A CLASS OF NONUNIQUE SOLUTIONS OF THE
SCHRÖDINGER EQUATION WITH THE POTENTIAL A
POSITIVE POWER OF THE DIRAC DELTA DISTRIBUTION

by

Elemer E. Rosinger

Technical Report No. 69
January 1976
A CLASS OF NONUNIQUE SOLUTIONS OF THE
SCHRODINGER EQUATION WITH THE POTENTIAL A
POSITIVE POWER OF THE DIRAC DELTA DISTRIBUTION

Elemer E. Rosinger

Abstract

In [12], function solutions $u(x) = u_-(x) + (u_+(x) - u_-(x)) \cdot H(x)$,
$x \in \mathbb{R}^1$, with $u_-, u_+ \in C^\infty(\mathbb{R}^1)$, $H$ the Heaviside function, were constructed
for the Schrödinger equation $(D^2 + k^2 + \alpha(\delta(x))^m)u(x) = 0$, $x \in \mathbb{R}^1$, in the
case of $m \in (0,1] \cup [2,\infty)$. The solutions were not unique for $m \in (2,\infty)$,
$\alpha \in (0,\infty)$. The present paper gives a wider class of nonunique solutions
of the above form, for the same values $m \in (2,\infty)$, $\alpha \in (0,\infty)$. For $m$
integer, the equation and the solutions are considered within the associative
and commutative algebras containing $D'(\mathbb{R}^1)$, introduced in [7].
For $m$ arbitrary, the construction is understood in the usual 'weak'
sense.

AMS (MOS) subject classification (1970). Primary 46F10 Keywords and Phrases.
Schrödinger equation, nonunique solutions.
1. INTRODUCTION

In [12], the following Schrödinger equation, with the potential any positive power of the Dirac $\delta$ distribution was considered.

(1) $$(D^2 + k^2 + \alpha(\delta(x))^m)u(x) = 0, \ x \in \mathbb{R}^1,$$

with the initial conditions,

(2) $$u(x_0) = y_0, \ \frac{Du(x_0)}{dx} = y_1,$$

where $k \in \mathbb{R}^1, \ m \in (0, \infty), \ \alpha \in \mathbb{R}^1, \ x_0 \in (-\infty, 0), \ y_0, y_1 \in \mathbb{R}^1.$

It was shown that function solutions of the form

(3) $$u(x) = u_-(x) + (u_+(x) - u_-(x)) \cdot H(x), \ x \in \mathbb{R}^1,$$

with $u_-, u_+ \in C^\infty(\mathbb{R}^1)$ and $H$ the Heaviside function, can be constructed.

More precisely, let $u_- \in C^\infty(\mathbb{R}^1)$ be the unique solution of

(4) $$(D^2 + k^2)u(x) = 0, \ x \in \mathbb{R}^1,$$

with the initial conditions (2), and suppose for $x \in \mathbb{R}^1$

\[ u_-(x) = \begin{cases} c_1 \cos kx + c_2 \sin kx & \text{if } k \neq 0 \\ c_1 + c_2x & \text{if } k = 0. \end{cases} \]

If $m \in (0, 1)$ and $\alpha \in \mathbb{R}^1$, then $u_+ = u_-.$

If $m = 1$ and $\alpha \in \mathbb{R}^1$, then for $x \in \mathbb{R}^1$

\[ u_+(x) = \begin{cases} c_1 \cos kx + (c_2 - \alpha c_1/k) \sin kx & \text{if } k \neq 0 \\ c_1 + (c_2 - \alpha c_1)x & \text{if } k = 0. \end{cases} \]

If $m = 2$, then $u_+$, therefore $u$ in (3), exists only for the
following 'discrete spectral' values $\alpha = (n\pi)^2$, with $n \in \mathbb{N}$ and in this case

$$u_+ = \begin{cases} 
  u_- & \text{if } \alpha = (2n\pi)^2, \text{ with } n \in \mathbb{N} \\
  -u_- & \text{if } \alpha = ((2n+1)\pi)^2, \text{ with } n \in \mathbb{N}.
\end{cases}$$

For $m \in (2,\infty)$ and $\alpha \in (0,\infty)$ it was shown that $u_-$ and therefore $u$ in (3) is not unique and either $u_+ = u_-$ or $u_+ = -u_-$. In the present paper it is shown that the nonuniqueness of the solution (3) in the case of $m \in (2,\infty)$ and $\alpha \in (0,\infty)$ is more wide, namely

$$\sigma c_1 \cos k x + (\sigma c_2 + Kc_1) \sin k x \quad \text{if } k \neq 0$$

$$\sigma c_1 + (\sigma c_2 + Kc_1) x \quad \text{if } k = 0$$

for any $\sigma \in \{-1,1\}$ and $K \in \mathbb{R}$ given. In particular, for $K = 0$ we obtain $u_+ = \pm u_-$. We recall that for $m$ positive integer, the equation (1), (2), containing the power $(\delta(x))^m$ of the Dirac $\delta$ distribution and the solution (3) can be considered within the associative and commutative algebras with unit element and containing the distributions in $D'(\mathbb{R})$, constructed in [7], [8], [10], [11], and [13]. In the case of $m \in (0,\infty)$ arbitrary, the equation (1), (2), the power $(\delta(x))^m$ and the solution (3) are considered in the usual 'weak' sense.

---

1) $N = \{0,1,2,\ldots\}$
3. THE CASE OF NONUNIQUE SOLUTIONS FOR \((m, \alpha) \in (2, \infty) \times (0, \infty)\)

Consider for \(h \in \mathbb{R}^1\), the differential equation

\[
(D^2 + h)y(x) = 0, \quad x \in \mathbb{R}^1,
\]

and for \(x \in \mathbb{R}^1\), the 2x2 matrix \(W(h, x) = \exp(xA_h)\) where

\[
A_h = \begin{pmatrix} 0 & 1 \\ -h & 0 \end{pmatrix}
\]

If \(v \in C^\infty(\mathbb{R}^1)\) is the unique solution of (6), with the initial conditions

\[v(a) = b, \quad Dv(a) = c, \quad \text{where} \quad a, b, c \in \mathbb{R}^1,\]

then

\[
\begin{pmatrix} D^p v(x) \\ D^{p+1} v(x) \end{pmatrix} = (-1)^p h^p W(h, x) W(h, a)^{-1} \begin{pmatrix} b \\ c \end{pmatrix}, \quad \forall p \in \mathbb{N}, \quad x \in \mathbb{R}^1.
\]

Now, the 'weak' approach of equation (1).

Suppose \(\omega > 0, \ K \in \mathbb{R}^1\) and define

\[
V(\omega, K, x) = \begin{cases} 0 & \text{if} \ x \in \mathbb{R}^1 \setminus (0, \omega) \\ 1 & \text{if} \ x \in (0, \omega). \end{cases}
\]

Assuming for \(\delta(x)\), the 'weak' representation \(\lim_{\omega \to 0} V(\omega, 1/\omega, \cdot)\),

we replace the equation (1) by

\[
(D^2 + k^2 + V(\omega, \omega^m, x))u(x) = 0, \quad x \in \mathbb{R}^1.
\]
It is important to mention that in the case of a positive integer, by considering (1), (2) and (3) within the mentioned algebras containing \( D'(\mathbb{R}^1) \), the solutions (3) will be independent of the particular 'weak' representation of \( \delta \). However, this representation has to be nonsymmetric since in the mentioned algebras, the Dirac distribution and its derivatives are nonsymmetric [10] due to the fact that relations as \( D^q\delta(-x) \neq c D^q\delta(x) \), \( \forall q \in \mathbb{N}, c \in \mathbb{C}^1 \), hold (for generalization, see [13]).

Denote not by \( u(\omega, \cdot) \in C^1(\mathbb{R}^1) \cap C^\infty(\mathbb{R}^1 \setminus \{0, \omega\}) \) the unique solution of (8), (2), on \( \mathbb{R}^1 \). Then, (7) results in

\[
\begin{bmatrix}
u(\omega, \omega) \\
Du(\omega, \omega)
\end{bmatrix} = \begin{bmatrix}
u(k^2 + \alpha/\omega^m, \omega) W(k^2, x_0) \\
W(k^2, x_0) - 1
\end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.
\]

Obviously, \( u(\omega, \cdot) = u_+ \) on \((0, \infty)\), for any \( \omega > 0 \). The problem is to establish the cases of \((m, \alpha)\) when

\[
\lim_{\omega \to 0} u(\omega, \cdot) = u_+ \quad \text{on } (0, \infty)
\]

and, first of all, the very meaning of that 'weak' limit.

Let us start with the meaning of (10).

Due to the fact that, for a positive integer (see [9], §2, Lemma 2) \( \text{supp}(\delta)^m \subset \{0\} \), one can expect that \( u_+ \) is a solution of (4) on \((0, \infty)\). Then, in order to obtain \( u_+ \) it is sufficient to determine

\[
z_0 = u_+(0), \quad z_1 = Du_+(0).
\]
We notice that \( u(\omega, \cdot) \) is also a solution of (4) on \( (\omega, \infty) \). Suppose, there exist \( (\omega_v \mid v \in \mathbb{N}) \subset (0, \infty) \), with \( \lim_{v \to \infty} \omega_v = 0 \) and \( [a, b] \subset (0, \infty) \), such that \( (u(\omega_v, \cdot) \mid v \in \mathbb{N}) \) is bounded on \( [a, b] \) and converges point wise to \( u_+ \) on an infinite and closed subset of \( [a, b] \). Then, it results that
\[
\lim_{v \to \infty} D^pu(\omega_v, \cdot) = D^p u_+ \quad \text{uniformly on each } [a', b'] \subset (0, \infty) \quad \text{and for each } p \in \mathbb{N}
\]
(actually, except the case \( k=0 \) and \( p=0 \), the above convergence will be uniform on each \( [a'', \infty) \), with \( a'' > 0 \)).

The reason we can replace the requirement of the 'continuous' limit \( \lim_{\omega \to 0} u(\omega, \cdot) \) by the weaker, 'sequential' one \( \lim_{v \to \infty} u(\omega_v, \cdot) \), is that the elements of the mentioned algebras containing the distributions in \( D'(\mathbb{R}) \) are classes of sequences of functions from \( \mathbb{R} \) to \( C^1 \). In that way, the sequence \((u(\omega_v, \cdot) \mid v \in \mathbb{N})\), if it converges in a appropriate sense, can still define an element of those algebras (see [12], §7).

Therefore, we can assume the following meaning for (10)

\[
\begin{align*}
(11) & \quad \exists (\omega_v \mid v \in \mathbb{N}) \subset (0, \infty) : \\
(11.1) & \quad \lim_{v \to \infty} \omega_v = 0 ; \\
(11.2) & \quad \lim_{v \to \infty} D^pu(\omega_v, \cdot) = D^p u_+ \quad \text{on each } [a, b] \subset (0, \infty) \quad \text{and for each } p \in \mathbb{N} ; \\
(11.3) & \quad \lim_{v \to \infty} \begin{pmatrix} u(\omega_v, \omega_v) \\ Du(\omega_v, \omega_v) \\ D^2u(\omega_v, \omega_v) \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \quad \text{exists and it is finite.}
\end{align*}
\]
According to (9), we have (11.3) only if

\[
\text{(12)} \quad \lim_{v \to \infty} Z(\omega_v) \text{ exists and it is finite, where we denote}\n\]

\[
\text{(13)} \quad Z(\omega) = W(k^2 + \alpha/\omega^m, \omega), \text{ for } \omega > 0.\n\]

Now, we shall show that for each \((m, \alpha) \in (2, \infty) \times (0, \infty)\) the condition (11) can be fulfilled.

**Theorem 1** Suppose \((m, \alpha) \in (2, \infty) \times (0, \infty)\) given. Then, for each \(\sigma \in \{-1, 1\}\) and \(K \in \mathbb{R}^1\), there exists \((\omega_v | v \in \mathbb{N}) \subset (0, \infty)\), with \(\lim_{v \to \infty} \omega_v = 0\), such that

\[
\text{(14)} \quad \lim_{v \to \infty} Z(\omega_v) = \begin{pmatrix} \sigma & 0 \\ K & \sigma \end{pmatrix}.
\]

**Proof** Obviously, \(k^2 + \alpha/(\omega_v)^m > 0\), \(\forall v \in \mathbb{N}\), therefore

\[
\text{(15)} \quad Z(\omega_v) = \begin{pmatrix} \cos L_v & \frac{1}{H_v} \sin L_v \\ -H_v \sin L_v & \cos L_v \end{pmatrix}
\]

where

\[
\text{(16)} \quad H_v = (k^2 + \alpha/(\omega_v)^m)^{1/2}, \quad L_v = \omega_v H_v.
\]

Since \(m \in (2, \infty)\), it results \(\lim_{v \to \infty} H_v = \lim_{v \to \infty} L_v = \infty\).

Define \(L : (0, \infty) \to (0, \infty)\) with \(L(\omega) = \omega(k^2 + \alpha/\omega^m)^{1/2}\).

There exists \(A > 0\) such that for each \(\alpha > A\), the equation \(L(\omega) = a\)
has exactly two solutions $0 < \tilde{\omega}_1(a) < \tilde{\omega}_2(a)$. Moreover,

\begin{equation}
(17) \quad \lim_{a \to \infty} \tilde{\omega}_1(a) = 0.
\end{equation}

Suppose $(n_v | v \in \mathbb{N}) \subseteq \mathbb{N}$ and $(e_{uv} | v \in \mathbb{N}) \subseteq \mathbb{R}$ such that \( \lim_{v \to \infty} n_v = \infty \), \( \lim_{v \to \infty} e_v = 0 \) and \((-1)^{n_v} = \sigma, \ n_v \pi + e_v > A, \ \forall v \in \mathbb{N} \).

Define \((\omega_v | v \in \mathbb{N}) \subseteq (0, \infty)\), with \( \omega_v = \tilde{\omega}_1(n_v \pi + e_v) \), then, due to (17), \( \lim_{v \to \infty} \omega_v = 0 \). Further, \( L_v = L(\omega_v) = n_v \pi + e_v \), hence \( \cos L_v = \sigma \cos e_v, \sin L_v = \sigma \sin e_v \), and \(-H_v \sin L_v = -\sigma e_v H_v \frac{\sin e_v}{e_v} \). Thus

\begin{equation}
(18) \quad \lim_{v \to \infty} (-H_v \sin L_v) = -\sigma \lim_{v \to \infty} e_v \left( k^2 + \alpha / (\tilde{\omega}_1(n_v \pi + e_v))^m \right)^{\frac{1}{2}}.
\end{equation}

One can notice that \( \tilde{\omega}_1 \in C^1(A, \infty) \) and \( \lim_{a \to \infty} \tilde{\omega}_1(a) = 0 \).

Therefore

\[
\lim_{v \to \infty} \frac{(e_v)^2}{(\tilde{\omega}_1(n_v \pi + e_v))^m} = \lim_{v \to \infty} \left( \frac{\tilde{\omega}_1(n_v \pi)}{|e_v|^{2/m}} + |e_v|^{1 - \frac{2}{m}} \tilde{\omega}_1(n_v \pi + \xi_v e_v) \right)^{-m}
\]

where \( \xi_v \in (0, 1) \), hence

\[
\lim_{v \to \infty} |e_v|^{1 - \frac{2}{m}} \tilde{\omega}_1(n_v \pi + \xi_v e_v) = 0.
\]

Now, (18) results in
\[
\lim_{v \to \infty} (-H_v \sin L_v) = -\lim_{v \to \infty} (\text{sign } e_v) \left( \frac{\alpha(e_v)^2}{(\bar{\omega}_v(n_v))^m} \right)^{\frac{1}{m}}.
\]

Obviously, by a proper choice of \( n_v \) and \( e_v \), the limit
\[
\lim_{v \to \infty} \frac{(e_v)^2}{(\bar{\omega}_v(n_v))^m}
\]
can assume any value in \([0, +\infty]\).

That ends the proof of Theorem 1.

**Theorem 2** Suppose \((\bar{\omega}_v|_{v \in \mathbb{N}})\) is a sequence as in Theorem 1 and define \( z_0, z_1 \in \mathbb{R}^1 \) by

\[
\begin{pmatrix}
  z_0 \\
  z_1
\end{pmatrix} =
\begin{pmatrix}
  \sigma & 0 \\
  K & \sigma
\end{pmatrix}
\begin{pmatrix}
  u_-(0) \\
  Du_-(0)
\end{pmatrix}
\]

Suppose \( u_+ \in C^\infty(\mathbb{R}^1) \) is the unique solution of (4), (11). Then,

\[
\lim_{v \to \infty} D^p u(\omega_v, \cdot) = D^p u_+, \forall p \in \mathbb{N},
\]
uniformly on each \([a, \infty)\), with \( a > 0 \), except for the case \( k = p = 0 \)
when the convergence is uniform only on each \([a, b]\) with \( 0 < a < b < \infty \).

**Proof.** It results from (7) and the proof of Theorem 1.

**Corollary.** Suppose \((m_0, \alpha) \in (2, \infty) \times (0, \infty)\) and \( \sigma \in \{-1, 1\}, K \in \mathbb{R}^1 \) given, then
$u_+$ can assume the form given in the expression in (5).

**Remark.** For $m = 3, 4, \ldots$, the solutions $u$ given by (3) and (5) can be still considered as solutions within the associative and commutative algebras containing the distributions in $D'(\mathbb{R}^1)$, since these solutions have the form (3) (see [12], §7, Theorem 6).
REFERENCES


