MAXIMAL SPANNING TREES FOR
CERTAIN PLANNAR GRAPHS

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Technical Report #68
December 1975
ABSTRACT

It is shown that a planar connected map with no 'islands' or 'island groups', with \( n \)-finite countries and such that every finite country has at least 3 bordering edges, can be spanned by a tree with number of leaves less or equal than \( \left\lfloor \frac{n}{2} \right\rfloor \). This bound is shown to be sharp.
INTRODUCTION

The problem of spanning a graph by a tree with minimal number of leaves has been studied by some graph theorists, see e.g. [5]. This problem is related to the problem of decomposing a partially ordered set into a minimal number of disjoint chains [3], [4], and to the Hamiltonian properties of graphs [5]. Using a recent lemma [2], we solve that problem for planar maps of a certain type (to be defined in the text). Our solution provides an additional approach to the above problem.

DEFINITIONS AND NOTATIONS

The objects we are dealing with in this note are maps. A map is a "planar topological graph" as defined in Berge [1]. The faces of the map will be called countries or cells. A country may be adjacent to another country through more than one boundary ("edge"). Two countries having at least one common edge are neighbors. (Two countries with no common edge are not neighbors even though they may have a common vertex.) Every vertex in a map has a degree of 3 or more. Every map has exactly one unbounded country called the infinite country or just "infinity".

Let \( M \) be a map. The map is connected if for every 2 countries in the map \( A \) and \( B \) there exists a sequence of finite countries in the map \( A = A_1, A_2, \ldots, A_k = B \) such that \( A_i \) and \( A_{i+1} \), for \( i = 1, \ldots, k-1 \),
are neighbors. An island in M is a country A in M which has exactly one neighbor. An island group C in M is a connected submap in M such that no country in C borders on infinity and there is exactly one country A in M-C such that if a member of C has a neighbor in M-C that neighbor is A.

Definition 1. A map M is legal if it is connected and contains no island groups.

Definition 2. If M is a legal map and A and B are two countries in M with a common edge then the pair (A,B) is legal if merging the pair into one country does not create an island group.

Definition 3. A blister in a map is a country with exactly two neighbors. The blister is external if one of its neighbors is the infinite country and is internal otherwise.

Definition 4. A map is k-blister if it has no internal blisters and has exactly k external blisters. It is acceptable if it is legal, has at least 2 finite countries, and is k-blister with k ≤ 2.

Definition 5. A spanning tree for a map is a tree such that: every country in the map except the infinite one corresponds to exactly one vertex in the tree; every vertex in the tree corresponds to exactly one finite country in the map; if two vertices in the tree are adjacent then the corresponding countries are neighbors. A leaf in a tree is a vertex of degree 1 in it.
**Notation:** As customary we shall use the notation \( \lfloor x \rfloor \) for the biggest integer less or equal than \( x \).

The following lemma was proved elsewhere [2] and will be used as a basis for the main result of this note.

**Lemma** Let \( M \) be a legal 0-blister map with at least 4 finite countries. There is a legal pair of countries whose removal from \( M \) splits \( M \) into acceptable components.

**MAIN RESULT**

The main result of this paper is the following

**Theorem 1.** Any legal 0-blister map with \( n \)-countries, \( n \geq 4 \), has a spanning tree with \( k \) leaves where \( k \leq \lfloor \frac{n}{2} \rfloor \).

**Proof.** We shall prove a slightly stronger result, namely that any legal \( i \)-blister map, \( i \leq 2 \), with \( n \)-countries, \( n \geq 4 \), has a spanning tree with \( k \) leaves where

\[
k \leq \lfloor \frac{n-i}{2} \rfloor + i.
\]

**Remarks:**

1. Notice that for \( i = 1,2 \) \( \lfloor \frac{n-i}{2} \rfloor + i \leq \lfloor \frac{n}{2} \rfloor + 1 \).

2. If the above statement is true for some \( n \geq 4 \) and \( i = 0 \) then it is also true for \( n+1 \) and \( i = 1 \) (remove the blister from the map, span the resulting map, with \( n \) countries by a tree with \( \lfloor \frac{n}{2} \rfloor \) leaves, connect the resulting tree to the removed blister to form a spanning
tree for the original map with at most \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) leaves), while the case \( n + 1(\geq 5), i = 2 \) for some \( n \) follows from the case \( n(\geq 4), i = 1 \) (using a similar argument).

3. It is clear that \( n \geq 4 \) is a necessary condition. Any tree has at least \( 2 \) leaves and \( 2 > \left\lfloor \frac{3}{2} \right\rfloor = 1 \).

Continuation of the Proof: By induction on \( n \). The theorem is trivially true for \( n = 4, i = 0,1,2 \) (the countries can be connected 2 by 2 and the two pairs are then connected by an edge, resulting in a spanning tree with 2 leaves. Notice that if a 4 countries map is 2-blister then the 2 blisters must be neighboring different countries or else we will have an additional 3rd blister - the remaining fourth country).

Assume now that we have already proved the theorem for some \( n_0 \geq 4 \) and \( i = 0,1,2 \) and for all \( n \) such that \( 4 \leq n \leq n_0 \) and \( i = 0,1,2 \). By Remark 2, it is enough to prove that the truth of the theorem for the case \( n_0 + 1, i = 0 \) follows, to satisfy the induction.

Assume then that we have a 0-blister map with \( n_0 + 1 \) countries. By the lemma in the previous section there is a legal pair of countries whose removal splits the map into acceptable components. Label the two countries of the pair by the labels a and b, denote the resulting components by \( C_1, C_2, \ldots, C_i, \ldots, C_k \); \( k \geq 1 \), and let \( n_i \) be the number of countries of the component \( C_i \). It follows from the fact that the components are acceptable that \( n_i \geq 2 \). Consider the following labeling procedure: Remove the common edge of a and b and consider the resulting
country as a single country, for the labeling procedure, labeled $ab$.
Start from some node on the contour of this country according to some
orientation of the plane. When passing an edge along the contour
(belonging to the contour) that borders with some country, in some
component $C_i$, then label that country be the label $c_i$ and the corres­
ponding edge by $e_i$ provided that no other country has been labeled
$c_i$ at an earlier stage of the process (it is possible that $C_i$ and $ab$
have more than one edge in common).

After the whole contour of $ab$ has been completed, every component
$C_i$ will have exactly one of its countries labeled $c_i$ (the original map
was connected and the components resulted from the removal of $ab$ from the
map), and each such country $c_i$ is connected to either $a$ or $b$ through
a common edge $e_i$, by construction.

We construct now a spanning tree for the given map as follows:
1. Connect the vertices $a$ and $b$. (Label vertices and corresponding
countries by the same label.)
2. If $C_i$ contains only 2 countries say $c_i$ and $d_i$ then connect
$c_i$ through $e_i$ to $a$ or to $b$, whatever is the case ($e_i$ is common
to either $a$ and $c_i$ or $b$ and $c_i$).
3. If $C_i$ has 3 countries then a spanning tree for $C_i$ exists with only
2 leaves such that one of its leaves corresponds to $c_i$ (trivial,
otherwise the original map would contain blisters). Construct that
spanning tree and connect $c_i$ through $e_i$ to either $a$ or $b$,
whatever is the case.
4. If $C_i$ has 4 or more countries then, by the induction hypothesis there exists a spanning tree for $C_i$ with at most $\left\lfloor \frac{n_i}{2} \right\rfloor$ leaves if $C_i$ is 0-blister, and at most $\left\lfloor \frac{n_i}{2} \right\rfloor + 1$ leaves if $C_i$ is 1-blister or 2-blister. If $C_i$ is 1-blister or 2-blister then one of its blisters must be $c_i$ ($C_i$ is connected to ab in the original map which was 0-blister) furthermore $c_i$, being a blister in $C_i$, must correspond to a leaf of the spanning tree of $C_i$.

Construct the spanning tree for $C_i$ and then connect $c_i$ through $e_i$ to either $a$ or $b$.

It is clear that the result is a spanning tree for the original map. Let us count now the number of leaves of this tree:

1) The pair of vertices $a, b$ contributes at most one leaf (either $a$ or $b$ is connected to some vertex $c_i$ in the tree).

2) If a component $C_i$ has only 2 or 3 countries then it contributes exactly one leaf to the tree (one of the 2-leaves of the component's tree $c_i$ ceases to be a leaf by connecting it to either $a$ or $b$, see parts 2 and 3 in the construction above).

3) If a component $C_i$ with 4 or more countries is 0-blister then it contributes $\left\lfloor \frac{n_i}{2} \right\rfloor$ leaves at most to the tree (induction hypothesis). If it is 1-blister or 2-blister then the spanning tree of the component has at most $\left\lfloor \frac{n_i}{2} \right\rfloor + 1$ leaves, but one of the leaves, that corresponding to $c_i$ (see part 4 of the construction above) ceases to be a leaf by connecting it to either $a$ or $b$. Therefore its contribution to the tree spanning the original map is not bigger than $\left\lfloor \frac{n_i}{2} \right\rfloor$, for this case too.
Counting all the contributions together we get

\[ 1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i}{2} \right\rfloor \leq 1 + \left\lfloor \frac{\sum_{i=1}^{k} n_i}{2} \right\rfloor = \left\lfloor \frac{n_0}{2} \right\rfloor \]

(notice that for \( n_i = 3 \), \( C_i \) contributes 1 leaf which is:

\( \left\lfloor \frac{n_i}{2} \right\rfloor = \left\lfloor \frac{3}{2} \right\rfloor \).

Q.E.D.

**Theorem 2**  The bound given by Theorem 1 is sharp.

**Proof.**  Consider a map as in Fig. 1.

Such a map has \( 2k+1 = n \) countries. Either \( a_i \) or \( b_i \) must correspond to a leaf in any spanning tree for it. It follows that the number of leaves of any spanning tree for it must have at least \( k \) leaves.

But \( k = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \)

Q.E.D.
BIBLIOGRAPHY


