AN ASSOCIATIVE, COMMUTATIVE DISTRIBUTION MULTIPLICATION

by

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See: Raju C.K. "Product and Compositions with the Dirac Delta function"
ABSTRACT

An associative and commutative multiplication is defined for the distributions in $D'(\mathbb{R}^n)$, by embedding that distribution space in associative and commutative algebras with unit element. The algebras, whose elements are classes of weakly convergent sequences of complex valued smooth functions, were first considered in [12] and [13].

The basic idea of the distribution multiplication consists in a special representation of the Dirac $\delta$ distribution, as a class of weakly convergent sequences of smooth functions satisfying a condition of "strong local presence". The associative and commutative distribution multiplication resulting, proves to be in a sense the best possible within weaker conditions than those given by L. Schwartz in [17], which made impossible any associative multiplication with unit element. Applications to one and three dimensional quantum particle motions in potentials positive powers of the Dirac distribution are presented.

Keywords and Phrases. Associative, commutative distribution multiplication.
INTRODUCTION

In [17], L. Schwartz proved that it is impossible to embed $D'(R^1)$ into an associative algebra $A$, under the following conditions:

1. the function $\psi(x) = 1, x \in R^1$, is the unit element of the algebra $A$;
2. there exists $F \subset C^0(R^1)$, $F = \{1, x, x(\ln|x| - 1)\}$, such that the multiplication in $A$ induces on $F$ the usual multiplication of functions;
3. there exists a linear mapping $D: A \rightarrow A$ such that
   a. $D$ satisfies on $A$ the "rule of product derivative":
      $D(a \cdot b) = (Da) \cdot b + a \cdot (Db)$, $\forall a, b \in A$;
   b. there exists $G \subset C^1(R^1)$, $G = \{1, x, x^2(\ln|x| - 1)\}$, such that $D$ is identical on $G$ with the usual derivative of functions;
4. there exists $\delta \in A, \delta \neq 0^1$, such that $x \cdot \delta = 0$.

In the present work, following the ideas in [12], [13] and [14], associative, commutative algebras with unit element and containing $D'(R^n)$ are constructed, in such a way that, under the corresponding form for $n \geq 1$:

a. the conditions (1) and (3.1) will be satisfied, (3.1) being supplemented by a property of the derivatives of arbitrary positive powers of certain elements in the algebras (see bellow),
b. the condition (4) will be satisfied under a strong form,
c. the condition (3) will be satisfied under a form reminding the following

---

1) $\delta$ represented in [17], the Dirac distribution and the relation $x \cdot \delta = 0$, a basic property of that distribution.
classical case the derivative operators are acting:

\[ \mathcal{C}^\infty(\mathbb{R}^n) \xrightarrow{d^p} \mathcal{C}^\infty(\mathbb{R}^n) \]

\[ \mathcal{C}^{p+q}(\mathbb{R}^n) \xrightarrow{d^p} \mathcal{C}^q(\mathbb{R}^n) \]

with \( p, q \in \mathbb{N}^n \),

(d) the conditions (2) and (3.2) will be satisfied under weaker forms, for instance, with at least \( F = G = \mathcal{C}^\infty(\mathbb{R}^n) \), and finally, an additional property, partially stronger than (2):

(e) for certain nonnegative elements in \( \mathcal{D}'(\mathbb{R}^n) \), in particular, the positive functions in \( \mathcal{C}^\infty(\mathbb{R}^n) \) and the Dirac \( \delta_{x_0} \) distributions, with \( x_0 \in \mathbb{R}^n \), arbitrary positive powers can be defined, possessing the usual properties:

\[ a^1 = a, \quad a^{\alpha + \beta} = a^\alpha a^\beta, \ (a^\alpha)^m = a^{\alpha m}, \quad \forall \alpha, \beta \in (0, \infty), m \in \mathbb{N}\setminus\{0\} \]

in the case of the nonnegative elements in \( \mathcal{D}'(\mathbb{R}^n) \) which are functions in \( \mathcal{C}^\infty(\mathbb{R}^n) \), the powers coincide with the usual powers of functions.

The interest in the possibility of defining in a suitable way arbitrary positive powers of the Dirac \( \delta \) distribution comes from the use of such powers in Quantum Mechanics (see [18] and the application given in Chapter 6 of the present work).

0.2. The construction is carried out within the "sequential approach" of the distributions due to J. Mikusiński, [9], [10], [11], in which a distribution is represented by a class of weakly convergent sequences of smooth functions.

Suppose \( \mathcal{W} \) is the set of all sequences of functions in \( \mathcal{C}^\infty(\mathbb{R}^n) \) and \( \mathcal{B} \) is the set of all sequences \( s \in \mathcal{W} \), weakly convergent to distributions in
$D'(\mathbb{R}^n)$. For $s \in S_o$, denote by $<s, \cdot> \in D'(\mathbb{R}^n)$ the weak limit of $s$. Denote by $V_o$ the kernel of the linear surjection

\[ S_o \ni s \quad \mapsto \quad <s, \cdot> \in D'(\mathbb{R}^n). \]

Then

\[ S_o / V_o \ni (s + V_o) \quad \mapsto \quad <s, \cdot> \in D'(\mathbb{R}^n) \]

is a vector space isomorphism.

Since $\mathcal{H}$ is in a natural way an associative and commutative algebra with unit element, the question arises whether it is possible to define a product of any two distributions $<s, \cdot>, <t, \cdot> \in D'(\mathbb{R}^n)$, by the product of the classes of sequences $s + V_o$ and $t + V_o$.

Unfortunately, that question cannot be answered in a simple way, since

\[ (V_o \cdot V_o) \cap S_o \neq V_o \]

therefore, it is impossible to construct diagrams

\[ I \quad \mapsto \quad A \quad \mapsto \quad \mathcal{H} \]

\[ \uparrow \quad \uparrow \]

\[ V_o \quad \mapsto \quad S_o \]

with $A$ subalgebra in $\mathcal{H}$, $I$ ideal in $A$ and $I \cap S_o = V_o$, which

\[ \text{2) } \quad \mapsto \quad \text{means the inclusion } \subset \]

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would generate the following linear embedding of \( D'(\mathbb{R}^n) \) into an associative and commutative algebra with unit element:

\[
\begin{array}{ccc}
D'(\mathbb{R}^n) & S_o/V_o & A/I \\
\psi & \psi & \psi \\
<s,*> & \text{isom} & s + V_o \xrightarrow{\text{lin, inj}} s + I
\end{array}
\]

However, in [12] and [13] it was shown that it is possible to construct diagrams

\[
\begin{array}{c}
I \quad \rightarrow \quad A \quad \rightarrow \quad W \\
\uparrow \\
V \quad \rightarrow \quad S \\
\downarrow \\
V_o \quad \rightarrow \quad S_o
\end{array}
\]

(6)

with \( A \) subalgebra in \( W \), \( I \) ideal in \( A \) and \( V, S \) vector subspaces in \( S_o \), such that

\[
\begin{align*}
(6.1) \\
I \cap S &= V_o \cap S = V \\
(6.2) \\
V_o + S &= S_o
\end{align*}
\]

A diagram of type (6) generates the following linear embedding of \( D'(\mathbb{R}^n) \) into an associative and commutative algebra with unit element

\[
\begin{array}{ccc}
D'(\mathbb{R}^n) & S_o/V_o & S/V & A/I \\
\psi & \psi & \psi & \psi \\
<s,*> & \text{isom} & s + V_o & \text{isom} & s + V \xrightarrow{\text{lin, inj}} s + I
\end{array}
\]
0.3. The intermediate quotient space \( S/V \) can be considered as a regularization of the natural representation of the distributions in \( D'(R^n) \) by classes of sequences in \( S_0/V_0 \), given in
\[ D'(R^n) \ni <s, \cdot> \left\langle \stackrel{\text{isom}}{=} (s + V_0) \in S_0/V_0 \right. \]

In constructing diagrams of type (6), the main problem is to choose the quotient space \( S/V \), that is, the pair of vector subspaces \( (V, S) \) in \( S_0 \). In this respect, the difficult task is to fulfill in (6.1), the inclusion
\[ (8) \quad I \cap S \subset V. \]

That problem is solved noticing the following two basic facts (see [14]).

First. It comes out that one of the "trouble sources" in distribution multiplication is the excessive freedom in choosing the weakly convergent sequences of smooth functions representing the Dirac \( \delta \) distribution. For instance, in the well known particular case [10], [11]:
\[ (9) \quad \delta(x) = \lim_{\nu \to \infty} \nu \psi(\nu x), \quad x \in R^1, \nu = 1, 2, 3, \ldots \]

the function \( \psi \in C^\infty(R^1) \) is supposed to have the following properties only:
\[ (9.1) \quad \text{supp } \psi \text{ compact}, \]
\[ (9.2) \quad \int_{R^1} \psi(x)dx = 1. \]

That freedom in choosing \( \psi \) can result in the vanishing of \( \psi \) in any given neighborhood of \( x = 0 \). In that case, none of the functions
\[ \chi_\nu(x) = \nu \psi(\nu x), \quad x \in R^1, \nu = 1, 2, 3, \ldots \]

of the weakly convergent sequence
defining $\delta$, will be "present" in the neighborhood of $x = 0$, a rather strange fact, if we think that $x = 0$ is the very point where $\delta$ is concentrated.

The choice of the weakly convergent sequences of smooth functions representing $\delta$ will be restricted by the "condition of the strong local presence in $x = 0$" (see the definition of $\mathcal{D}_{\delta}$, Chapter 1, §1.2.2., (16)).

The meaning of that condition is illustrated in the particular case of (9), (9.1), (9.2), as follows: $\psi \in \mathcal{C}^\infty(R^1)$ will give a valid representation of $\delta$, only if it satisfies the additional condition

$$(9.3) \quad D^p\psi(0) \neq 0, \quad \forall p = 0,1,2,\ldots$$

It results obviously that $\psi$ cannot be symmetric. In general, the Dirac $\delta$ distribution will not be symmetric:

$$(10) \quad D^q\delta(-x) \neq cD^q\delta(x), \quad x \in R^1, \ c \in C^1, \ q = 0,1,2,\ldots$$

Actually, the condition of the strong local presence in $x = 0$, has an even wider impact on the way the Dirac $\delta$ distribution appears within the algebras containing the distributions in $D'(R^n)$. For instance, in the case of $n = 1$, if $a_o,\ldots,a_m \in R^1 \setminus \{0\}$ are pair wise different, then $D^q\delta(a_0 x),\ldots,$ $D^q\delta(a_m x)$ are linear independent, for any given $q \in N$. In particular, if $0 < a < a' < \infty$, then $\delta(ax)$ and $\delta(a'x)$ are linear independent and, in a sense, $\delta(a'x)$ will be "thinner" or "more concentrated in $x = 0$" than $\delta(ax)$. That possibility of distinguishing between the same type of "generalized functions" concentrated in a single point $x_o \in R^1$, reminds the "infinitely small" entities of the nonstandard models of $R^1$. (Compare with [18] and the references mentioned there.)
Second. It results easily that \( I \) can be the vector subspace generated in \( W \) by \( V \cdot A \), thus, (8) will be satisfied for the smallest choice of \( V \):

\[
(11) \quad V = 0 \quad \text{(the null space of} \quad W) \]

and, due to (5), will not be satisfied for the biggest choice:

\[
(12) \quad V = V_0 .
\]

But, (11) leads to a rather trivial distribution multiplication, therefore, one can think of choosing \( V \) still "near" to (11) but not to (12).

On the other hand, the fulfilment of (4), more precisely, of its stronger form (formulated here for the sake of simplicity, for \( n = 1 \) only)

\[
(4*) \quad (x-x_0)^r \cdot D^q \delta_{x_0} = 0, \quad D^q \delta_{x_0} \neq 0, x, x_0 \in \mathbb{R}_1, q, r = 0, 1, 2, \ldots, r > q,
\]

requires that \( V \) contains all the sequences of the form

\[
(13) \quad (x-x_0)^r \cdot D^q \delta_{x_0}, x, x_0 \in \mathbb{R}_1, q, r = 0, 1, 2, \ldots, r > q
\]

where \( s \) \( x_0 \) are the valid sequences representing \( \delta_{x_0} \).

The above facts suggest that \( V \) be chosen as the "minimal" vector subspace in \( V_0 \), containing the sequences in (13).

In the case of \( n = 1 \), several applications are given.

The following formulas in Quantum Mechanics, [11], involving the Dirac and Heisenberg distributions are proved:

\[
(1/x) \cdot \delta(x) = -D\delta(x)/2
\]
\[(\delta(x))^2 - \frac{(1/x)^2}{\pi^2} = -(1/x^2)/\pi^2 \]

\[(\delta_+(x))^2 = -\frac{D\delta(x)}{4\pi i} - \frac{(1/x^2)}{4\pi^2} \]

\[(\delta_-(x))^2 = \frac{D\delta(x)}{4\pi i} - \frac{(1/x^2)}{4\pi^2} \]

An other application consists in obtaining the wave functions for one, respectively three dimensional motions of quantum particles subjected to potentials arbitrary positive powers of the Dirac \(\delta\) distribution.

The following one dimensional and stationary case of the Schrödinger equation is considered:

\[ (D^2 + k + \alpha(\delta(x))^m)\psi(x) = 0, \quad x \in \mathbb{R}, \]

with \(k, \alpha \in \mathbb{R}, m \in (0,\infty)\).

It is proved that within the algebras containing \(D'(\mathbb{R}^1)\), there exist function solutions of the form

\[ \psi(x) = \begin{cases} 
\psi_-(x) & \text{for } x < 0 \\
\psi_+(x) & \text{for } x > 0 
\end{cases} \]

where \(\psi_-, \psi_+ \in C^\infty(\mathbb{R}^1)\).

For instance, in the case of \(m = 1\), when the potential is \(U(x) = -\alpha\delta(x)\), the solutions \(\psi\) obtained within the algebras containing the distributions, agree with the well known results in Quantum Mechanics (see [20], pp. 40-41).

In the case of \(m = 2\), when the potential is \(U(x) = -\alpha(\delta(x))^2\), it results that only for discrete levels of the potential, corresponding to
\[ \alpha = (\nu \pi)^2, \quad \nu = 0, 1, 2, \ldots \]

one can obtain solutions \( \psi \).

The above one dimensional Schrödinger equation will also give the solutions of the radial wave equation for three dimensional motions of a particle subjected to a potential given by any positive power of the Dirac \( \delta \) distribution on a spherical cavity of nonzero radius, provided the angular momentum is zero (see [21], p. 83, [20], p. 225).

Further details see in Chapter 6, §6.1.1 and 6.1.2.

0.5. The associative and commutative multiplication was first presented in [14], in the case of the distributions in \( D'(R^1) \). There and in [15], it was shown that the extension to the case of the distributions in \( D'(R^n) \), \( n \geq 2 \), could be done easily, the only difficulty being to prove that the set \( Z_0^\delta \) of the weakly convergent sequences representing \( \delta \) and satisfying the "condition of strong local presence" was not void for \( n \geq 2 \). The proof of that fact could be obtained from a conjecture generalizing a well known property of the Vandermonde determinants.

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CHAPTER I - ASSOCIATIVE, COMMUTATIVE ALGEBRAS WITH UNIT ELEMENT, CONTAINING THE DISTRIBUTIONS IN $D'(R^n)$

1.1 Notations

1.1.1 We denote

$$N = \{0, 1, 2, \ldots\}, \quad \bar{N} = N \cup \{\infty\}.$$  

Throughout, we shall suppose given $n \in N$, $n \geq 1$.

For $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in \bar{N}^n$, we define

$$p + q = (p_1 + q_1, \ldots, p_n + q_n)$$

with the usual assumptions that $a + \infty = \infty + a = \infty$, $\forall a \in \bar{N}$.

We denote $p \leq q$, only if $p_i \leq q_i$, $\forall 1 \leq i \leq n$, with the usual assumptions that $a \leq \infty$, $\forall a \in \bar{N}$.

The basic classes of sequences of complex valued smooth functions and operations on them are the following ones :

(1) $\mathcal{W} = \mathcal{W}(R^n)$

thus, if $s \in \mathcal{W}$, $\nu \in N$, $x \in R^n$, then $s(\nu) \in C^\infty(R^n)$ and $s(\nu)(x) \in C^1$.

For $\psi \in C^\infty(R^n)$, we denote by $u(\psi) \in \mathcal{W}$ the corresponding constant sequence, that is, $u(\psi)(\nu) = \psi$, $\forall \nu \in N$. We also denote

(2) $U = \{u(\psi) \mid \psi \in C^\infty(R^n)\}$.

In a natural way, $\mathcal{W}$ is an associative and commutative algebra\(^3\) with

\(^3\) The vector spaces and algebras considered, are over the field $C^1$ of the complex numbers.
the unit element $u(1)$ and null element $u(0)$. The null space in $\mathcal{W}$ is therefore, $O = \{u(0)\}$.

For given $p \in \mathbb{N}^n$, the derivative operator $D^p: \mathcal{W} \rightarrow \mathcal{W}$, is defined by $(D^p s)(v)(x) = (D^p (s(v)))(x)$, $\forall s \in \mathcal{W}, v \in \mathbb{N}, x \in \mathbb{R}^n$.

For given $x_o \in \mathbb{R}^n$, the translation operator $\tau_{x_o}: \mathcal{W} \rightarrow \mathcal{W}$ is defined by $(\tau_{x_o}(s))(v)(x) = s(v)(x-x_o)$, $\forall s \in \mathcal{W}, v \in \mathbb{N}, x \in \mathbb{R}^n$.

We consider

(3) $S_o = \{s \in \mathcal{W} | s \text{ weakly convergent in } D'(\mathbb{R}^n)\}$ and for $s \in S_o$, denote its weak limit by $<s, \cdot>$. The kernel of the linear surjection

$$S_o \ni s \overset{\omega}{\longrightarrow} <s, \cdot> \in D'(\mathbb{R}^n)$$

is denoted by $V_o$.

Therefore,

(3.1) $S_o/V_o \ni (s + V_o) \overset{\omega}{\longrightarrow} <s, \cdot> \in D'(\mathbb{R}^n)$

is a vector space isomorphism.

1.1.2 The classes of sequences in $\mathcal{W}$, necessary in the construction of the vector subspaces $\mathcal{V}$ in the diagrams of type (6), introduction, are defined as follows:

(4) $\mathcal{W}^0_\delta = \{s \in \mathcal{W} \mid \forall \varepsilon > 0: \exists \nu \in \mathbb{N}, \forall \nu \in \mathbb{N}, x \in \mathbb{R}^n : \nu \geq \nu, \|x\| \geq \varepsilon \Rightarrow s(v)(x) = 0\}$
For $p \in \mathbb{N}^n$, we denote

\[
\mathcal{W}_p^0 = \{ s \in \mathcal{W} \mid \forall q \in \mathbb{N}^n, q \leq p: \exists \nu_q \in \mathbb{N}: \forall \nu \in \mathbb{N}, \nu \geq \nu_q \implies D^q_s(\nu)(0) = 0 \}
\]

as well as

\[
V^0_\delta, p = V_o \cap \mathcal{W}_\delta \cap \mathcal{W}_p^0
\]

and finally, the vector subspaces of $V_o$, which will be chosen for $V:

\[
V_\delta, p \text{ the vector subspace in } V_o \text{ generated by } \bigcup_{x \in \mathbb{R}^n} \tau_x V^0_\delta, p .
\]

1.2 The Class of Sequences representing the Dirac $\delta$ Distribution

1.2.1 First, we need several auxiliary notions

Suppose $m \in \mathbb{N}$ given and denote

\[
P(n,m) = \{ p = (p_1, \ldots, p_n) \in \mathbb{N}^n \mid |p| = p_1 + \ldots + p_n \leq m \}
\]

and

\[
\ell(n,m) \text{ the number of elements in } P(n,m).
\]

One can see that $\ell(1,m) = m + 1$, $\forall m \in \mathbb{N}$ and $\ell(n+1,m) = \sum_{0 \leq k \leq m} \ell(n,k)$, $\forall m \in \mathbb{N}$.

One can also see that there exists a total order $\rightarrow$ on $\mathbb{N}^n$ such that

\[
\mathbb{N}^n = \{ p(1), p(2), \ldots \}
\]
(11) \( p(1) \rightarrow p(2) \rightarrow \cdots \)

(12) \( \forall m \in \mathbb{N} : P(n,m) = \{p(1), \ldots, p(k(n,m))\} \).

Given \( s \in \mathcal{W} \), we consider the following Wronskian type infinite matrix of functions

\[
W(s)(x) = \begin{pmatrix}
D^{p(1)}s(0)(x) & \cdots & D^{p(\mu)}s(0)(x) & \cdots \\
\vdots & & \vdots & \\
D^{p(1)}s(\nu)(x) & \cdots & D^{p(\mu)}s(\nu)(x) & \cdots
\end{pmatrix}, \quad x \in \mathbb{R}^n.
\]

Denote by \( M \) the set of all infinite vectors of complex numbers

\( X = (x_{\mu} \mid \mu \in \mathbb{N}), x_{\mu} \in \mathbb{C}, \forall \mu \in \mathbb{N}, \) with a finite number of non-zero components \( x_{\mu} \).

An infinite matrix of complex numbers

\[
A = (a_{\nu \mu} \mid \nu, \mu \in \mathbb{N}), a_{\nu \mu} \in \mathbb{C}, \forall \nu, \mu \in \mathbb{N},
\]

called column wise nonsingular, only if:

(14) \( \forall X \in M: AX \in M \implies X = 0. \)

A characterization of column wise nonsingularity which results easily, is given in:

Lemma 1: The infinite matrix of complex numbers \( A = (a_{\nu \mu} \mid \nu, \mu \in \mathbb{N}) \) is column wise nonsingular, only if
1.2.2 And now, the definition of the class $Z^0_\delta$ of the sequences representing the Dirac $\delta$ distribution:

\[(16) \quad Z^0_\delta = \{ s \in S_0 \cap W^0_\delta \mid \begin{array}{c}
\ast) \quad \langle s, \cdot \rangle = \delta \\
\ast\ast) \quad W(s)(0) \text{ is column wise nonsingular}
\end{array} \} \]

We also need

\[(17) \quad Z^0_\delta = \bigotimes_{x \in \mathbb{R}^n} Z^0_\delta \]

The condition $\ast\ast)$ in (16) can be called "strong local presence" of the sequence $s$ in $x = 0 \in \mathbb{R}^n$, due to its meaning in the following particular case of $s$, which for $n = 1$ gives the well known representation [10], [11] of $\delta$, mentioned already in (9), Introduction.

Suppose $\psi \in D(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx = 1$. and define $s_\psi \in \mathcal{W}$, such that

\[(18) \quad s_\psi(v)(x) = \mu_1(v) \cdot \ldots \cdot \mu_n(v) \cdot \psi(\mu_1(v)x_1, \ldots, \mu_n(v)x_n) , \]

\[\forall v \in \mathbb{N}, x = (x_1, \ldots, x_n) \in \mathbb{R}^n :\]
where the mapping

\[(18.1) \quad n \in \nu \rightarrow \mu(n) = (\mu_1(n), \ldots, \mu_n(n)) \in N^n\]

is constructed in (18.2) - (18.5).

Define \(k(\nu) \in N\), with \(\nu \in N\), by

\[k(0) = 0 \quad \text{and} \quad k(\nu + 1) = k(\nu) + l(\nu, \nu + 1), \forall \nu \in N.\]

Define \(h(\nu) \in N\), with \(\nu \in N\), by

\[h(0) = 1 \quad \text{and} \quad h(\nu + 1) = h(\nu) + \nu + 1, \forall \nu \in N.\]

Define \(e(\nu) \in N^n\), with \(\nu \in N\), by

\[(18.4) \quad e(\nu) = (h(\nu), \ldots, h(\nu)), \forall \nu \in N.\]

Finally, define \(\mu(\nu) \in N^n\), with \(\nu \in N\), by

\[\mu(0) = (1, \ldots, 1) \in N^n \quad \text{and} \quad \mu(\nu + 1) = \mu(\nu) + P(\nu, \nu + 1), \forall \nu \in N.\]

The construction of the mapping

\[N \ni \nu \rightarrow \mu(\nu) \in N^n\]
In Fig. 1, the set denoted by $M_4$ can be written, in the terms of (18.5), as

$$M_4 = \{\mu(k(3)+1), \ldots, \mu(k(4))\} = e(4) + p(2,4).$$

Then, obviously

$$s_\psi \in S \cap w^0_{\delta} \quad \text{and} \quad <s_\psi, \ast> = \delta,$$

since $\lim_{n \to \infty} \mu_i(n) = \infty$, $\forall i \in \{1, \ldots, n\}$.

The basic property of the sequences defined in (18), is given in

Proposition 1. The following three conditions are equivalent:

*) $s_\psi \in Z^0_{\delta}$,

**) $W(s_\psi)(0)$ is column wise nonsingular,

***) $\forall p \in N^n: D^p\psi(0) \neq 0$.

Proof. The conditions *) and **) are equivalent, due to (19). We establish now, the equivalence between **) and ***)

Suppose $q \in N^n$, then

$$D^q s_\psi(n)(0) = (\mu(n))^{q + e}D^q\psi(0), \forall n \in N, \text{ where } e = (1, \ldots, 1) \in N^n.$$

Therefore

$$W(s_\psi)(0) = A \cdot B$$

where
(21.1) \[ A = \begin{pmatrix} (\mu(0))^{P(1)} + e & \cdots & (\mu(0))^{P(\sigma)} + e \\ \vdots & \ddots & \vdots \\ (\mu(\nu))^{P(1)} + e & \cdots & (\mu(\nu))^{P(\sigma)} + e \end{pmatrix} \]

and

(21.2) \[ B = \begin{pmatrix} D^{P(1)}\psi(0) & 0 & \cdots & 0 \\ 0 & D^{P(2)}\psi(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{P(3)}\psi(0) \end{pmatrix} \]

According to Theorem VAN (this Chapter, §1.10), A is column wise nonsingular. Indeed, it suffices to show that A satisfies (15) in Lemma 1. Suppose given \( \bar{v}, \bar{u} \in N \). We choose \( m \in N \), such that \( \bar{u} = \ell(n,m+1)-1 \geq \bar{\mu} \) and \( k(m) \geq \bar{v} \). Now, we choose

\[ v_0 = k(m)+1, \ v_1 = k(m)+2, \ldots, \ v_{\bar{\mu}} = k(m+1). \]

Then, the conditions *) and **) in (15) are obviously satisfied. The relation (18.5) and the mentioned Theorem VAN will imply directly the condition ***) in (15). Now, taking once more into account Lemma 1, \( W(s^\psi)(0) \) will be column wise nonsingular, only if \( D^{P(\nu)}\psi(0) \neq 0, \ \forall \nu \in N \), which ends the proof.

The first of the two sentences upon which the associative and commutative distribution multiplication is based, is given in:

**Corollary 1.** \( Z^o_\delta \neq \emptyset \) and \( Z_\delta \neq \emptyset \).

Proof. According to Proposition 1, in order to prove that $Z_\delta \neq \emptyset$, it suffices to show that there exists $\psi \in D(R^n)$, with $\int R^n \psi(x)dx = 1$ such that $D^p \psi(0) \neq 0$, $\forall p \in \mathbb{N}^n$.

Define $\alpha: R^n \rightarrow R$ with $\alpha(x_1, ..., x_n) = \exp(x_1 + ... + x_n)$ and suppose $\beta \in D(R^n)$ such that $\beta \geq 0$ and $\beta = 1$ in a certain neighborhood of $x = 0 \in R^n$.

Then $K = \int R^n \alpha(x)\beta(x)dx > 0$.

Now, we can define $\psi = \alpha \beta / K$. Indeed, it results then $D^p \psi(0) = 1/K \neq 0$, $\forall p \in \mathbb{N}^n$. From its definition, it follows that $Z_\delta \neq \emptyset$.

1.3 Further Classes of Sequences

We denote

(22) $\mathcal{S}_\delta^0 = \{s \in \mathcal{S}_\delta \mid \text{supp } \langle s, * \rangle \subset \{0\} \subset R^n\}$

and

(23) $\mathcal{S}_\delta$, the vector subspace generated in $\mathcal{S}_\delta^0$ by $\bigcup_{x \in R^n} x \mathcal{S}_\delta^0$.

For $s \in \mathcal{W}$, we denote

(24) $\mathcal{S}(s)$ the vector subspace generated in $\mathcal{W}$ by $\{D^p s \mid p \in \mathbb{N}^n\}$.

Suppose $\Sigma = (s_x \mid x \in R^n) \in Z_\delta$, then we denote

(25) $\mathcal{S}(\Sigma)$ the vector subspace generated in $\mathcal{S}_\delta$ by $\bigcup_{x \in R^n} \mathcal{S}(s_x)$. 

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We shall also need

(26) \( W_{\delta} \) the vector subspace generated in \( W \) by \( \bigcup_{x \in \mathbb{R}^n} x^0 \).

and, for \( p \in \mathbb{N}^n \)

(27) \( W_{\delta,p} \) the vector subspace generated in \( W \) by \( \bigcup_{x \in \mathbb{R}^n} (x^p \cap W_{\delta,p}) \).

Proposition 2

1) For \( p,q \in \mathbb{N}^n \), \( p \leq q \), the following inclusions hold

\[
\begin{align*}
S_\delta \cap W_\delta &= S_o \cap W_\delta \\
S_\delta &\rightarrow S_o \\
V_\delta,q &\rightarrow V_\delta,p & V_o
\end{align*}
\]

2) If \( p \in \mathbb{N}^n \), \( q \in \mathbb{N}^n \), then

\[
D_p(V_\delta,q) \subseteq V_\delta,q .
\]

3) If \( \Sigma \in \mathbb{Z}_\delta \), then \( S(\Sigma) \subseteq S_\delta \cap W_\delta = S_o \cap W_\delta \) and \( S_\delta = V_o \bigoplus S(\Sigma) \).

4) There exists \( S_1 \) vector subspace in \( S_o \), such that

\[
S_o = U \bigoplus S_\delta \bigoplus S_1 .
\]

Proof: 1), 2) and 3) result easily. 4) It is sufficient to notice that

\[
U \cap S_\delta = 0 .
\]
1.4 Regular Pairs of Vector Subspaces in Associative Algebras

In order to construct the algebras containing $D'(R^n)$ by using the diagrams of type (6) in Introduction, we need the notion of regular pairs of vector subspaces in associative algebras.

That notion is useful in constructing the upper part

$$\begin{align*}
I & \longrightarrow A \longrightarrow W \\
V & \longrightarrow S
\end{align*}$$

Of the mentioned diagrams and in satisfying the condition (see (8), Introduction)

$$I \cap S \subseteq V.$$

Suppose $W$ is an associative algebra and $V \subseteq S$ are vector subspaces in $W$. We denote

$$\begin{align*}
A(S) & \text{ the subalgebra generated in } W \text{ by } S, \\
I(V,S) & \text{ the vector subspace generated in } W \text{ by } V \cdot A(S).
\end{align*}$$

We shall try to construct (28) under the form

$$\begin{align*}
I(V,L) & \longrightarrow A(L) \longrightarrow W \\
V & \longrightarrow S
\end{align*}$$

where $L$ is a vector subspace in $W$ and $L \supset S$. 
In that case (29) will become

\[(33) \quad I(V,L) \cap S \subseteq V.\]

The main reason for preferring diagrams of type (32), is the facility in their construction, resulting from the way \(I(V,L)\) can be obtained according to the definition in (31).

Using \(L\) in (32), not necessarily equal to \(S\), offers the possibility of obtaining for the algebras \(A(L)\), properties which the algebra \(A(S)\) does not always possess.

And now, several results which can easily be established.

**Lemma 2.** 1) \(I(V,S)\) is a right ideal in \(A(S)\).

2) The following inclusions hold

\[
\begin{align*}
I(V,S) & \subseteq A(V) \rightarrow A(S) \rightarrow W \\
V & \subseteq S 
\end{align*}
\]

3) if \(W\) has a unit element \(1\) and \(1 \in S\), then

\[(34) \quad A(V) \rightarrow A(S) \rightarrow W \\
V \rightarrow S \]

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Lemma 3  Suppose $A$ is a subalgebra in $W$ and $I$ is a right ideal in $A$, such that

\[ I \rightarrow A \rightarrow W \]

(35)

\[ V \rightarrow S \]

Then, the following inclusions hold

1) $I \rightarrow A \rightarrow W$

2) If $W$ has a unit element $I$ and $I \in S$, then

Suppose that in addition $I \cap S = V$, that is, the mapping

\[ S/V \ni (s + V) \rightarrow (s + I) \in A/I \]

is linear and injective, then also $I(V,S) \cap S = V$, thus the mapping

\[ S/V \ni (s + V) \rightarrow (s + I(V,S)) \in A(S)/I(V,S) \]

is linear and injective.
The inclusions (34) in Lemma 2 justify the attempt in (32) of constructing (28) if \( V \) and \( S \) are given.

The inclusions (36) in Lemma 3 show that the diagrams of type (34) are "minimal" among all the diagrams of type (35). Moreover, if any diagram (35) satisfies (29), then the "minimal" diagram (34) will also satisfy (29). Therefore, in constructing (28), we can limit ourselves to the "minimal" diagrams (32).

In order to obtain conditions for satisfying (29), we need the following notion.

The pair \((V,S)\) of vector subspaces in the associative algebra \( W \) is called \underline{regular in} \( W \), only if

\[
I(V,W) \cap S = V.
\]

**Lemma 4.** Suppose \( W \) has a unit element \( 1 \) and \( 1 \in S \), then

1) \[
\begin{array}{c}
I(V,W) \quad \xrightarrow{\rightarrow} \quad W \\
V \quad \xrightarrow{\rightarrow} \quad S
\end{array}
\]

thus, \( V \subseteq I(V,W) \cap S \),

therefore, \((V,S)\) is \underline{regular in} \( W \), only if

\[
I(V,W) \cap S \subseteq V.
\]

2) \((V,S)\) is \underline{regular in} \( W \) only if for each \( L \) vector subspace in \( W \), the inclusion \( S \subseteq L \) implies \( I(V,L) \cap S \subseteq V \).
Proposition 3  Suppose \( W \) has a unit element \( 1 \) and \( 1 \in S \). If the pair \((V,S)\) is regular in \( W \), then, for every \( L \) vector subspace in \( W \), \( L \supset S \), there exists the following linear injective mapping

\[
S/V \ni (s + V) \rightarrow (s + I(V,L)) \in A(L)/I(V,L).
\]

If \( W \) is commutative, then, the above mapping is a linear embedding of the vector space \( S/V \) into the associative, commutative algebra \( A(L)/I(V,L) \), with the unit element \( 1 + I(V,L) \).

1.5 Regular Pairs in \( W \)

Before we prove Theorem 1, the second result upon which the associative, commutative distribution multiplication is based, several preliminary properties are given. Some of them will also be used later.

Lemma 5. Suppose \( s \in W \), then \( s \in W_0 \) only if there exists \( X \subset \mathbb{R}^n \) finite, such that

\[
\forall \varepsilon > 0: \exists \nu_\varepsilon \in \mathbb{N}: \forall v \in \mathbb{N}, v \geq \nu_\varepsilon:
\]

\[
D^{q_s(v)}(x) = 0, \forall q \in \mathbb{N}^n, x \in \mathbb{R}^n \setminus X(\varepsilon)
\]

where we denoted

\[
X(\varepsilon) = \{ y \in \mathbb{R}^n | \exists x \in X: \| y - x \| < \varepsilon \}.
\]

Proof. It results easily.

Proposition 4. \( W_0 \) is an ideal in \( W \).

Proof. It results from Lemma 5.
Lemma 6. Suppose \( s \in \mathcal{W} \) and \( p \in \mathbb{N}^n \), then \( s \in \mathcal{W}_{\delta, p} \) only if there exists \( X \subset \mathbb{R}^n \) finite, such that

1) \( \forall q \in \mathbb{N}^n, q \leq p; \exists v \in \mathbb{N}: \forall v \in \mathbb{N}, v \geq v_q: \)
\[ D^{q_s}(v)(x) = 0, \forall x \in X, \]

2) \( \forall \epsilon > 0; \exists v \in \mathbb{N}: \forall v \in \mathbb{N}, v \geq v_\epsilon: \)
\[ D^{q_s}(v)(y) = 0, \forall q \in \mathbb{N}, y \in \mathbb{R}^n \setminus X(\epsilon). \]

Proof. It results easily, taking into account Lemma 5 and the inclusion \( \mathcal{W}_{\delta, p} \subset \mathcal{W}_{\delta} \).

Proposition 5. Suppose \( p \in \mathbb{N}^n \), then \( \mathcal{W}_{\delta, p} \) is an ideal in \( \mathcal{W} \).

Proof. It results from Lemma 6.

Proposition 6. Suppose \( p \in \mathbb{N}^n \) and \( \Sigma \in \mathcal{Z}_\delta \), then
\[ \mathcal{W}_{\delta, p} \cap \mathcal{S}(\Sigma) = 0. \]

Proof. Assume \( t \in \mathcal{W}_{\delta, p} \cap \mathcal{S}(\Sigma) \) and \( \Sigma = (s_x \mid x \in \mathbb{R}^n) \).

Since \( t \in \mathcal{W}_{\delta, p} \), there exists \( X \subset \mathbb{R}^n \) finite, satisfying 1) and 2) in Lemma 6.

In the same time, \( t \in \mathcal{S}(\Sigma) \) implies
\[ t = \Sigma \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq k} c_{i j} p^j(x) s_x, \]
with \( m, k \in \mathbb{N} \setminus \{0\}, c_{i j} \in \mathbb{C} \) (possibly, some of them zero) \( p(1), \ldots, p(k) \in \mathbb{N} \)
pair wise different, and \( x_1, \ldots, x_m \in \mathbb{R}^n \) also pair wise different.
We are going to prove that

\[ c_{i1} = \ldots = c_{ik} = 0, \quad \forall 1 \leq i \leq m. \]

Indeed, suppose \( 1 \leq i \leq m \) given.

Assume \( \varepsilon > 0 \) such that

\[ \|x_{i1} - x_{i2}\| > 2\varepsilon, \quad \forall 1 \leq i_1 < i_2 \leq m. \]

Then, the inclusion (see 3) in Proposition 2) \( S(\Sigma) \subset S_\delta \cap W_\delta \subset W_\delta \)

and Lemma 5 will imply

\[ \exists \forall \varepsilon \in N: \forall \varepsilon \in N, \forall \geq \varepsilon : \]

\[ D^q_s (\nu)(x_{i1}) = 0, \quad \forall q \in N^m, \quad 1 \leq i_1 \leq m, \quad i_1 \neq i. \]

**Case 1:** \( x_{i1} \notin X. \)

We can also assume that \( x_{i1} \notin X(\Sigma) \), then, 2) in Lemma 6 results in

\[ \exists \forall \varepsilon \in N: \forall \varepsilon \in N, \forall \geq \varepsilon : \]

\[ D^q_t (\nu)(x_{i}) = 0, \quad \forall q \in N^m. \]

We can conclude

\[ 0 = t(\nu)(x_{i}) = \sum_{1 \leq j \leq k} c_{ij} \sigma^{(j)}(\nu)(x_{i1}), \quad \forall \nu \in N, \forall \geq \nu_0. \]

Since \( s_{x_{i1}} \in \tau x_{i1} z^*_\delta \), the condition ***) in the definition of \( z^*_\delta \) will imply

\[ c_{i1} = \ldots = c_{ik} = 0. \]
Case 2: \( x_i \in X \).

Denote \( q = (0, \ldots, 0) \in N^n \), then \( q \leq p \) and \( t \in W_{\delta,p} \) implies

\[
t(v)(x_i) = D^q t(v)(x_i) = 0, \quad \forall v \in N, \quad v \geq v_q.
\]

Thus, the above relation (37) results again, and it again implies (38), which completes the proof.

And now, the main result in this section 1.5, offering sufficient condition - see also Remark 1 below - for the regularity of the inclusion diagrams upon which the construction of the associative and commutative algebras with unit element and containing the distributions in \( D'(\mathbb{R}^n) \), is based.

**Theorem 1.** Suppose \( \Sigma \in Z_\delta \) and \( S_1 \) is a vector subspace in \( S_0 \), such that

\[
(R) \quad (U \oplus S(\Sigma) \oplus (S_\delta \cap W_{\delta,0})) \cap S_1 = 0,
\]

then, for each \( p \in \mathbb{N}^n \), the pair

\[
(V_\delta, p \oplus U \oplus S(\Sigma) \oplus S_1)
\]

is regular in \( W \).

**Proof.** It suffices to prove that

\[
(39) \quad l(V_\delta, p, W) \cap (V_\delta, p \oplus U \oplus S(\Sigma) \oplus S_1) \subseteq V_\delta, p.
\]

Since \( V_\delta, p \subseteq W_{\delta, p} \) and \( W_{\delta, p} \) is an ideal in \( W \), it results \( l(V_\delta, p, W) \subseteq W_{\delta, p} \).
In the same time \( V_{\delta, p} \oplus U \oplus S(\Sigma) \oplus S_1 \subseteq S_0 \).

Assume that \( t \) belongs to the left term of the inclusion (39), then

\[(40) \quad t \in S_0 \cap \tilde{W}_{\delta, p} \subseteq S_\delta \cap \tilde{W}_{\delta, p} .\]

Now, due to the condition (REG), the relation

\[t \in V_{\delta, p} \oplus U \oplus S(\Sigma) \oplus S_1\]

results in \( t \in V_{\delta, p} \oplus S(\Sigma) \), since (40).

Thus, \( t = v + s \), with \( v \in V_{\delta, p} \), \( s \in S(\Sigma) \).

We obtain

\[s = t - v \in \tilde{W}_{\delta, p} - V_{\delta, p} \subseteq \tilde{W}_{\delta, p} ,\]

since \( t \in l(V_{\delta, p}, \tilde{W}) \subseteq \tilde{W}_{\delta, p} \).

Finally, \( s \in \tilde{W}_{\delta, p} \cap S(\Sigma) \), which due to Proposition 6, results in \( s = u(0) \).

Therefore, \( t = v \in V_{\delta, p} \) and the proof is completed.

**Remark 1.** The condition (REG) in Theorem 1 is essential in defining (see (48.1) in §1.6.2) the family of associative, commutative algebras with unit element and containing the distributions in \( D'(\mathbb{R}^n) \).

Since it seems to be rather laborious to check whether a given vector subspace \( S_1 \) in \( S_0 \) satisfies the condition (REG), we shall give two stronger conditions, which are easier to handle, as they do not contain \( \Sigma \in \mathbb{Z}_\delta :\)

 Regel1 \( (U \oplus S_\delta) \cap S_1 = 0 \),

 Regel2 \( (U \oplus (S_\delta \cap \tilde{W}_\delta)) \cap S_1 = 0 \).
One can easily see that, for a given vector subspace $S_1$ in $S_0$, the following implications hold:

$$(\text{REG}1) \implies (\text{REG}2) \implies ((\text{REG}), \forall \xi \in \mathbb{Z}_0).$$

1.6 The Algebras Containing $D'(R^n)$

1.6.1 First, an auxiliary notion.

A property $P$, valid for certain subalgebras in $\mathcal{W}$, will be called admissible on $\mathcal{W}$, only if

(41) $\mathcal{W}$ has the property $P$,

(42) any intersection of subalgebras in $\mathcal{W}$, having the property $P$, will also have that property.

Obviously, if $P_1, \ldots, P_\alpha$ are admissible, then, their conjunction

(43) $P = P_1 \land \ldots \land P_\alpha$

is also admissible.

Sometimes, in defining the algebras containing $D'(R^n)$, several (finite) sets of admissible properties will be used. Each set of such properties - replaceable according to (43), by their conjunction - will lead to a specific family of algebras containing $D'(R^n)$.

Three of the more frequently used admissible properties of subalgebras $\mathcal{A}$ in $\mathcal{W}$, are the following ones:
1) \( A \) is "derivative invariant":

\[
D^p A \subseteq A, \quad \forall p \in \mathbb{N},
\]

where \( D^p : \mathcal{W} \to \mathcal{W} \) was defined in §1.1.1;

2) \( A \) is "cofinal invariant":

\[
\forall t \in \mathcal{W}:
\]

\[
\exists s \in A, \mu \in \mathbb{N} : \begin{cases} 
\forall v \in \mathbb{N}, v \geq \mu : \quad t(v) = s(v) 
\end{cases} \Rightarrow t \in A
\]

and

3) \( A \) is "positive power invariant":

\[
\forall s \in A : \begin{cases} 
\text{(*)} \quad s(v)(x) \geq 0, \quad \forall v \in \mathbb{N}, x \in \mathbb{R}^n 
\text{(**)} \quad s^{\alpha} \in \mathcal{W}, \quad \forall \alpha \in (0,\infty)
\end{cases} \Rightarrow (s^{\alpha} \in A, \forall \alpha \in (0,\infty))
\]

where \( s^{\alpha}(v)(x) = (s(v)(x))^\alpha, \quad \forall \alpha \in (0,\infty), v \in \mathbb{N}, x \in \mathbb{R}^n \).

When the results concerning the algebras containing \( D'(\mathbb{R}^n) \) are true in the most general setting, no admissible property will be used. In order to obtain a unified approach, we shall speak in that case about the maximal admissible property on \( \mathcal{W} \).

1.6.2 And now, the definition of the algebras containing \( D'(\mathbb{R}^n) \).

We denote by \( \Lambda \) the set of all \( \lambda = (\Sigma, S_1) \) where
(47) \[ \Sigma \in \mathbb{Z}_\delta, \]

(48) \[ S_1 \text{ is a vector subspace in } S_\circ, \text{ such that} \]

(48.1) \[ (U \oplus S(\Sigma) \oplus (S_\delta \cap W_\delta, 0)) \cap S_1 = 0, \]

(48.2) \[ S_\circ = U + S_\delta + S_1. \]

**Remark 2.** One can see that \( \Lambda \neq \emptyset \). Indeed, \( \mathbb{Z}_\delta \neq \emptyset \), according to Corollary 1, §1.2.2. Further, the possibility to choose \( S_1 \) as required, results for instance, from 4) in Proposition 2, §1.3 as well as the condition (REG1) and the implication (REG1) \( \implies \) (REG), in Remark 1, §1.5.

Suppose, \( P \) is a given admissible property on \( W \) (eventually, the conjunction of several admissible properties on \( W \) or, just the maximal admissible property on \( W \) — see §1.6.1.)

Suppose \( p \in \bar{N}^n \) and \( \lambda = (\Sigma, S_1) \in \Lambda. \)

We denote

(49) \[ S_{p, \delta} = V_{p, \delta} U \oplus S(\Sigma) \oplus S_1, \text{ (see 3) and 4) in Proposition 2, §1.3)} \]

(50) \[ A_{p, \lambda} \text{ the smallest subalgebra in } W, \text{ satisfying } P, \text{ as well as} \]

(50.1) \[ S_{p, \lambda} \subseteq A_{p, \lambda}, \]

(51) \[ I_{p, \lambda} = I(V_{\delta, p}, A_{p, \lambda}). \]

The associative, commutative algebras with unit element and containing \( D'(\mathbb{R}^n) \), will be

(52) \[ A_{p, \lambda} = A_{p, \lambda} / I_{p, \lambda}. \]
Remark 3. 1) The algebras $A_{p,\lambda}$, with $p \in \mathbb{N}^n$, $\lambda \in \Lambda$, depend obviously on the given admissible property $P$ on $\mathbb{W}$.

For convenience of expression, we shall call the resulting algebras $A_{p,\lambda}$, with $p \in \mathbb{N}^n$, $\lambda \in \Lambda$, as follows:

1.1) **derivative algebras**, if $P$ implies (44);
1.2) **cofinal algebras**, if $P$ implies (45), and
1.3) **positive power algebras**, if $P$ implies (46).

When no case is mentioned, it will be assumed that the admissible property $P$ on $\mathbb{W}$, can be arbitrary.

2) If $P$ is replaced by a stronger admissible property $P'$ on $\mathbb{W}$, that is,

$$(53) \quad P' \Rightarrow P,$$

then, $A_{p,\lambda}$ and therefore $I_{p,\lambda}$ will increase, since less subalgebras in $\mathbb{W}$, will still satisfy $P'$.

3) Actually, the above method of defining algebras containing the distributions could possibly give additional such algebras. Indeed, denote by $\Lambda'$ the set of all $\lambda = (\Sigma, S_1)$ satisfying (47), (48), (48.2) and, instead of (48.1), the following weaker condition:

$$(\text{REG}') \quad (V_{\delta,0} \uplus U \uplus S(\Sigma)) \cap S_1 = \emptyset,$$

and the pair

$$(V_{\delta,0}, V_{\delta,0} \uplus U \uplus S(\Sigma) \uplus S_1)$$

is regular in $\mathbb{W}$.

Then, $\Lambda \subset \Lambda'$ and, using the above method, one can still define algebras for $p \in \mathbb{N}^n$ and $\lambda \in \Lambda'$.
We have preferred the smaller family, corresponding to $\Lambda$, due to the fact that the second part of the condition (REG'), unlike the conditions (REG), (REG2) and especially (REG1) (see Theorem 1 and Remark 1, §1.5), does not offer an easy way of constructing $S_1$ in different specific instances, met both in theory and applications. Moreover, in most of such cases, $S_1$ satisfying the strongest and simplest condition (REG1) will be still satisfactory.

It remains as an open question, whether indeed $\Lambda'$ is strictly greater than $\Lambda$.

1.6.3 The next five theorems present the basic properties of the algebras containing the distributions.

The results in §1.4 and Theorem 1 in §1.5, lead to the characterization of the $A_{p,\lambda}$ algebras, valid for any admissible property $P$, presented in:

**Theorem 2** Suppose $p \in \mathbb{N}^n$ and $\lambda \in \Lambda$ given. Then,

1) $I_{p,\lambda} \rightarrow A_{p,\lambda} \rightarrow \hat{W}$

and

1.1) $I_{p,\lambda} \cap S_{p,\lambda} = V_o \cap S_{p,\lambda} = V_{\delta,\lambda}, \quad V_o + S_{p,\lambda} = S_o$,

therefore, the following linear mappings exist:
1.2) \[ S/V_0 \xrightarrow{\alpha_{p,\lambda}} S/V_0 \xrightarrow{\beta_{p,\lambda}} A_{p,\lambda} \]

with

1.2.1) \[ \alpha_{p,\lambda}(s + V_{\delta,p}) = s + V_0 \]

1.2.2) \[ \beta_{p,\lambda}(s + V_{\delta,p}) = s + I_{p,\lambda} \]

2) \( A_{p,\lambda} = A_{p,\lambda}/I_{p,\lambda} \) is an associative, commutative algebra, with the unit element \( u(1) + I_{p,\lambda} \).

3) \( D'(\mathbb{R}^n) \xrightarrow{\varepsilon_{p,\lambda}} A_{p,\lambda} \)

with \( \varepsilon_{p,\lambda} = \beta_{p,\lambda} \circ \alpha_{p,\lambda}^{-1} \circ \omega^{-1} \) (for \( \omega \), see (3.1) in §1.1.1) is a linear embedding of \( D'(\mathbb{R}^n) \) into \( A_{p,\lambda} \).

4) The multiplication in \( A_{p,\lambda} \) induces on \( C^\infty(\mathbb{R}^n) \) the usual multiplication of functions.

We define now, the positive powers of certain non-negative elements in \( D'(\mathbb{R}^n) \). The positive integer powers will coincide with the repeated multiplication in the algebras containing \( D'(\mathbb{R}^n) \).

Denote

(54) \[ C^\infty(\mathbb{R}^n)^+ = \{ \psi \in C^\infty(\mathbb{R}^n) \mid ^* \psi(x) \geq 0, \forall x \in \mathbb{R}^n \}

\[ ^{**} \psi^\alpha \in C^\infty(\mathbb{R}^n), \forall \alpha \in (0,\infty) \}

Obviously, if \( \psi \in C^\infty(\mathbb{R}^n) \) and \( \psi(x) > 0, \forall x \in \mathbb{R}^n \), then \( \psi \in C^\infty(\mathbb{R}^n)^+ \).

But, there exists \( \psi \in C^\infty(\mathbb{R}^n) \), with \( \psi(x) \geq 0, \forall x \in \mathbb{R}^n \), such that \( \psi \not\in C^\infty(\mathbb{R}^n)^+ \), for instance, \( \psi(x) = x_1^2 \cdots x_n^2, \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).
Nevertheless, defining

\[ \psi(x) = \begin{cases} 
\exp\left(-\sum_{1 \leq i \leq n} 1/x_i\right) & \text{if } x_i > 0, \ \forall 1 \leq i \leq n \\
0 & \text{if } \exists 1 \leq i \leq n: x_i \leq 0
\end{cases} \]

it results \( \psi \in C^\infty(R^n)^+ \).

Denote further

\( W^+ = \{ s \in W \mid s(v) \in C^\infty(R^n)^+, \ \forall v \in N \}. \)

It follows that for \( s \in W^+ \), any positive power \( s^\alpha \) can be defined by

\( s^\alpha(v)(x) = (s(v)(x))^\alpha, \ \forall \alpha \in (0, \infty), \ \forall v \in N, \ x \in R^n, \)

and it will always result \( s^\alpha \in W^+ \).

Moreover,

\( R_+^1 \cdot W^+ \subset W^+ \) and \( W^+ \cdot W^+ \subset W^+ \).

Suppose now, given \( \lambda = (E, S_1) \in \Lambda \) and denote

\( S_\lambda^+ = (U \oplus S(E) \oplus S_1) \cap W^+. \)

The distributions in

\( D_\lambda^+(R^n)^+ = \{ \langle s, \cdot \rangle \mid s \in S_\lambda^+ \} \)

will be called \( \lambda \) nonnegative and they will possess arbitrary positive powers.
Obviously,

\[ C^\infty(\mathbb{R}^n)^+ \subset D^1(\mathbb{R}^n)^+ \].

An additional property of \( D^1(\mathbb{R}^n)^+ \), granting the existence of arbitrary positive powers of the Dirac \( \delta \) distributions, will be now given. In Chapter 6, as an application, function solutions will be constructed for a Schrödinger type equation with the potential any positive power of the Dirac \( \delta \) distribution. In [18], details about the interest in Quantum Mechanics of the arbitrary positive powers of the Dirac \( \delta \) distribution, are presented.

**Proposition 7** There exists \( \lambda \in \Lambda \), such that

\[ \delta_{x_0} \in D^1(\mathbb{R}^n)^+ , \forall x_0 \in \mathbb{R}^n. \]

**Proof.** According to Lemma 7 below, we can choose \( s \in Z_\delta^o \cap \bar{\mathbb{N}}^+ \). Defining \( \Sigma = (\tau x s \mid x \in \mathbb{R}^n) \), it results \( \Sigma \in Z_\delta^o \). Further, 3) and 4) in Proposition 2, §1.3 result in the existence of a vector subspace \( S_1 \in S_o \) such that for \( \lambda = (\Sigma, S_1) \) we obtain \( \lambda \in \Lambda \).

Now, obviously, \( \tau x s \in S(\Sigma) \cap \bar{\mathbb{N}}^+ \subset S_\lambda^+ \), \( \forall x_0 \in \mathbb{R}^n \), therefore,

\[ \delta_{x_0} = \langle \tau x s, s \rangle \in D^1(\mathbb{R}^n)^+ , \forall x_0 \in \mathbb{R}^n, \]

which completes the proof.

**Lemma 7.** \( Z_\delta^o \cap \bar{\mathbb{N}}^+ \neq \emptyset \).

**Proof.** Define \( \alpha \in C^\infty(\mathbb{R}^1) \) by

\[ \alpha(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \]
then, obviously \( \alpha \in C(R^1)^+ \).

Define \( \beta \in C(R^1) \) by
\[
\beta(x) = \frac{\alpha(x)}{\alpha(x) + \alpha(1-x)}, \quad \forall x \in R^1.
\]

We prove that \( \beta \in C^\infty(R^1)^+ \). Define \( \beta_1 \in C^\infty(R^1) \) by
\[
\beta_1(x) = \alpha(x) + \alpha(1-x), \quad \forall x \in R^1.
\]
Then, \( \beta_1(x) > 0, \quad \forall x \in R^1 \), hence \( 1/\beta_1 \in C^\infty(R^1) \), \( 1/\beta_1(x) > 0, \quad \forall x \in R^1 \), which implies \( 1/\beta_1 \in C^\infty(R^1)^+ \).

Since \( \beta = \alpha \cdot (1/\beta_1) \) and \( \alpha \in C^\infty(R^1)^+ \), the required relation follows.

Define now, \( \gamma \in C^\infty(R^1) \) by
\[
\gamma(x) = \beta(x+2)\beta(2-x), \quad \forall x \in R^1,
\]
then \( \gamma \in C^\infty(R^1)^+ \), since \( \beta \) has that property.

Moreover,

\begin{align*}
(61) \quad & \gamma(x) = 0, \quad \forall x \in R^1 \setminus (-2,2), \\
(62) \quad & \gamma(x) = 1, \quad \forall x \in [-1,1] \quad \text{and} \\
(63) \quad & \gamma(x) > 0, \quad \forall x \in (-2,2).
\end{align*}

Define \( \eta \in C^\infty(R^n) \) by
\[
\eta(x_1, \ldots, x_n) = \gamma(x_1) \ldots \gamma(x_n) \exp(x_1 + \ldots + x_n)
\]
then \( \eta \in D(R^n) \cap C^\infty(R^n)^+ \) and
\[
K = \int_{R^n} \eta(x) \, dx > 0.
\]
Finally, define \( \psi \in C^\infty(\mathbb{R}^n) \) by \( \psi = \eta/K \), then

\[
(64) \quad \psi \in D(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)^+ .
\]

\[
(65) \quad \int_{\mathbb{R}^n} \psi(x) dx = 1
\]

\[
(66) \quad D^p \psi(0) = 1/K, \forall p \in \mathbb{N}^n .
\]

Therefore, \( s_\psi \) defined by (18), §1.2.2, belongs to \( Z^O_\delta \), according to Proposition 1, §1.2.2. Since \( \psi \in C^\infty(\mathbb{R}^n)^+ \), we obtain \( s_\psi \in \nu^+ \) and the proof is completed.

The definition and the properties of arbitrary positive powers of the elements in \( D^+(\mathbb{R}^n)^+ \), are given in :

**Theorem 3.** In the case of positive power algebras, suppose \( p \in \mathbb{N}^n \) and \( \lambda \in \Lambda \) given.

Then,

1) For every \( \alpha \in (0,\infty) \), one can define

\[
D^+(\mathbb{R}^n)^+ \ni S \quad \mapsto \quad S^\alpha \in A_{p,\lambda}
\]

by \( S = \langle s, \ast \rangle \)

\[
\quad \mapsto \quad s^\alpha + I_{p,\lambda} \in A_{p,\lambda}
\]

where \( s \in S^+_{\lambda} \).

2) Suppose \( S \in D^+(\mathbb{R}^n)^+ \), then

2.1) \( S^1 = S \);  
2.2) \( S^{\alpha + \beta} = S^\alpha \cdot S^\beta \), \( \forall \alpha, \beta \in (0,\infty) \);  
2.3) \( (S^\alpha)^m = S^{\alpha \cdot m} \), \( \forall \alpha \in (0,\infty) \), \( m \in \mathbb{N} \setminus \{0\} \).
3) The mapping in 1) coincides on $C^\infty(R^n)^+$ with the usual power of functions.

**Remark 4.** The relations 2.1) and 2.2) result in:

\[(67) \quad \forall s \in D'(R^n)^+, \quad \alpha \in (0,\infty) :\]

\[\alpha \in \mathbb{N} \implies s^\alpha = s \cdot \ldots \cdot s,\]

\[\alpha \text{ times}\]

therefore, the positive integer powers are the usual repeated multiplication within the algebras containing the distributions.

**Proof of Theorem 3.** 1) Suppose $s \in S^\lambda_\lambda$. Then $s \in A_{p,\lambda} \cap \mathcal{H}^+$, since $S^\lambda_\lambda \subset \mathcal{H}^+$ and obviously, $S^\lambda_\lambda \subset S_{p,\lambda} \subset A_{p,\lambda}$.

By the hypothesis, $P \implies (46)$, thus,

\[s^\alpha \in A_{p,\lambda} \cap \mathcal{H}^+ \subset A_{p,\lambda}, \quad \forall \alpha \in (0,\infty).\]

Now, due to the fact that the mapping

\[U \oplus S(\Sigma) \oplus S_1 \ni s \quad \xrightarrow{\langle \cdot, \cdot \rangle} s \in D'(R^n)\]

is a vector space isomorphism (see (48.2), §1.6.2) the definition of the mapping $S \mapsto s^\alpha$ results to be correct.

2) and 3) result easily.

The relationship between the algebras $(A_{p,\lambda} | \ p \in \mathbb{N})$, for a given $\lambda \in \Lambda$, results in:
Theorem 4. Suppose $\lambda \in \Lambda$ given. Then,

1) For each $p, q, r \in \mathbb{N}^n$, $p \leq q \leq r$, the following diagram of homomorphism of algebras is commutative:

\[
\begin{array}{c}
\gamma_{r,p,\lambda} \\
\downarrow \\
A_{r,\lambda} \\
\gamma_{r,q,\lambda} \\
\downarrow \\
A_{q,\lambda} \\
\gamma_{q,p,\lambda} \\
\downarrow \\
A_{p,\lambda}
\end{array}
\]

where $\gamma_{q,p,\lambda}(s+I_{q,\lambda}) = s + I_{p,\lambda}$.

2) For each $p, q \in \mathbb{N}^n$, $p \leq q$, the following diagram is commutative:

\[
\begin{array}{c}
A_{q,\lambda} \\
\gamma_{q,p,\lambda} \\
\downarrow \beta_{q,\lambda} \\
S_{q,\lambda} / V_{q,\lambda} \\
\omega \circ \alpha_{q,\lambda} \\
D'(R^n) \\
\downarrow \\
D(R^n)
\end{array} \quad \begin{array}{c}
A_{p,\lambda} \\
\gamma_{q,p,\lambda} \\
\downarrow \beta_{p,\lambda} \\
S_{p,\lambda} / V_{p,\lambda} \\
\omega \circ \alpha_{p,\lambda} \\
D'(R^n) \\
\downarrow \\
D(R^n)
\end{array}
\]

where $\eta_{q,p,\lambda}(s+V_{q,\lambda}) = s + V_{p,\lambda}$ is a vector space isomorphism.

3) For each $p, q \in \mathbb{N}^n$, $p \leq q$, the following diagram is commutative:

\[
\begin{array}{c}
A_{q,\lambda} \\
\gamma_{q,p,\lambda} \\
\downarrow \epsilon_{q,\lambda} \\
D'(R^n) \\
\downarrow \\
D'(R^n)
\end{array} \quad \begin{array}{c}
A_{p,\lambda} \\
\gamma_{q,p,\lambda} \\
\downarrow \epsilon_{p,\lambda} \\
D'(R^n) \\
\downarrow \\
D'(R^n)
\end{array}
\]
Proof. 1) \( V_\delta, q \subset V_\delta, p \), since \( p \leq q \). Thus, \( S_{q, \lambda} \subset S_{p, \lambda} \),

\[
A_{q, \lambda} \subset A_{p, \lambda}
\]
and finally, \( I_{q, \lambda} \subset I_{p, \lambda} \).

2) It suffices to prove that \( \eta_{q, p, \lambda} \) is a vector space isomorphism. Indeed, suppose \( \lambda = (\Sigma, S_1) \), then, the relations \( V_\delta, q \subset V_\delta, p \) and

\[
S_{q, \lambda} = V_\delta, q \oplus U \oplus S(\Sigma) \oplus S_1
\]

\[
S_{p, \lambda} = V_\delta, p \oplus U \oplus S(\Sigma) \oplus S_1
\]
result in the inclusions

\[
\begin{array}{ccc}
V_\delta, p & \rightarrow & S_{p, \lambda} \\
\downarrow & & \downarrow \\
V_\delta, q & \rightarrow & S_{q, \lambda}
\end{array}
\]

and in \( V_\delta, p \cap S_{q, \lambda} = V_\delta, q \), \( V_\delta, p + S_{q, \lambda} = S_{p, \lambda} \), which imply the required statement.

3) It results from 2) above and 3) in Theorem 2.

The definition and the properties of the derivative operators in the algebras containing \( D'(\mathbb{R}^n) \), are given in :

**Theorem 5.** In the case of derivative algebras, suppose \( \lambda \in \Lambda \) given. Then,

1) For each \( p \in \mathbb{N}^n, q \in \mathbb{N}^n \), there exists the linear mapping

\[
d^p_{p+q, \lambda} : A_{p+q, \lambda} \rightarrow A_{q, \lambda}
\]
defined by

\[ D^{p+q, \lambda}(s+t) = D^p s + D^q t \]

2) For each \( p \in \mathbb{N}^n, q, r \in \mathbb{N}^n, q \leq r \), the following diagram is commutative:

\[
\begin{array}{ccc}
A_{p+r, \lambda} & \xrightarrow{D^p} & A_{r, \lambda} \\
\downarrow & & \downarrow \\
\gamma_{p+r, p+q, \lambda} & & \gamma_{r, q, \lambda} \\
A_{p+q, \lambda} & \xrightarrow{D^p} & A_{q, \lambda}
\end{array}
\]

3) The mapping \( D^{p+q, \lambda} \) satisfies the "Leibnitz rule of product derivative":

for any \( S, T \in A_{p+q, \lambda} \), the relation holds

\[ D^{p+q, \lambda}(S \cdot T) = \sum_{r \in \mathbb{N}^n} C^r_p (\gamma_{p+q-r, q, \lambda}(D_r^r S)) \cdot (\gamma_{r+q, q, \lambda}(D^{p+q-r, q, \lambda} T)) , \]

which due to 2) above, can be written as

\[ D^{p+q, \lambda}(S \cdot T) = \sum_{r \in \mathbb{N}^n} C^r_p (D_r^r \gamma_{p+q+r, q, \lambda}(S)) \cdot (D^{p+q-q-r, q, \lambda} T) . \]

In both formulas, the products in the left terms are in \( A_{p+q, \lambda} \), while the products in the right terms are in \( A_{q, \lambda} \).

In particular, if \( p = (p_1, \ldots, p_n) \) and \( |p| = p_1 + \ldots + p_n = 1 \), then, the usual "rule of product derivative" results:

\[ D^{p+q, \lambda}(S \cdot T) = (D^{p+q, \lambda} S) \cdot \gamma_{p+q, q, \lambda}(T) + \gamma_{p+q, q, \lambda}(T) \cdot D^{p+q, \lambda} T , \]
which, due to 1) in Theorem 4, can be written under the following classical form:

\[ D^p_{p+q,\lambda}(S \cdot T) = (D^p_{p+q,\lambda}S) \cdot T + S \cdot D^p_{p+q,\lambda}T, \]

the product in the left term being in \( A_{p+q,\lambda} \), while the ones in the right term, in \( A_{q,\lambda} \).

4) For the functions in \( C^\infty(R^n) \) and the distributions in \( D'(R^n) \) with finite support, the mapping \( D^p_{p+q,\lambda} \) is identical with the usual derivative \( D^p \).

5) Suppose that in addition, the algebras are also positive power algebras. If \( p = (p_1, \ldots, p_n) \in \mathbb{N}^n \) and

\[ |p| = p_1 + \ldots + p_n = \lambda, \text{ then, for any } S \in D'(R^n)^+, \text{ the relation holds in } A_{q,\lambda} : \]

\[ D^p_{p+q,\lambda}(S^\alpha) = \alpha S^\alpha D^p_{p+q,\lambda}S, \quad \forall \alpha \in (1, \infty). \]

In particular, for the Dirac \( \delta_{x_o} \) distributions it results in any of the algebras:

\[ D^p(\delta_{x_o})^\alpha = \alpha(\delta_{x_o})^\alpha D^p_{p+q,\lambda} \delta_{x_o}, \quad \forall x_o \in R^n, \quad \alpha \in (1, \infty). \]

**Proof.** It results directly.

In the case of \( n = 1 \), it is possible to improve 4) in Theorem 5.
Indeed, denote

\[ D'_0(R^1) = \{ S \in D'(R^1) \mid \exists p \in N, a_0, \ldots, a_p \in C^1, a_p \neq 0: \Sigma a_i d^q S \text{ has finite support} \} \]

Then, in the case of derivative algebras, we obtain:

**Theorem 6.** Suppose \( S \in D'(R^1) \setminus (C^\infty(R^1) + D'_0(R^1)) \), then, there exists \( S_1 \) vector subspace in \( S_0 \), with \((U \oplus S_0) \cap S_1 = 0\), such that for each \( \Sigma \in Z_6 \) and \( p \in N, q \in \bar{N} \), the mapping \( d^p S, d^q S \rightarrow A_{p+q, \lambda} \), where \( \lambda = (\Sigma, S_1) \), is identical with the usual distribution derivative \( D^p \), on the set of distributions \( \{ S, DS, D^2 S, \ldots \} \).

**Proof.** Suppose \( S = <s, \cdot > \) for a certain \( s \in S_0 \) and denote \( S' = S(s) \). Then, according to Lemma 8, below, \((U \oplus S_0) \cap S_1 \neq 0\), thus, \( S' \) can be extended to a vector subspace \( S_1 \) of \( S_0 \), such that \((U \oplus S_0) \cap S_1 = 0\) and \( S_0 = U \oplus S_0 \oplus S_1 \). Then, \( \lambda = (\Sigma, S_1) \in A, \forall \Sigma \in Z_6 \). Suppose now, \( \Sigma \in Z_6 \) and \( p \in N, q \in \bar{N} \), then

\[ \{ s, Ds, D^2 s, \ldots \} = S(s) = S' \subset S_1 \subset S_{p+q, \lambda} \subset A_{p+q, \lambda} \]

Suppose given \( r \in N \), then \( D^r S \in D'(R^1) \) and \( D^r S = D^r s + I_{p+q, \lambda} \in A_{p+q, \lambda} \), since \( D^r S = <D^r s, \cdot > \) and \( D^r s \in S(s) \subset S_1 \).

Hence, \( d^p S, d^q S \rightarrow A_{p+q, \lambda} \) and \( D^p S + I_{p+q, \lambda} = D^{p+r} S + I_{q, \lambda} = D^{p+r} S \).
Lemma 8. Suppose \( s \in S_0 \) and \( s \neq u(0) \), then

\[(U \oplus S_\delta) \cap S(s) = 0 \iff <s, \cdot> \in C^\infty(R^1) + D_\delta^I(R^1).\]

Proof. First, the \( \iff \) implication. Suppose, it is false and let \( t \in (U \oplus S_\delta) \cap S(s) \) be such that \( t \neq u(0) \). Then, \( t = u(\psi) + s_1 = \sum a_q D^q s, \) with \( \psi \in C^\infty(R^1), \) \( s_1 \in S_\delta, p \in N, a_o, \ldots, a_p \in C^1 \) and \( a_p \neq 0 \). Denote \( P(D) = \sum a_q D^q. \) Let \( \chi \in C^\infty(R^1) \) be such that \( P(D) \chi = \psi, \) then \( s_2 = s - u(\chi) \in S_0 \) and \( P(D) s_2 = s_1, \) hence,

\[<s_2, \cdot> \in D_\delta^I(R^1), \]

since \( s_1 \in S_\delta. \) One can conclude that \( <s, \cdot> = <u(\chi), \cdot> + <s_2, \cdot> \in C^\infty(R^1) + D_\delta^I(R^1), \) which contradicts the initial assumptions.

The implication \( \implies \). Suppose, it is false and \( <s, \cdot> \in C^\infty(R^1) + D_\delta^I(R^1). \) Then, \( <s, \cdot> = <u(\psi), \cdot> + <s_1, \cdot>, \) with \( \psi \in C^\infty(R^1) \) and \( s_1 \in S_\delta \) such that \( \exists p \in N, a_o, \ldots, a_p \in C^1, a_p \neq 0: t = \sum a_q D^q s_1 \in S_\delta. \)

With the above notation for \( P(D) \), we obtain \( <P(D)s, \cdot> = <u(P(D)\psi), \cdot> + <t, \cdot>, \) hence \( P(D)s = u(P(D)\psi) + t + v, \) with \( v \in V_\delta. \) It results

\[P(D)s \in U \oplus S_\delta. \]

Now, if \( P(D)s \neq u(0) \), then \( (U \oplus S_\delta) \cap S(s) \ni P(D)s \neq u(0), \) which contradicts the initial assumptions. On the other side, if \( P(D)s = u(0), \) then \( \exists \chi \in C^\infty(R^1): s = u(\chi) + w, \) with \( w \in V_\delta. \) Thus, \( s \in U \oplus V_\delta \subseteq U \oplus S_\delta. \)

But, \( s \in S(s), \) hence \( s \in (U \oplus S_\delta) \cap S(s) = 0, \) that means, \( s = u(0), \) which again contradicts the initial assumptions.
1.7 Products with Dirac Distributions

In this section, properties of products with Dirac distributions are presented. The relations in Theorems 8, 9, 10, 11 and Corollary 2, are proper for the distribution multiplication presented here, since they involve products which cannot be given sense in \( D'(\mathbb{R}^n) \).

First, the properties of the type mentioned in (4) and b) in 0.1, Introduction.

**Theorem 7.** Suppose given \( x_0 \in \mathbb{R}^n \) and \( q \in \mathbb{N}^n \). Then,

1) \( D^q \delta_{x_0} \neq 0 \in A_p, \lambda, \forall p \in \mathbb{N}^n, \lambda \in \Lambda. \)

2) \( \psi(x-x_0) \cdot D^q \delta_{x_0} = 0 \in A_p, \lambda, \forall p \in \mathbb{N}^n, \lambda \in \Lambda, \) for each \( \psi \in \mathcal{C}^\infty(\mathbb{R}^n) \), such that

\[
(69) \quad D^r \psi(0) = 0, \forall r \in \mathbb{N}^n, r \leq p \text{ or } r \leq q.
\]

3) \( (x-x_0)^r \cdot D^q \delta_{x_0} = 0 \in A_p, \lambda, \forall p \in \mathbb{N}^n, \lambda \in \Lambda \) if \( r \in \mathbb{N}^n, r \geq p + e \) and \( r \geq q + e \), where \( e = (1, \ldots, 1) \in \mathbb{N}^n. \)

**Proof.** 1) Suppose \( \lambda = (\Sigma, \Sigma') \) and \( \Sigma = (s_x \mid x \in \mathbb{R}^n). \) According to 3) in Theorem 2, §1.6.3,

\[
D^q \delta_{x_0} = D^q s_{x_0} + \mathcal{I}_p, \lambda \in A_p, \lambda.
\]

Therefore, \( D^q \delta_{x_0} = 0 \in A_p, \lambda, \) only if \( D^q s_{x_0} \in \mathcal{I}_p, \lambda. \)

But \( D^q s_{x_0} \in S(\Sigma), \) hence \( D^q \delta_{x_0} = 0 \in A_p, \lambda \) implies \( D^q s_{x_0} \in \mathcal{I}_p, \lambda \cap S(\Sigma) \subset \mathcal{C}^\infty(\mathbb{R}^n). \)
$W_{\delta,p} \cap B(\Sigma) = \emptyset$, the last equality resulting from Proposition 6, §1.5, while the inclusion $I_{p,\lambda} \subseteq W_{\delta,p}$ resulting from Lemma 6, §1.5, the definition (51), §1.6.2 and the obvious inclusion $V_{\delta,p} \subseteq W_{\delta,p}$.

Now, the relation $D^q s_{x_0} \in I$ is absurd, since $s_{x_0} \in \tau_{x_0} Z^0_{\delta}$.

2) The following relations hold in $A_{p,\lambda}$ (see Theorem 2, §1.6.3).

$\beta_{p,\lambda}(\psi(x-x_0)) = \tau_{x_0} u(\psi) + I_{p,\lambda}$

$\beta_{p,\lambda}(D^q \delta_{x_0}) = D^q s_{x_0} + I_{p,\lambda}$

Thus, we obtain in $A_{p,\lambda}$

$\beta_{p,\lambda}(\psi(x-x_0)) \cdot \beta_{p,\lambda}(D^q \delta_{x_0}) = (\tau_{x_0} u(\psi)) \cdot D^q s_{x_0} + I_{p,\lambda}$.

Denote $v = (\tau_{x_0} u(\psi)) \cdot D^q s_{x_0}$, then obviously, $v \in W$, and

$v(v)(x) = \psi(x-x_0) \cdot D^q s_{x_0} (v)(x), \forall v \in N, x \in R^n$.

According to (69), $D^r \psi(0) = 0$, $\forall r \in N, r \leq q$, therefore, $v \in V_0$. Since $s_{x_0} \in \tau_{x_0} Z^0_{\delta}$, it results $v \in \tau_{x_0} W^0_{\delta}$. Finally, (69) implies

$D^r \psi(0) = 0, \forall r \in N, r \leq p$. Thus, $v \in \tau_{x_0} W^0_{p}$. Concluding, we obtain

$v \in V_0 \cap \tau_{x_0} W^0_{\delta} \cap \tau_{x_0} W^0_{p} \subseteq \nu_{\delta,p} \subseteq I_{p,\lambda}$, which implies

$\beta_{p,\lambda}(\psi(x-x_0)) \cdot \beta_{p,\lambda}(D^q \delta_{x_0}) = 0 \in A_{p,\lambda}$.
3) It results from 2), taking \( \psi(x) = x^r \), since both \( p \) and \( q \) are finite.

**Remark 5.** In the case of \( n = 1 \), the relations in 3), Theorem 7, result in:

\[
\forall p \in N, \lambda \in \Lambda, x_o \in \mathbb{R}^1:
(70) \quad (x-x_o)^{p+1} \cdot \delta^p_{x_o} = (x-x_o)^{p+1} \cdot D_\delta x_o \ldots = (x-x_o)^{p+1} \cdot D^{p+1} \delta_{x_o} = 0 \in A_p, \lambda,
\]

\[
(71) \quad (x-x_o)^{q+1} \cdot D^q \delta_{x_o} = 0 \in A_p, \lambda, \forall q \in N, q \geq p.
\]

If \( p \geq 1 \), taking the derivative \( D^1_{p, \lambda} \) in (71), it results:

\[
(72) \quad (q+1)(x-x_o)^{q} \cdot D^q \delta_{x_o} + (x-x_o)^{q+1} \cdot D^{q+1} \delta_{x_o} = 0 \in A_{p-1}, \lambda, \forall q \in N, q \geq p.
\]

An expected property of the product of two Dirac distributions is given in:

**Theorem 8.** In each algebra \( A_{p, \lambda} \), with \( p \in \mathbb{N}^n, \lambda \in \Lambda \), the relations hold:

\[
D^q x \cdot D^r y = 0 \in A_{p, \lambda}, \forall x, y \in \mathbb{R}^n, x \neq y, q, r \in \mathbb{N}^n.
\]

**Proof.** Suppose \( \lambda = (\Sigma, \Sigma') \) and \( \Sigma = (s_x \ | \ x \in \mathbb{R}^n) \), then

\[
D^q x = D^q s_x + I_{p, \lambda}, \quad D^r y = D^r s_y + I_{p, \lambda} \in A_{p, \lambda}.
\]

Hence,

\[
D^q x \cdot D^r y = D^q s_x \cdot D^r s_y + I_{p, \lambda}.
\]

But, \( x \neq y \), \( s_x \in \mathcal{T}^o_x \delta, s_y \in \mathcal{T}^o_y \delta \), imply

\[
\exists \nu_o \in N: \forall \nu \in N, \nu \geq \nu_o : D^q s_x(\nu) \cdot D^r s_y(\nu) = u(0).
\]

Thus, \( D^q s_x \cdot D^r s_y \in \mathcal{V}_{\delta, p} \subset I_{p, \lambda} \), which implies the required relation.
The next two theorems show the nontriviality of the product in the case of the Dirac distributions concentrated in the same point of \( R^n \).

**Theorem 9.** In each algebra \( A_p, \lambda \), with \( p \in \bar{N}^n \), \( \lambda \in \Lambda \), the relations hold:

\[
(D^{r-\delta}_x)q \neq 0 \in A_p, \lambda, \quad \forall x_0 \in R^n, \quad r \in N^n, \quad q \in N\backslash\{0\}.
\]

**Proof.** Suppose \( \lambda = (\Sigma, s_1) \) and \( \Sigma = (s_x \mid x \in R^n) \), then,

\[
(D^{r-\delta}_x)q = (D^{r-\delta}_x s_x)q + I_p, \lambda \in A_p, \lambda.
\]

Thus, \( (D^{r-\delta}_x)q = 0 \in A_p, \lambda \), only if \( (D^{r-\delta}_x)q \in I_p, \lambda \). Suppose, \( (D^{r-\delta}_x)q \in I_p, \lambda \), then:

\[
\forall h \in N^n, h \leq p: \exists \forall_h \in N: \forall v \in N, v \geq \forall_h : (D^h((D^{r-\delta}_x s_x)q(v))(x_0)) = 0.
\]

since \( I_p, \lambda \subset W_\delta, p \) and from \( s_x \in \tau_x \delta^0 \subset \tau_x p_0 \delta \) it results

\[
(D^{r-\delta}_x)q \in \tau_x p_0 \delta, \text{ thus, finally}
\]

\[
(D^{r-\delta}_x)q \in W_\delta, p \cap \tau_x p_0 \delta \subset \tau_x p_0.
\]

Since \( q \geq 1 \), (73) will result, for \( h = 0 \), in

\[
(D^{r-\delta}_x)q(v)(x_0) = 0, \quad \forall v \in N, v \geq \forall_0.
\]

That relation implies that the elements of the \( r \)-th column in \( W(s_x)(x_0) \) (see (13) in §1.2.1) will be zero, for \( v \in N, v \geq \forall_0 \). Then, according to Lemma 1, §1.2.1, \( W(s_x)(x_0) \) will not be column wise non-singular, which contradicts the initial assumption that \( s_x \in \tau_x \delta^0 \).
In special cases, a stronger property of the product nontriviality for the Dirac distributions concentrated in the same point of $\mathbb{R}^n$, can be obtained as follows:

**Theorem 10.** There exists $\Sigma \in \mathbb{Z}_\delta$, such that in each algebra $A_p,\lambda$, with $p \in \mathbb{N}^n$, $\lambda = (\Sigma, S_1) \in \Lambda$, the relations hold:

$$(D^r \delta_x)_0 \ldots (D^q \delta_x)_0 \neq 0 \in A_p,\lambda, \forall x_0 \in \mathbb{R}^n, q \in \mathbb{N}, r, \ldots, r \in \mathbb{N}^n.$$ 

**Proof.** We consider $s \in Z_\delta$ defined in the proof of Lemma 7, §1.6.3, and define $\Sigma \in Z_\delta$ by $\Sigma = (\tau_x s \mid x \in \mathbb{R}^n)$. Suppose now, $p \in \mathbb{N}^n$ and $\lambda = (\Sigma, S_1) \in \Lambda$. Then, in the algebra $A_p,\lambda$, the relation holds:

$$\prod_{0 \leq i \leq q} D^r s(x_0) = \prod_{0 \leq i \leq q} D^r \tau_x s + I_p,\lambda \in A_p,\lambda.$$ 

Denote $v = \prod_{0 \leq i \leq q} D^r \tau_x s$, then $\prod_{0 \leq i \leq q} (D^r \delta_x)_0 = 0 \in A_p,\lambda$, only if $v \in I_p,\lambda$. Suppose $v \in I_p,\lambda$, then, according to an argument similar to that used to establish (74) in the proof of Theorem 9, it results $v \in \tau_x w_{x_0}^p$. Since $p \geq 0 \in \mathbb{N}^n$, we can conclude that

$$\exists v_0 \in \mathbb{N} : \forall v \in \mathbb{N}, v \geq v_0 : v(v)(x_0) = 0.$$ 

But, $v(v)(x_0) = \prod_{0 \leq i \leq q} D^r s(v)(0)$ and, according to (20) in the proof of Proposition 1, §1.2.2, and (66) in the proof of Lemma 7, §1.6.3, it results $v(v)(x_0) \neq 0, \forall v \in \mathbb{N}$, therefore, a contradiction.

And now, two properties of the Dirac distributions, specific for the distribution multiplication presented in this work. For the sake of simplicity, we suppose $n=1$. In both, Th.11 and Cor.2, the case of derivative algebras is assumed.
Theorem 11. Suppose given $p \in \mathbb{N}\{0\}$ and $\lambda \in \Lambda$, then, the relations hold:

$$D_{p,\lambda}^{1}(x-x_o)^k \cdot (D^{r}\delta_{x_o}^r)^q = k(x-x_o)^{k-1} \cdot (D^{r}\delta_{x_o}^r)^q$$

for $x_o \in \mathbb{R}^l$, $k, q, r \in \mathbb{N}$, $k > \max\{r, p-1\}$, $q \geq 2$, where the product in the left term is in $A_{p,\lambda}$, while the one in the right term is in $A_{p-1,\lambda}$.

Proof. According to 3) and 4) in Theorem 5, §1.6.3,

$$D_{p,\lambda}^{1}(x-x_o)^k \cdot (D^{r}\delta_{x_o}^r)^q = k(x-x_o)^{k-1} \cdot (D^{r}\delta_{x_o}^r)^q + (x-x_o)^k \cdot (D^{r}\delta_{x_o}^r)^q - 1 \cdot D^{r+1}\delta_{x_o}^{r+1}.$$ 

According to 3) in Theorem 7,

$$(x-x_o)^k \cdot D^{r}\delta_{x_o}^r = 0 \in A_{p-1,\lambda}$$

since $k \geq \max\{p-1, r\} + 1$.

Since the product is associative and $q - 1 \geq 1$, the first relation will imply the required result.

Corollary 2. Suppose given $p \in \mathbb{N}$ and $\lambda \in \Lambda$, then, the relations hold:

$$(x-x_o)^r \cdot (D^{r}\delta_{x_o}^r)^q = 0 \in A_{p,\lambda}$$

for $x_o \in \mathbb{R}^l$, $q, r \in \mathbb{N}$, $q \geq 2$, $r \geq p+1$.

Proof. Since $r + 1 > \max\{r, p\}$, Theorem 11 implies

$$D_{p+1,\lambda}^{1}(x-x_o)^r \cdot (D^{r}\delta_{x_o}^r)^q = (r + 1) \cdot (x-x_o)^r \cdot (D^{r}\delta_{x_o}^r)^q.$$ 

According to 3) in Theorem 7,

$$(x-x_o)^r + 1 \cdot D^{r}\delta_{x_o}^r = 0 \in A_{p+1,\lambda}$$

which completes the proof.
1.8 Remark on a Multiplication Theory with Stronger Conditions for Derivatives

1.8.1 It will be shown that even in the case of $n = 1$, a certain stronger condition - which could seem to be natural - on the way the derivative is defined, will lead to a trivial distribution multiplication.

Suppose $A$ is an associative, commutative algebra, such that:

A1) $\mathcal{P}(R^1) \oplus \mathcal{D}'(R^1) \subseteq A$,

where $\mathcal{P}(R^1)$ is the set of the complex valued polynomials on $R^1$ and $\mathcal{D}'(R^1)$ is the set of the distributions in $\mathcal{D}'(R^1)$, with finite support;

A2) the multiplication in $A$ induces on $\mathcal{P}(R^1)$ the usual multiplication of polynomials and the polynomial $\psi(x) = 1, \forall x \in R^1$, is the unit element of the algebra $A$;

A3) there exists a linear mapping $D: A \rightarrow A$, such that:

A3.1) $D$ is identical on $\mathcal{P}(R^1) \oplus \mathcal{D}'(R^1)$ with the usual derivative;

A3.2) $D$ satisfies on $A$ the "rule of product derivative":

$$D(a \cdot b) = (Da) \cdot b + a \cdot (Db), \forall a, b \in A;$$

A4) $(x-x_0) \cdot \delta_{x_0} = 0 \in A, \forall x_0 \in R^1$.

Then, it results easily (see §1.8.2) that:

(75) $(x-x_0)^p \cdot D^q \delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p, q \in N, p \geq q+1$;

(76) $(p+1) \cdot D^p \delta_{x_0} + (x-x_0) \cdot D^{p+1} \delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p \in N;$
\[(77) \quad (x-x_0)^p \cdot (D^p \delta_{x_0})^q = 0 \in A, \forall x_0 \in R^1, p,q \in N, q \geq 2.\]

The last relation implies:

\[(78) \quad (\delta_{x_0})^2 = \delta_{x_0} \cdot D_{x_0} = 0 \in A, \forall x_0 \in R^1.\]

Taking into account the usual interpretations of \(\delta^2\) (see [1], [2], [3], [4], [5], [6], [11], [18] and [19]) it results that a multiplication under the conditions A1) - A4) - if there exists such one - is not of significant interest.

In this respect, the distribution multiplication presented in this work, with the derivative being defined - in the case of \(n=1\) - as follows:

\[D^p_{\infty,\lambda}: A_{\infty,\lambda} \rightarrow A_{\infty,\lambda}, \quad \forall p \in N,\]

\[D^p_{p+q,\lambda}: A_{p+q,\lambda} \rightarrow A_{q,\lambda}, \quad \forall p,q \in N,\]

reminding the way the usual derivatives act on the spaces \(C^p(R^1), p \in \bar{N}\), can be considered, due to Theorem 9 and 10, §1,7, the best possible one.

1.8.2 Here, the proofs for the relations (75), (76), (77) and (78) are given.

Applying \(D\) to A4), it results, due to A3.1) and A3.2) that

\[(76.1) \quad \delta_{x_0} + (x-x_0) \cdot D\delta_{x_0} = 0 \in A, \forall x_0 \in R^1,\]

which multiplied by \((x-x_0)\), gives according to A4)

\[(x-x_0)^2 \cdot D\delta_{x_0} = 0 \in A, \forall x_0 \in R^1.\]
Applying $D$ to the last relation and then, multiplying by $(x-x_0)$, we obtain

$$(x-x_0)^3 \cdot D^2 \delta_{x_0} = 0 \in A, \forall x_0 \in R^1.$$ 

Repeating the procedure, it results (75).

The relation (76) results applying repeatedly $D$ to (76.1) above.

Multiplying (76) by $(x-x_0)^p$, we obtain

$$(p+1)(x-x_0)^p \cdot D^p \delta_{x_0} + (x-x_0)^{p+1} \cdot D^{p+1} \delta_{x_0} = 0 \in A, \forall x_0 \in R^1, p \in N.$$ 

Now, multiplying that relation by $(D^p \delta_{x_0})^{q-1}$, we obtain (77), due to (75).

The relation $(\delta_{x_0})^2 = 0 \in A, \forall x_0 \in R^1$, in (78) results from (77), for $p = 0, q = 2$. Applying $D$ to the last relation, the proof of (78) is completed.

1.9 Applications

1.9.1 Formulas with the Heisenberg distributions.

Certain formulas in Quantum Mechanics, involving the Heisenberg distributions:

$$\delta_+ = (\delta_0 + (1/x)/\pi i)/2$$

$$\delta_- = (\delta_0 - (1/x)/\pi i)/2$$

will be proved within the distribution multiplication presented.

**Proposition 8.** There exists $\lambda \in A$ such that in each algebra $A_{p,\lambda}$, with $p \in \tilde{N}$, the relations hold, [11]:

$$(79) \quad (\delta_+)^2 = -D \delta_0/4\pi i - (1/x^2)/4\pi^2$$
\begin{equation}
\delta_0 \cdot (1/x) = -D\delta_0 / 2
\end{equation}

\begin{equation}
(\delta_0)^2 - (1/x)^2/\pi^2 = -(1/x^2)/\pi^2.
\end{equation}

**Proof.** Consider, in the case of \( n = 1 \), \( \psi \in \mathcal{D}(R_1) \) and \( s \in \mathcal{Z}_0^\delta \) defined in the proof of Lemma 7, §1.6.3.

Suppose \( L > 0 \), such that \( \text{supp} \ \psi \subset [-L,L] \). Denote \( M_p = \sup\{|D^p\psi(x)|, x \in R_1\} \), with \( p \in \mathbb{N} \).

Then,
\[
|x^{p+1} \cdot D^p \mathcal{s}(\nu)(x)| \leq M_p \cdot L^p, \quad \forall p, \nu \in \mathbb{N}, \ x \in R_1,
\]
therefore, \( s \) is a "delta sequence" according to [10], [11].

Denote \( \Sigma = (\tau_s \mid x \in R_1) \), then \( \Sigma \in \mathcal{Z}_0^\delta \), since \( s \in \mathcal{Z}_0^\delta \).

Defining \( t(\nu) = s(\nu) \ast (1/x) \), with \( \nu \in \mathbb{N} \), it results \( t \in \mathcal{Z}_0 \) and \( <t, \cdot> = (1/x) \). According to Lemma 9 below and Lemma 8, §1.6.3, there exists a vector subspace \( \mathcal{S}_1 \) in \( \mathcal{S}_0 \), such that \( \mathcal{S}(t) \subset \mathcal{S}_1 \), \( (U \oplus \mathcal{S}_\delta) \cap \mathcal{S}_1 = 0 \) and \( \mathcal{S}_0 = U \oplus \mathcal{S}_\delta \oplus \mathcal{S}_1 \). Denoting \( \lambda = (\Sigma, \mathcal{S}_1) \), it results \( \lambda \in \Lambda \).

From [11] it results that
\[
t_1 = (s+t/\pi i)^2, \ t_2 = (s-t/\pi i)^2, \ t_3 = s \cdot t \in \mathcal{S}_0 \text{ and}
\]
\[
<t_1, \cdot> = -D\delta_0 / \pi i - (1/x^2)/\pi^2
\]
\[
<t_2, \cdot> = D\delta_0 / \pi i - (1/x^2)/\pi^2
\]
\[
<t_3, \cdot> = -D\delta_0 / 2.
\]
But, for each $p \in \tilde{N}$, we obtain in $A_{p, \lambda}$

$$
\delta_+ = (s + t/\pi^2)/2 + I_{p, \lambda},
$$
$$
\delta_- = (s - t/\pi^2)/2 + I_{p, \lambda},
$$
$$
\delta_o \cdot (1/x) = s \cdot t + I_{p, \lambda}
$$

hence, the relations (79), (80) and (81). The relation (82) results, for instance, from (79) and (81), since we can write in $A_{p, \lambda}$:

$$(\delta_+^2 = ((\delta_o^2 - (1/x)^2/\pi^2 + 2\delta_o \cdot (1/x)/\pi^2)/4 .
$$

**Lemma 9.** Suppose $m \in N \setminus \{0\}$, then

$$(1/x^m) \notin C^\infty(R^1) + D^\prime_\sigma(R^1) .$$

**Proof.** Suppose, it is false, then

$$
\exists \psi \in C^\infty(R^1), T \in D^\prime_\sigma(R^1) : (1/x^m) = \psi + T .
$$

But, $T = (1/x^m) - \psi$ implies that

$$
\exists \chi \in C^\infty(R^1 \setminus \{0\}) : T \mid_{R^1 \setminus \{0\}} = \chi .
$$

Since $T \in D^\prime_\sigma(R^1)$, it follows that

$$
\exists p \in N, a_0, \ldots, a_p \in C^1, a_p \neq 0, S \in D^\prime_\sigma(R^1) : \sum_{0 \leq q \leq p} a_q T = S .
$$

Denote $S_1 = S \mid_{R^1 \setminus \{0\}} \in D^\prime_\sigma(R^1 \setminus \{0\})$, then

$$
S_1 = \sum_{0 \leq q \leq p} a_q T \chi , \text{ hence } S_1 \in C^\infty(R^1 \setminus \{0\}) , \text{ thus finally, } S_1 = 0 .
$$
since by its definition $S_1 \in D^i_\delta(R^1 \backslash \{0\})$.

Now, we obtain $P(D)(1/x^m) = P(D)\psi$ on $R^1 \backslash \{0\}$, where

$$P(D) = \sum_{0 \leq q \leq p} a^D_q \cdot q^q.$$  

We can conclude, differentiating in the last relation:

$$\sum_{0 \leq q \leq p} (-1)^q \frac{(m+q-1)!}{(m-1)!} \cdot a^q \cdot x^{p-q} = x^{m+p} P(D)\psi(x), \forall x \in R^1 \backslash \{0\}.$$  

Taking the limit for $x \to 0$, we obtain

$$(-1)^p \frac{(m+p-1)!}{(m-1)!} \cdot a_p = 0$$

which contradicts the assumption $a_p \neq 0$.

1.9.2 Application to a Riccati Differential Equation

Proposition 9. The Riccati differential equation

$$y' = x^{p+q} \cdot y \cdot (y + 1) + D^q \delta_\delta, \ p, q \in \mathbb{N}, \ p \geq 1,$$

has in any algebra $A_\delta, \lambda$, with $\lambda \in \Lambda$, the general solution

$$y(x) = 1/(c \exp(-x^{p+q+1}/(p+q+1))-1) + D^q \delta_\delta(x), \ x \in R^1, \ c \in (-\infty, 0],$$

which belongs to $C^\infty(R^1) \oplus D^i_\delta(R^1)$.

Proof. Suppose $c \in R^1$ and define $\psi \in C^\infty(R^1)$, by

$$\psi(x) = c \exp(-x^{p+q+1}/(p+q+1))-1, \ \forall x \in R^1.$$  

Then, $c \in (-\infty, 0]$ implies $1/\psi \in C^\infty(R^1)$. 
Therefore, \( c \in (-\infty, 0] \) implies

\[
T = \frac{1}{\psi} + D^q \delta_o \in C^\infty(R^1) \bigoplus D^q_o(R^1) \subset A_{p, \lambda}, \quad \forall p \in \bar{N}, \lambda \in \Lambda.
\]

Since \( A_{p, \lambda} \) is associative, commutative and with the unit element \( 1 \in C^\infty(R^1) \), the following holds for \( c \in (-\infty, 0] \):

\[
(83) \quad x^{p+q} \cdot T \cdot (T+1) = x^{p+q} \cdot (D^q \delta_o)^2 + 2 \cdot x^{p+q} \cdot (D^q \delta_o) \cdot (1/\psi) + x^{p+q} \cdot (1/\psi)^2 + x^{p+q} \cdot (D^q \delta_o) + x^{p+q} \cdot (1/\psi).
\]

According to 3) in Theorem 7, §1.7, we obtain in the algebras \( A_{o, \lambda} \), with \( \lambda \in \Lambda \):

\[
x^{p+q} \cdot (D^q \delta_o) = 0 \in A_{o, \lambda}
\]

since \( p+q \geq \max\{0, q\} + 1 \), as \( p \geq 1 \).

Now, within \( A_{o, \lambda} \), with \( \lambda \in \Lambda \), (83) will become:

\[
x^{p+q} \cdot T \cdot (T+1) = x^{p+q} \cdot (1/\psi) \cdot (1/\psi + 1), \quad \forall c \in (-\infty, 0]
\]

relation implying that \( T \) is the solution of the considered Riccati equation.

\[\textbf{1.9.3 Constructing Algebras Containing given Weakly Convergent Sequences of Smooth Functions}\]

In certain applications (see for instance, Theorem 6, §1.6.3 or Theorem 8, §6.7, Chapter 6) the following problem arises:

\[
(84) \quad \text{given a vector subspace } S \text{ in } S_o, \text{ to find out whether there exist algebras } A_{p, \lambda}, \text{ with } p \in \bar{N}^n, \lambda \in \Lambda, \text{ such that}
\]
(84.1) \[ s + I_{p, \lambda} \in A_{p, \lambda}, \forall s \in S. \]

A sufficient condition for an affirmative answer to (84), is given in:

**Theorem 12.** Suppose, for a certain \( \Sigma \in Z_\delta \), the relation holds:

\[
(U \oplus S(\Sigma) \oplus (S_\delta \cap W_\delta, 0)) \cap S = (V_\delta, \infty \oplus U \oplus S(\Sigma)) \cap S.
\]

Then, there exists a vector subspace \( S_1 \) in \( S_\circ \), such that:

1) \( \lambda = (\Sigma, S_1) \in \Lambda, \)
2) \( s \in S_{p, \lambda} = V_\delta, p \oplus U \oplus S(\Sigma) \oplus S_1 \in A_{p, \lambda}, \forall p \in \tilde{N}^n. \)

**Proof.** Since \( \tilde{V}_\delta, \infty \subset S_\delta \cap W_\delta, 0 \), the hypothesis results in the equality

\[
(85) \quad (U \oplus S(\Sigma) \oplus (S_\delta \cap W_\delta, 0)) \cap S = (V_\delta, \infty \oplus U \oplus S(\Sigma)) \cap S.
\]

Denote by \( S' \) the vector subspace in \( S_\circ \) defined by (85) and consider \( S'' \) a vector subspace in \( S_\circ \), such that

\[
(86) \quad S = S' \oplus S''.
\]

Further, consider \( S''' \) a vector subspace in \( S_\circ \), such that

\[
(87) \quad S_\circ = (U + S_\delta + S) \oplus S'''.
\]

We show that one can take

\[
(88) \quad S_1 = S'' \oplus S'''.
\]

In this respect, the relations (48.1) and (48.2) in §1.6.2 have to be proved.
First, the relation (48.1).

Obviously,

\[(U \oplus S(\Sigma) \oplus (S_\delta \cap \bar{W}_\delta, \alpha)) \cap S_1 = (U \oplus S_\delta) \cap S_1\]

but, due to (86), (87) and (88)

\[(U \oplus S_\delta) \cap S_1 \subset S'' \subset S.\]

Therefore,

\[(U \oplus S(\Sigma) \oplus (S_\delta \cap \bar{W}_\delta, \alpha)) \cap S_1 \subset (U \oplus S(\Sigma) \oplus (S_\delta \cap \bar{W}_\delta, \alpha)) \cap S = S'.\]

Now, (89) and (90) imply

\[(U \oplus S(\Sigma) \oplus (S_\delta \cap \bar{W}_\delta, \alpha)) \cap S_1 \subset S' \cap S'' = \emptyset\]

since (86).

The relation (48.2) follows easily from (86), (87) and (88).

2) Obviously, \(V_{\delta, p} \subset S_\delta \cap \bar{W}_\delta, \alpha\), \(\forall p \in \bar{N}\), hence, the equality (85) results in

\[S' = (V_{\delta, p} \cup U \oplus S(\Sigma)) \cap S, \forall p \in \bar{N}.\]

Then, taking into account (86) and (88), the proof is completed.

Two, useful corollaries result.

**Corollary 3.** Suppose \(s \in S_\alpha\), such that for a certain \(\Sigma \in \mathcal{Z}_\delta\), either \(s \in V_{\delta, \alpha} \cup U \oplus S(\Sigma)\), or \(s \notin U \oplus S(\Sigma) \oplus (S_\delta \cap \bar{W}_\delta, \alpha)\).
Then, there exists a vector subspace $S_1$ in $\mathcal{S}_0$, such that

1) $\lambda = (\Sigma, S_1) \in \Lambda$

2) $s \in S_{p, \lambda} = V_{\delta, p} \bigoplus U \bigoplus S(\Sigma) \bigoplus S_1 = A_{p, \lambda}$, $\forall p \in \tilde{N}$.

**Proof.** It results, applying Theorem 12 to the vector subspace $S = \mathcal{C}_s$ in $\mathcal{S}_0$.

And now, an extension of Theorem 6, §1.6.3.

**Corollary 4.** In the case of $n = 1$, suppose $s \in \mathcal{S}_0$, such that for a certain $\Sigma \in Z_\delta$, the inclusion holds:

$$(U \bigoplus S(\Sigma) \bigoplus (S_{\delta} \cap W_{\delta, 0})) \cap S(s) = (V_{\delta, \infty} \bigoplus U \bigoplus S(\Sigma)) \cap S(s).$$

Denote $S = \langle s, \cdot \rangle \in D'(R^1)$.

Then, there exists a vector subspace $S_1$ in $\mathcal{S}_0$, such that:

1) $\lambda = (\Sigma, S_1) \in \Lambda$,

2) the mapping $\mathcal{D}^{p+q, \lambda}_p : A_{p+q, \lambda} \rightarrow A_{q, \lambda}$, with $p \in \mathbb{N}$, $q \in \tilde{N}$, is identical on $\{S, DS, D^2S, \ldots \} \subset D'(R^1)$, with the usual distribution derivative $\mathcal{D}^p$.

**Proof.** It results, applying Theorem 12 to the vector subspace $S = S(s)$ in $\mathcal{S}_0$. 

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1.10 Appendix

We present a theorem, partially generalizing a well known property of the Vandermonde determinants, used in the proof of Proposition 1, §1.2.2. The theorem was conjectured in [15], under a weaker form, still sufficient for the proof of the mentioned Proposition 1, §1.2.2.

R. C. King offered the proof of the following stronger property (see the notations in §1.2.1):

**Theorem VAN** Suppose \( n \in \mathbb{N}\backslash\{0\} \) given. Then, for each \( a \in \mathbb{N}^n \), \( a \geq e = (1, \ldots, 1) \in \mathbb{N}^n \) and \( t \in \mathbb{N}\backslash\{0\} \), the relation holds

\[
\begin{vmatrix}
(a+p(1))p(1) & \cdots & (a+p(1))p(t) \\
\vdots & & \vdots \\
(a+p(l))p(1) & \cdots & (a+p(l))p(t)
\end{vmatrix} = \prod_{1 \leq i \leq t} \prod_{1 \leq j \leq n} (p_j(i))! > 0,
\]

where \( p(i) = (p_1(i), \ldots, p_n(i)) \), \( \forall 1 \leq i \leq l \).

**Remark.** The value of the determinant depends only on \( n, l, p(1), \ldots, p(l) \) and does not depend on \( a \).

**Proof.** Let us consider the determinant

\[
\Delta_l = |(a + p(\sigma))p(\tau)|, \text{ with } 1 \leq \sigma, \tau \leq \ell.
\]

For \( 1 \leq \tau \leq \ell \), the \( \tau \)-th column in \( \Delta_l \) is

\[
C_l(\tau) = \begin{pmatrix}
p_1(\tau) \\
\vdots \\
p_n(\tau)
\end{pmatrix}
\]

if \( a = (a_1, \ldots, a_n) \).
Denote for \(1 \leq \tau \leq \ell\), the column
\[
C_2(\tau) = \left( \begin{array}{c}
p(1)p(\tau) \\
\vdots \\
p(\ell)p(\tau)
\end{array} \right)
\]
considering wherever, \(0^0 = 1\).

For \(1 \leq \tau \leq \ell\), we obtain
\[
C_2(\tau) = C_1(\tau) + \sum_{\rho} (p(\tau))(-a)p(\tau)-p(\rho)c_1(p(\rho)).
\]
where the sum \(\sum\) is taken for all \(1 \leq \rho \leq \ell\), such that \(|p(\rho)| < |p(\tau)|\).

Denoting now
\[
\Delta_2 = |p(\sigma)p(\tau)|, \text{ with } 1 \leq \sigma, \tau \leq \ell,
\]
the previous relations imply \(\Delta_2 = \Delta_1\), since \(C_2(\tau)\) is the \(\tau\)-th column in \(\Delta_2\).

In order to simplify \(\Delta_2\), we need the following function \(F: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\), defined by
\[
F(h,k) = \begin{cases} 
1 & \text{if } k = 0 \\
h(h-1)\ldots(h-k+1) & \text{if } k \geq 1
\end{cases}
\]

One can notice, that \(F(h,k) = 0 \iff h-k+1 \leq 0 \iff h \leq k\); moreover,
\[
F(h,h) = h!
\]

For \(1 \leq \tau \leq \ell\), we define:
\[
C_3(\tau) = \begin{pmatrix}
F(p_1(1), p_1(\tau)) \times \cdots \times F(p_n(1), p_n(\tau)) \\
\vdots \\
F(p_1(\ell), p_1(\tau)) \times \cdots \times F(p_n(\ell), p_n(\tau))
\end{pmatrix}
\]

then,

\[
C_3(\tau) = C_2(\tau) + \sum\left(\sum_{j \in J_1 \cup \cdots \cup J_n} (-j) \right) \begin{pmatrix} p(1)q(\tau) \\
\vdots \\
p(\ell)q(\tau) \end{pmatrix}
\]

where *) the sum \( \Sigma \) is taken for all

\[
J_1 \subseteq \{1, \ldots, p_1(\tau) - 1\}, \ldots, J_n \subseteq \{1, \ldots, p_n(\tau) - 1\}
\]
such that \( J_1 \cup \cdots \cup J_n \neq \emptyset \),

**) \( q(\tau) = p(\tau) - (|J_1|, \ldots, |J_n|) \), where \( |J_i| \) is the number of elements in \( J_i \).

Therefore, with the same meaning of the sum \( \Sigma \), we can write

\[
C_3(\tau) = C_2(\tau) + \sum\left(\sum_{j \in J_1 \cup \cdots \cup J_n} (-j) \right) C_2(p(\tau, J_1, \ldots, J_n))
\]

where \( p(\tau, J_1, \ldots, J_n) \in \mathbb{N} \) is defined uniquely by

\[
p(p(\tau, J_1, \ldots, J_n)) = p(\tau) - (|J_1|, \ldots, |J_n|)
\]

and thus, \( 1 \leq p(\tau, J_1, \ldots, J_n) \leq \ell \) and \( |p(p(\tau, J_1, \ldots, J_n))| < |p(\tau)| \).

Denoting by \( \Delta_3 \) the determinant with the columns \( C_3(1), \ldots, C_3(\ell) \), it results \( \Delta_3 = \Delta_2 \) and therefore, \( \Delta_3 = \Delta_1 \).
The proof of Theorem VAN is completed, noticing that for 
1 ≤ σ, τ ≤ l, the implications hold:

\[
\begin{align*}
\int \mathbf{F}(p_i(σ), p_i(τ)) = 0 \iff & (\exists 1 ≤ i ≤ n : p_i(σ) < p_i(τ)) \iff \\
& \iff \text{non } (p(τ) ≤ p(σ)).
\end{align*}
\]

**NOTICE**

The Chapters 2-6 will be presented in a second paper.
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