^2 = 0 \in A, \forall x_0 \in R^1$. Applying $D$ to that relation, the proof of (54) is completed.

Remark 5. Taking into account the usual interpretations of $(\delta_{x_0})^2$, $x_0 \in R^1$ (see [1], [2], [4], [5], [7], [8], [17], [27]; [28]) a distribution multiplication implying (54), is not of interest.
1.8. APPENDIX

The proof of Lemma 1 in §1.1, is given here.

1) It results easily.

2) For $A \in \mathbb{R}^1$, $\nu \in \mathbb{N}$ denote

$$E(A,\nu) = \{ x \in \mathbb{R}^n | s(\nu)(x) \geq A \}.$$ 

First, we prove the relation

$$\lim_{\nu \to \infty} \int_{E(A,\nu)} s(\nu)(x) \, dx \geq 1, \forall A \in \mathbb{R}^1.$$ 

Suppose, it is false. Then

$$\exists A \in \mathbb{R}^1, \epsilon > 0, \mu' \in \mathbb{N}: \forall \nu \in \mathbb{N}, \nu \geq \mu': \int_{E(A,\nu)} s(\nu)(x) \, dx < 1 - \epsilon.$$ 

But, due to 1)

$$\exists \mu'' \in \mathbb{N}: \forall \nu \in \mathbb{N}, \nu \geq \mu'': 1 - \epsilon/2 \leq \int_{\mathbb{R}^n} s(\nu)(x) \, dx \text{ and for } \nu \in \mathbb{N} \text{ the relations hold}$$ 

$$\int_{\mathbb{R}^n} s(\nu)(x) \, dx = \int_{E(A,\nu)} s(\nu)(x) \, dx + \int_{B(0,a_{\nu}) \setminus E(A,\nu)} s(\nu)(x) \, dx \leq \int_{E(A,\nu)} s(\nu)(x) \, dx + A \cdot \int_{B(0,a_{\nu})} \, dx.$$ 

We obtain for $\nu \in \mathbb{N}, \nu \geq \mu = \max\{\mu',\mu''\}$ the inequality $1 - \epsilon/2 \leq 1 - \epsilon + A \cdot \int_{B(0,a_{\nu})} \, dx$ which is absurd since $\lim_{\nu \to \infty} a_{\nu} = 0.$
We prove now that for \( v \in \mathbb{N} \) there exists \( A_v \in [0, \infty) \), with
\[
\lim_{v \to \infty} A_v = \infty,
\]
and such that
\[
(57) \quad \lim_{v \to \infty} \int_{E(A_v, v)} s(v)(x) \, dx \geq 1.
\]
Indeed, according to (56), for \( \mu \in \mathbb{N} \) there exists \( v_\mu \in \mathbb{N} \), with
\[
v_0 < v_1 < \ldots < v_\mu < \ldots
\]
and such that
\[
(58) \quad 1 - 1/(\mu+1) \leq \int_{E(\mu, v_\mu)} s(v_\mu)(x) \, dx, \quad \forall \mu \in \mathbb{N}.
\]
Define now \( A_v = \inf\{ \mu \in \mathbb{N} \mid v \leq v_\mu \} \) then \( A_v = \mu, \mu \in \mathbb{N} \), therefore (58) will imply (57).

Suppose now a sequence \( A_v \in \mathbb{R}^1 \), with \( v \in \mathbb{N} \), as given by (57), then
\[
(59) \quad \lim_{v \to \infty} \int_{E(A_v, v)} (s(v)(x))^2 \, dx = \infty.
\]
Indeed, \( \forall v \in \mathbb{N} : (s(v))^2 \geq A_v s(v) \) on \( E(A_v, v) \) since \( 0 \leq A_v \leq s(v) \) on \( E(A_v, v) \). Therefore,
\[
\forall v \in \mathbb{N} : \int_{E(A_v, v)} (s(v)(x))^2 \, dx \geq A_v \cdot \int_{E(A_v, v)} s(v)(x) \, dx.
\]
The relation (59), will result now, taking into account (57) and the fact that \( \lim_{v \to \infty} A_v = \infty \). Obviously, (59) implies
\[
\lim_{v \to \infty} \int_{\mathbb{R}^n} (s(v)(x))^2 \, dx = \infty
\]
which, due to the fact that \( \text{supp } s(v) \subset B(0, a_v) \) and \( \lim_{v \to \infty} a_v = 0 \), will result in \( s^2 \in S_0 \).
2. DIRAC ALGEBRAS CONTAINING $D'(\mathbb{R}^n)$

2.1 Introduction

In this chapter, specific instances of the algebras defined in Chap. 1 are constructed. These algebras will be used in Chap. 3, in the study of shock wave solutions of the nonlinear partial differential equations:

\begin{align*}
    (1.1) & \quad u_t(x,t) + a(u(x,t)) \cdot u_x(x,t) = 0, \quad x \in \mathbb{R}^1, \quad t > 0, \\
    (1.2) & \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}^1,
\end{align*}

where $a$ is a polynomial

It is known, [14], [25], [3], that under rather general conditions, the solutions $u$ of (1.1) and (1.2), possess the properties:

\begin{align*}
    (2.1) & \quad u \in C^\infty((\mathbb{R}^1 \times (0,\infty)) \setminus \Gamma) ; \\
    (2.2) & \quad u \text{ satisfies (1.1) on } (\mathbb{R}^1 \times (0,\infty)) \setminus \Gamma ; \\
    (2.3) & \quad \Gamma \text{ is a locally finite collection of smooth curves.}
\end{align*}

Therefore, the singularities of $u$ occur on the smooth curves $\Gamma$, which are closed subsets in $\mathbb{R}^2$, with no interior, and the only problem is to give a meaning to the nonlinear equations (1.1) in the neighbourhood of those singularities. As noticed in 1) in Remark 4, §1.6, Chap. 1, the above problem reduces to a proper choice of the ideals $I^Q(V(p),S')$, defining the algebras $A^Q(V,S',p)$, which contain the distributions in $D'(\mathbb{R}^n)$ (see (25) in §1.4, Chap. 1).

The mentioned ideals are chosen so that they satisfy a local vanishing property stronger than in Theorem 6, §1.6, Chap. 1 (see also 2) in Remark 4, §1.6, Chap. 1).
2.2 Ideals and Compatible Vector Spaces of Smooth Functions.

The algebras containing $D'(R^n)$ were constructed through the inclusion diagrams (23), §1.4, Chap. 1, which depend on the existence of pairs $(V, S') \in R(P)$ of vector subspaces in $S_0$ (see §§1.3, 1.4, Chap. 1). A way of constructing pairs $(V, S')$ which belong to $R(P)$, for any admissible property $P$ on $W$, is presented in the sequel.

An ideal $I$ in $W$ and a vector subspace $T$ in $S_0$ are called compatible, only if:

1. $I \cap T = V_0 \cap T = 0$

2. $I \cap S_0 \subseteq V_0 \oplus T$.

Theorem 1. Suppose given an ideal $I$ in $W$ and a compatible vector subspace $T$ in $S_0$. Assume $V$ is a vector subspace in $I \cap V_0$ and $S$ is a vector subspace in $S_0$ satisfying:

3. $V_0 \oplus T \oplus S = S_0$

4. $U \subseteq T \oplus S$.

Then $(V, T \oplus S) \in R(P)$ for any admissible property $P$ on $W$.

Proof. Denote $S' = T \oplus S$. It suffices to prove the relation (see (20.3), §1.4, Chap. 1):

5. $I(V, W) \cap (V \oplus S') \subseteq V$.

First, we notice that $I(V, W) \subseteq I$, since $V \subseteq I$ and $I$ is an ideal in $W$. Therefore:

6. $I(V, W) \cap (V \oplus S') \subseteq I \cap (V \oplus S')$. 
But

(7) \( I \cap S' = 0 \).

Indeed, (2) results in

\[ I \cap S' \subseteq (I \cap S_o') \cap S' \subseteq (V_o \oplus T) \cap (T \oplus S) = T, \]

therefore, \( I \cap S' \subseteq I \cap T = 0 \), according to (1). The relations (6) and (7) imply (5) and the proof is completed.

Specific instances of compatible ideals \( I \) and vector spaces \( T \), called respectively locally vanishing and local classes, will be constructed in Proposition 7 and Theorem 2, §2.4.

2.3 Locally Vanishing Ideals, Singularity Systems

For \( p \in \mathbb{N}^n \) denote by \( \mathcal{W}_p \) the set of all \( w \in W \) satisfying the local vanishing property:

\[ \forall x \in \mathbb{R}^n, q \in \mathbb{N}^n, q \leq p: \]

\[ \exists \mu \in \mathbb{N}: \]

\[ \forall v \in \mathbb{N}, v \geq \mu: \]

\[ D^q_w(v)(x) = 0. \]

Proposition 1 \( \mathcal{W}_p \), with \( p \in \mathbb{N}^n \), are ideals in \( W \).

Proof. It results easily.

An ideal \( I \) in \( W \) is called locally vanishing, only if

(9) \( I \subseteq \mathcal{W}_o \).
Remark 1  The significance of local vanishing conditions was discussed in § 2), Remark 4, § 1.6, Chap. 1.

Examples of locally vanishing ideals, used in Chap. 3, in the study of nonlinear shock waves are constructed now.

**A singularity system on** $\mathbb{R}^n$ **is a set** $\Gamma$ **of mappings** $\gamma: \mathbb{R}^n \to \mathbb{R}^m$, $\gamma \in C^\infty$, where $m, n \in \mathbb{N}$.

Denote by $F_{\Gamma}$, the set of all $F_{\Delta} = \bigcup_{\gamma \in \Delta} F_{\gamma}$, where $\Delta \subset \Gamma$:

$F_{\gamma} = \{x \in \mathbb{R}^n \mid \gamma(x) = 0\}$ and $(F_{\gamma} \mid \gamma \in \Delta)$ is locally finite. Obviously, $F_{\Gamma}$ is a set of closed subsets in $\mathbb{R}^n$.

For $G \subset F_{\Gamma}$ and $p \in \mathbb{N}$, denote by $I_{G,p}$ the ideal in $\mathbb{W}$ generated by all the sequences $w \in \mathbb{W}$ satisfying

\[(10) \quad \exists G \in G:
\]

\[(10.1) \quad \forall q \in \mathbb{N}^n, \ q \leq p:
\]

\[\exists \mu_1 \in \mathbb{N}:
\]

\[\forall v \in \mathbb{N}, \ v \geq \mu_1, \ x \in G:
\]

\[D^q w(v)(x) = 0
\]

\[(10.2) \quad \forall K \subset \mathbb{R}^n \setminus G, \ K \text{ compact}:
\]

\[\exists \mu_2 \in \mathbb{N}:
\]

\[\forall v \in \mathbb{N}, \ v \geq \mu_2, \ x \in K:
\]

\[w(v)(x) = 0.
\]

**Proposition 2.** $I_{G,p}$ is locally vanishing and $I_{G,p} \subseteq \mathbb{W}_p$.

**Proof.** It suffices to show that $w \in \mathbb{W}_p$ whenever $w \in \mathbb{W}$ satisfies (10). Indeed, assume $w \in \mathbb{W}$ satisfies (10) for a certain $G \in G$. Take $x \in \mathbb{R}^n$.

If $x \in \mathbb{R}^n \setminus G$ then (10.2) will imply (8) with $p = 0$, since $\mathbb{R}^n \setminus G$ is open. If $x \in G$ then (10.1) implies (8) with $p = 0$ and the proof is completed.
Denote by \( J_{G,p} \) the set of all the sequences \( w \in \mathcal{W} \) satisfying (10).

One obtains easily:

**Proposition 3:** \( I_{G,p} \) is the set of all finite sums of elements in \( J_{G,p} \).

Two useful types of elements in \( J_{G,p} \) are constructed now. For \( w \in \mathcal{W}, \gamma \in \Gamma \) and \( \alpha \in \mathcal{C}^{\infty}(\mathbb{R}^{m_{\gamma}}) \), define \( w_{\gamma,\alpha} \in \mathcal{W} \) by

\[
w_{\gamma,\alpha}(v)(x) = \alpha((v+1)\gamma(x)) \cdot w(v)(x), \quad \forall v \in \mathbb{N}, x \in \mathbb{R}^{n}.
\]

**Proposition 4:** If \( \alpha \in D(\mathbb{R}^{m_{\gamma}}) \) satisfies for a given \( k \in \bar{\mathbb{N}} \) the condition

\[
D^{r}\alpha(0) = 0, \quad \forall r \in \mathbb{N}^{m_{\gamma}}, \quad |r| \leq k,
\]

then \( w_{\gamma,\alpha} \in J_{G,p} \) for any \( G \subseteq F_{\Gamma} \) with \( F_{\gamma} \in G \) and \( p \in \mathbb{N}^{n}, |p| \leq k \).

**Proof.** It can be seen that \( w_{\gamma,\alpha} \) and \( F_{\gamma} \in G \) satisfy (10).

For \( \gamma \in \Gamma \), with \( m_{\gamma} \neq 1 \), denote by \( \delta_{\gamma} \) the Dirac \( \delta \) distribution of the surface \( F_{\gamma} \) and suppose \( s_{\gamma} \in S_{O}^{\circ} \) such that \( <s_{\gamma}, \cdot> = \delta_{\gamma} \). For \( \alpha, \beta \in \mathcal{C}^{\infty}(\mathbb{R}^{1}), q \in \mathbb{N}^{n} \), define \( s_{\gamma,\alpha} \in \mathcal{W} \) by

\[
s_{\gamma,\alpha}(v)(x) = \alpha((v+1)\gamma(x)) \cdot \beta(\gamma(x)) \cdot D^{q}s_{\gamma}(v)(x), \quad \forall v \in \mathbb{N}, x \in \mathbb{R}^{n}.
\]

**Proposition 5:** If \( \alpha \in D(\mathbb{R}^{1}), \alpha = 1 \) in a neighbourhood of \( 0 \in \mathbb{R}^{1} \), and \( \beta \) satisfies for a given \( k \in \bar{\mathbb{N}} \) the condition

\[
D^{r}\beta(0) = 0, \quad \forall r \in \mathbb{N}, r \leq k,
\]

then

1) \( s_{\gamma,\alpha} \in J_{G,p} \cap S_{O}^{\circ} \) for any \( G \subseteq F_{\Gamma} \) with \( F_{\gamma} \in G \) and \( p \in \mathbb{N}^{n}, |p| \leq k \).

2) if \( k > |q| \) then \( s_{\gamma,\alpha} \in V_{\lambda}^{\circ} \).
Proof. 1) It can be seen that \( s, q \) and \( F \in G \) satisfy (10), therefore \( s, q \in \mathcal{G}_p \). The relation \( s, q \in \mathcal{S}_o \) results easily.

2) It follows easily.

A relevant information about the sequences \( s \in I_{G, p} \cap \mathcal{S}_o \) is given in:

Proposition 6: If \( s \in I_{G, p} \cap \mathcal{S}_o \) then

\[ \exists G_1, \ldots, G_h \in G: \text{supp} \langle s, \cdot \rangle \subset \text{fr} \ G_1 \cup \ldots \cup \text{fr} \ G_h, \]

therefore \( \text{int} \ \text{supp} \langle s, \cdot \rangle = \emptyset. \)

(fr \( A \) and \( \text{int} \ A \) denote respectively the frontier and interior of the subset \( A \subset \mathbb{R}^n \)).

Proof. According to Proposition 3, there exist \( w_1, \ldots, w_h \in I_{G, p} \) and \( G_1, \ldots, G_h \in G \) such that

\[ s = w_1 + \ldots + w_h \]

and \( w_i, G_i \) satisfy (10). Since \( G_1 \cup \ldots \cup G_h \) is closed, the relations

(10.2), (15) and \( s \in \mathcal{S}_o \) imply

\[ \text{supp} \langle s, \cdot \rangle \subset G_1 \cup \ldots \cup G_h. \]

Take now \( 1 \leq i \leq h, x \in \text{int} \ G_i \) and denote

\[ I = \{ 1 \leq j \leq h \mid x \in \text{int} \ G_j \}, \quad J = \{ 1 \leq j \leq h \mid x \in \text{fr} \ G_j \}, \]

\[ K = \{ 1 \leq j \leq h \mid x \in G_j \}. \]

Since \( G_1, \ldots, G_h \) are closed, the relations result

\[ I \cap J = \emptyset, \quad J \cap K = \emptyset, \quad I \cup J \cup K = \{ 1, \ldots, h \}. \]

For \( j \in I \) take \( V_j \subset G_j, V_j \) neighbourhood of \( x \). For \( j \in K \) take

\[ V_j \subset \mathbb{R}^n \setminus G_j, V_j \) compact neighbourhood of \( x \). Denote \( V = \bigcap_{j \in \text{JUK}} V_j \). Then

(10.1) applied to \( V_j \) with \( j \in I \) and (10.2) applied to \( V_j \), with \( j \in K \)
will result in:

\[ \exists \mu \in \mathbb{N} : \forall \nu \in \mathbb{N}, \nu \geq \mu, \gamma \in V: s(\nu)(\gamma) = \sum_{j \in J} w_j(\nu)(\gamma). \]

If \( J = \emptyset \), the above relation implies \( x \in \text{supp} \ <s, \cdot> \). Suppose \( J \neq \emptyset \) and \( j \in J \), then \( x \in \text{fr} \ G_j \). Taking now into account (16), the proof is completed.

**Corollary** If \( s \in I_{G,p} \cap S_o \) then, there exists \( \Delta \subset \Gamma \), such that

1) \( \text{supp} \ <s, \cdot> \subset \bigcup_{\gamma \in \Delta} \text{fr} \ F_{\gamma} \)

2) \( (F_{\gamma} \upharpoonright \gamma \in \Delta) \) locally finite.

**Proof.** For each \( G_i \in G \) in Proposition 6, there exists \( \Delta_i \subset \Gamma \) such that \( G_i = \bigcup_{\gamma \in \Delta} F_{\gamma} \) and \( (F_{\gamma} \upharpoonright \gamma \in \Delta_i) \) is locally finite. Choosing \( \Delta = \Delta_1 \cup \cdots \Delta_h \), the proof is completed.

### 2.4 Local Classes, Compatibility

A vector subspace \( T \in S_o \) is called a **local class**, only if

\[ (17) \quad T \cap V'_o = \emptyset, \]

and

\[ \forall t \in T, t \neq u(0) : \]

\[ \exists x \in \mathbb{R}^n : \]

\[ (18) \quad \forall \mu \in \mathbb{N} : \]

\[ \exists v \in \mathbb{N}, v \geq \mu : \]

\[ t(v)(x) \neq 0. \]
Proposition 7: A locally vanishing ideal $I$ and a local class $T$ are compatible only if

$$I \cap S_o = V_o \oplus T.$$

Proof. It results easily.

An important property of the local classes is given in:

Theorem 2: For any vector subspace $S_1$ in $W \cap S_o$ there exists local classes $T$, such that $S_1 \subset V_o \oplus T$.

Proof. Assume $(e_i \mid i \in I)$ is a Hamel base in the vector space $E = S_1/(S_1 \cap V_o)$. Then $e_i = s_i + (S_1 \cap V_o)$, where $s_i \in S_1$. Suppose $a: I \to (R^1 \setminus [-1,1])$ is injective and $\psi \in D(R^n)$, $\psi(0) \neq 0$. For $i \in I$ define $v_i \in V_o$ by

$$v_i(v)(x) = (a(i))^{(v+1)} \cdot \psi((a(i))^{2(v+1)} x), \forall v \in N, x \in R^n.$$

Denote by $T$ the vector subspace generated in $S_o$ by $(s_i + v_i \mid i \in I)$. Obviously, $S_1 \subset V_o + T$. It remains only to prove that $T$ is a local class.

First, the relation (17). Suppose $t \in T$, then

$$t = \sum_{i \in J} c_i (s_i + v_i).$$

where $J \subset I$, $J$ finite and $c_i \in C$. Obviously, $t \in V_o$, only if $\sum_{i \in J} c_i s_i \in V_o$,

which is equivalent to $\sum_{i \in J} c_i e_i = 0 \in E$. Therefore, $t \in V_o$ only if $c_i = 0$, $\forall i \in J$, that is, $t = u(0)$, if one takes into account (20). Now, we prove that the relation (18) holds for $x = 0 \in R^n$. Indeed, suppose $t \in T$, $t \neq u(0)$, such that

$$\exists \mu \in N: \forall v \in N, v \gg \mu: t(v)(0) = 0.$$
Suppose \( t \) given by (20). Since \( J \) is finite and \( s_i \in S_i \subset \mathcal{W}_o \), \( \forall i \in J \), the relation (8) with \( x = 0 \), \( p = 0 \), will imply:

(22) \( \exists \mu' \in N: \forall v \in N, v \geq \mu', i \in J: s_i(v)(0) = 0. \)

The relations (21), (22) and (20) give

(23) \( \exists \mu'' \in N: \forall v \in N, v \geq \mu'' \colon \sum_{i \in J} c_i(v)(0) = 0. \)

Taking into account (19) and \( \psi(0) \neq 0 \), we obtain from (23) the relation

(24) \( \sum_{i \in J} c_i(a(i))v = 0, \forall v \in N, v > \mu''. \)

Since \( a \) is injective, (24) implies \( c_i = 0 \), \( i \in J \), therefore \( t = u(0) \), according to (20). The contradiction obtained ends the proof.

Proposition 7 and Theorem 2 result in:

Theorem 3. For each locally vanishing ideal \( I \) there exist compatible local classes \( T \).

Remark 2. Taking into account Theorems 1 and 3, in order to prove the existence of pairs \( (V,S') \) belonging to \( R(P) \), for any admissible property \( P \) on \( \mathcal{W} \), it remains to show the existence of vector subspaces \( S \) in \( S_o \), satisfying (3) and (4).

A local class \( T \) is called Dirac class, only if

(25) \( \forall t \in T: \text{int supp } \langle t, \cdot \rangle = \emptyset. \)

Proposition 8 Given a Dirac class \( T \), there exist vector subspaces in \( S \) in \( S_o \), satisfying (3) and the following stronger version of (4):

(26) \( U \subset S. \)
Proof. It suffices to notice that $(V \oplus T) \cap U = \emptyset$.

A locally vanishing ideal $I$ is called a Dirac ideal, only if

(27) $\forall s \in I \cap S_o : \text{int supp } \langle s, \cdot \rangle = \emptyset$.

Proposition 9 Given a singularity system $I$, $G \subseteq F_T$ and $p \in \overline{N}^n$, $I_{G,p}$ is a Dirac ideal.

Proof. It results from Proposition 6.

Theorem 4. For each Dirac ideal $I$ there exist compatible Dirac classes $T$.

Proof. Taking $S_1 = I \cap S_o$ and constructing $T$ as in the proof of Proposition 7, one obtains a Dirac class.

2.5 Dirac Algebras

Suppose given an admissible property $Q$ on $W$, a Dirac ideal $I$ and a compatible Dirac class $T$. For $V$, vector subspace in $I \cap V_o$ and $S$, vector subspace in $S_o$ satisfying (3)' and (4), denote (see (25), §1.4, Chap. 1):

(28) $A^Q_p = A^Q(V \oplus T \oplus S, p)$, with $p \in \overline{N}^n$, and call them Dirac algebras.

Remark 3: The existence of the Dirac algebras results from Theorem 1, §2.2, Proposition 9, Theorem 4 and Proposition 8, §2.4. These algebras possess the properties presented in Theorems 2, 3 and 4, §1.5, Chap. 1.
CHAPTER 3. SHOCK WAVES AND DIRAC ALGEBRAS

3.1 Piecewise Smooth Weak Solutions

Consider the partial differential equation

\begin{align*}
(1.1) & \quad u_t(x,t) + a(u(x,t)) \cdot u_x(x,t) = 0, \quad x \in \mathbb{R}, \quad t > 0; \\
(1.2) & \quad u(x;0) = u_0(x), \quad x \in \mathbb{R},
\end{align*}

where \( a: \mathbb{R} \rightarrow \mathbb{R} \) is a polynomial and \( u_0: \mathbb{R} \rightarrow \mathbb{R} \) is integrable.

Suppose, \( u \) is a piecewise smooth solution of (1.1), (1.2), with the properties:

\begin{align*}
(2.1) & \quad u \in \mathcal{C}^\infty((\mathbb{R} \times [0,\infty)) \setminus \Gamma), \\
(2.2) & \quad u \text{ is locally bounded on } \mathbb{R} \times [0,\infty), \\
(2.3) & \quad \int_{\mathbb{R} \times [0,\infty)} (u(x,t)\psi_t(x,t) + f(u(x,t))\psi_x(x,t))dxdt = 0, \text{ for } \psi \in \mathcal{D}(\mathbb{R} \times [0,\infty)), \text{ where } f \text{ is a primitive of } a, \\
(2.4) & \quad \Gamma \text{ is a locally finite (in } \mathbb{R} \times [0,\infty)) \text{ set of smooth curves } \gamma(x,t) = 0, \quad \gamma: \mathbb{R} \times [0,\infty) \rightarrow \mathbb{R}, \quad \gamma \in \mathcal{C}^\infty.
\end{align*}

The abuse of notation in (2.1) to denote the subset \( \{(x,t) \in \mathbb{R} \times [0,\infty) \mid \exists \gamma \in \Gamma: \gamma(x,t) = 0\} \) also by \( \Gamma \), will be repeated due to its convenience.

Remark 1. According to [14], [25], [3], in case of smooth or even piecewise smooth initial data \( u_0 \), the shock wave solutions of (1.1), (1.2), possess under rather general conditions, the properties (2.1)-(2.4). Usually \( \Gamma \) consists of a finite number of smooth curves. Therefore, the following assumption on \( \Gamma \), will not be restrictive:

\begin{align*}
(3) & \quad (\gamma^{-1}(B) \mid \gamma \in \Gamma) \text{ is locally finite in } \mathbb{R} \times [0,\infty), \text{ for any bounded } B \subset \mathbb{R}.
\end{align*}
Suppose now, given \( \alpha \in C^\infty(\mathbb{R}^1) \), such that

\[(4.1) \quad \alpha = 0 \text{ on a neighbourhood } V \text{ of } x = 0 \in \mathbb{R}^1, \]

\[(4.2) \quad \alpha = 1 \text{ outside of a bounded neighbourhood } B \text{ of } x = 0 \in \mathbb{R}^1. \]

Define \( s_{u,\alpha} \) and \( v_{u,\alpha} \), two sequences of functions on \( H \), by

\[(5.1) \quad s_{u,\alpha}(v)(x,t) = u(x,t) \prod_{\gamma \in \Gamma} \alpha((v+1)\gamma(x,t)), \text{ for } v \in \mathbb{N}, (x,t) \in H \setminus \Gamma, \]

\[(5.2) \quad s_{u,\alpha}(v)(x,t) = 0, \text{ for } v \in \mathbb{N}, (x,t) \in \Gamma \cap H, \]

where \( H = \mathbb{R}^1 \times (0,\infty), D_t = D(0,1), D_x = D(1,0). \)

Lemma 1. \( s_{u,\alpha} \) and \( v_{u,\alpha} \) are well defined and \( s_{u,\alpha}, v_{u,\alpha} \in \mathcal{H}(H). \)

Proof. First, we prove that \( s_{u,\alpha} \) is well defined. Suppose given \( v \in \mathbb{N} \) and \( (x,t) \in H \setminus \Gamma \). Due to (3) and (4.2) the set \( \{ \gamma \in \Gamma \mid ((v+1)\gamma(x,t)) \notin B \} \) is finite, therefore \( \{ \gamma \in \Gamma \mid ((v+1)\gamma(x,t)) \neq 1 \} \) is finite and (5.1) is correct. We prove now, that \( s_{u,\alpha} \in \mathcal{H}(H) \). Suppose given \( v \in \mathbb{N} \). Since \( H \setminus \Gamma \) is open, (5.1) implies that \( s_{u,\alpha}(v) \in C^\infty \) on \( H \setminus \Gamma \). Suppose given \( (x,t) \in \Gamma \cap H \). Then \( \gamma(x,t) = 0 \) for a certain \( \gamma \in \Gamma \). Assume \( W \subseteq H \) is an open neighbourhood of \( (x,t) \), such that \( (v+1)\gamma(W) \subseteq V \). Then \( s_{u,\alpha}(v) = 0 \) on \( W \), due to (5). Therefore \( s_{u,\alpha}(v) \in C^\infty \) on \( W \). Now, it results obviously that \( v_{u,\alpha} \) is well defined and \( v_{u,\alpha} \in \mathcal{H}(H). \)

Lemma 2. Given \( K \subseteq H \setminus \Gamma \), \( K \) compact, there exists \( v \in \mathbb{N} \) such that \( s_{u,\alpha}(v) = u \) and \( v_{u,\alpha}(v) = 0 \) on \( K \), for \( v \in \mathbb{N}, v \geq \mu \).
Proof. Denote \( \Gamma_0 = \{ \gamma \in \Gamma \mid \gamma(K) \cap B \neq \emptyset \} \), then \( \Gamma_0 \) is finite, due to (3) and (4.2). Denote now \( \varepsilon = \inf \{ |\gamma(x,t)| \mid \gamma \in \Gamma_0, (x,t) \in K \} \), then \( \varepsilon > 0 \), since \( \Gamma_0 \) is finite, \( K \subseteq H\setminus \Gamma \) and \( K \) compact. Take \( u \in \mathbb{N} \) such that \( B \subseteq [-u\varepsilon,u\varepsilon] \), then \( v \in \mathbb{N}, v \geq u \) implies \( (v+1)\gamma(K) \subseteq \mathbb{R}^{-1}B \), for any \( \gamma \in \Gamma \). Taking into account (5.1), the proof is completed.

Lemma 3. For any \( v \in \mathbb{N}, p \in \mathbb{N}^2 \) it results

\[
D^p s_{u,\alpha}(v) = D^p v_{u,\alpha}(v) = 0 \quad \text{on} \quad \Gamma \cap H.
\]

Proof. It follows from the last part of the proof of Lemma 1.

Proposition 1 \( s_{u,\alpha} \in S_o(H) \) and \( \langle s_{u,\alpha}, \cdot \rangle = u \).

Proof. It follows from (2.1), (2.2) and Lemma 2.

Proposition 2 \( v_{u,\alpha} \in V_o(H) \cap \bigcap_{p \in \mathbb{N}^2} W_p(H) \) (see § 2.2, Chap.2).

Proof. The relation \( v_{u,\alpha} \in \bigcap_{p \in \mathbb{N}^2} W_p(H) \) results from Lemmas 2 and 3 and the fact that \( H\setminus \Gamma \) is open. It remains to prove that \( v_{u,\alpha} \in V_o(H) \). Suppose \( \psi \in D(H), v \in \mathbb{N} \), then (6) and (2.3) result in

\[
| \int_H v_{u,\alpha}(v)(x,t)\psi(x,t)dxdt | =
\]

\[
= | \int_H (s_{u,\alpha}(v)(x,t)D_t^\psi(x,t) + f(s_{u,\alpha}(v)(x,t))D_x^\psi(x,t))dxdt | =
\]

\[
= | \int_H ((s_{u,\alpha}(v)(x,t)-u(x,t))D_t^\psi(x,t) + (f(s_{u,\alpha}(v)(x,t)) - f(u(x,t)))D_x^\psi(x,t))dxdt | \leq
\]

\[
\leq \int_{H \setminus \Gamma} \Pi \alpha((v+1)\gamma(x,t))|u(x,t)| |D_t^\psi(x,t)|dxdt +
\]

\[
+ \int_{H \setminus \Gamma} \Pi \alpha((v+1)\gamma(x,t))|u(x,t)| |\varphi(r(v,x,t))| |D_x^\psi(x,t)|dxdt
\]
where \( r(v,x,t) \) is a real number between \( s_{u,a}(v)(x,t) \) and \( u(x,t) \).

Taking now into account (2.2), the above inequality gives

\[
\lim_{v \to 0} \int_{\mathcal{H}} v_{u,a}(v)(x,t) \psi(x,t) dx dt = 0
\]

and the proof is completed.

### 3.2 Dirac Algebras

Based on Theorem 1, §2.2, Chap. 2, (see also Remark 3, §2.5, Chap. 2) Dirac algebras are constructed (see (28), §2.5, Chap. 2) in which the piecewise smooth solutions (2) will satisfy (1.1), considered with the multiplication and derivative within those algebras.

Denote by \( I_{u,a} \) the ideal generated in \( \mathcal{H}(H) \) by \( v_{u,a} \).

**Proposition 3** \( I_{u,a} \) is a Dirac ideal (see §2.4, Chap. 2) and \( I_{u,a} \subseteq \cap_{p \in \mathbb{N}^2} \mathcal{H}_p(H) \).

**Proof.** It follows from Lemmas 2, 3 and Proposition 2 in §3.1.

Suppose now, given any Dirac ideal \( I \), such that

\[
I_{u,a} \subseteq I
\]

and a vector subspace \( V \), with

\[
V_{u,a} \in V \subseteq I \cap \mathcal{V}_o(H).
\]

According to Theorem 4, §2.4, Chap. 2, there exists a Dirac class \( T \), compatible with \( I \). Due to Proposition 8, §2.4, Chap. 2, there exists a vector subspace \( S \) in \( S_o \), satisfying
Proposition 4 Suppose, \( u \) is not identically zero on \( H \setminus \Gamma \). Then \( S \) can be chosen to satisfy besides (9) and (10), also

\[
(11) \quad s_{u, \alpha} \in S.
\]

Proof. According to Lemma 2, \( s_{u, \alpha} \in V(H) \), since \( u \) is not identically zero on \( H \setminus \Gamma \). Therefore, \( s_{u, \alpha} \in (V(H) \oplus T) \) due to the fact \( T \) is a Dirac class. Taking now into account Proposition 1, the proof is completed.

In the case of derivative algebras (see §1.4, Chap.1), suppose given an admissible property \( Q \) on \( \mathcal{H}(H) \).

Theorem 1. Any piece wise smooth solution \( u \) of (1.1), (1.2) satisfying (2.1) - (2.4), belongs to each of the Dirac algebras \( A_Q^p \), with \( p \in \mathbb{R}^2 \), and satisfies in each of them the equation

\[
(12) \quad D_t u + a(u) \cdot D_x u = 0 \in A_Q^p,
\]

where the polynomial \( a(u) \) and the product \( a(u) \cdot D_x u \) are computed according to the multiplication in \( A_Q^p \), while

\[
D_t = D_{(0,1)}^{(0,1)} : A_Q^p \oplus (0,1) \to A_Q^p
\]

and

\[
D_x = D_{(1,0)}^{(1,0)} : A_Q^p \oplus (0,1) \to A_Q^p
\]

are the derivative operators in those algebras (see (27) in Theorem 3, §1.5, Chap. 1).
Proof. Suppose, \( u \) is not identically zero on \( H \setminus \Gamma \). According to Theorem 1, §2.2, Chap.2, \((V,T \oplus S) \in R(Q)\). The relation (11) implies

\[
(13) \quad s_{u,\alpha} \in V(p) \oplus T \oplus S, \ \forall p \in \tilde{N}^2 \quad \text{(see (22), §1.4, Chap.1)}.
\]

Therefore, 2) and 3) in Theorem 2, §1.5, Chap.1, and Proposition 1, §3.1, result in \( u \in A_p^Q, \ \forall p \in \tilde{N}^2 \). The above relations (13) and (6) as well as 2) and 3) in Theorem 2 and 1) in Theorem 3, both in §1.5. Chap.1, will give:

\[
(14) \quad \tilde{a} \cdot \tilde{u} + a(u) \cdot \tilde{x} \tilde{u} - \tilde{u} + I^Q(V(p),T \oplus S) \in A_p^Q, \ \forall p \in \tilde{N}^2.
\]

Now, Proposition 2, §3.1 and (8) imply

\[
(15) \quad v_{u,\alpha} \in V(p), \ \forall p \in \tilde{N}^2 \quad \text{(see (22), §1.4, Chap.1)}.
\]

But \( V(p) \subseteq I^Q(V(p),T \oplus S) \) due to 1) in Theorem 1, §1.4, Chap.1. Therefore, (14) and (15) result in (12) and the proof is completed.

Remark 2. Theorem 1 remains still valid if in (1.1) the polynomial \( a \) will have coefficients smooth functions of \( t \in (0,\infty) \):

\[
a(t;u(x,t)) = \sum_{0 \leq i \leq k} a_i(t) \cdot (u(x,t))^i, \quad a_0, \ldots, a_k \in C^\infty(0,\infty),
\]

provided that \( Q = \tilde{P} \) is the strongest admissible property on \( K(H) \) (see §1.6, Chap.1).
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REFERENCE (cont'd)


