INCREASING THE FAMILY OF ASSOCIATIVE AND COMMUTATIVE ALGEBRAS CONTAINING THE DISTRIBUTIONS IN $D'(R^1)$

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ABSTRACT

In [13], [15] and [18] associative and commutative algebras with unit element and containing the distributions in $\mathcal{D}'(R^1)$ were constructed. In the present paper a way is given the amount of those algebras can be increased. A purely algebraic condition is given in order to extend the construction of the mentioned algebras for the case of the distributions in $\mathcal{D}'(R^n)$, with $n \geq 2$.

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1. INTRODUCTION

The associative and commutative multiplication theory for the distributions in $D'(R^1)$, presented in [13], [14], [15], [16], [17], [18], [19] and [20], is based on the construction of diagrams of the following form, first presented in [11] and [12]:

\[ I \rightarrow A \rightarrow W \]

(A)

\[ V \rightarrow S \rightarrow S_0 \]

where

1) $W$ is the set of all the sequences $s: N \rightarrow C^\infty(R^1)$ of complex valued functions: $s(v) \in C^\infty(R^1)$ and $s(v)(x) \in C^1$, $\forall v \in N, x \in R^1$;

2) $A$ is a subalgebra in $W$, $I$ is an ideal in $A$;

3) $S_0$ is the set of all $s \in W$, $s$ weakly convergent in $S'(R^1)$; $V_0$ is the set of all $v \in S_0$, $v$ weakly convergent in $S'(R^1)$ to $0$;

4) $S$ is a vector subspace of $S_0$, $V$ is a vector subspace of $V_0$;

5) the following relations hold:

\[ I \cap S = V_0 \cap S = V \]

\[ V_0 + S = S_0 \]

---

1) each --- means the inclusion $\subset$.
Given any diagram (A), one can obviously construct the following linear embedding of \( D'(R^1) \) into an associative and commutative algebra with unit element

\[
\begin{array}{cccc}
D'(R^1) & S_o/V_o & S/V & A/I \\
\cup & \text{lsom} & \text{lsom} & \text{lin} \to \text{inj} \\
<s, *> & s + V_o & s + V & s + I \\
\text{vect} & \text{vect} & \\
\end{array}
\]

where \( <s, *> \in D'(R^1) \) is the distribution generated by \( s \in S_o \).

The main task in constructing a diagram of the type (A) is the choice of the quotient space \( S/V \), that is, of the pair \((V, S)\).

The vector space \( S \) is constructed ([13], §7) using a special subset of sequences in \( S_o \), called "\( \delta \) sequences", and given by ([13], §5):

\[
Z_o^\delta = \left\{ s \in S_o \right\}
\]

1) \( <s, *> = \delta_o \)

2) \( \forall \varepsilon > 0: \exists \nu \in \mathbb{N}: \forall \varepsilon \in \mathbb{N}, \nu \geq \varepsilon, x \in R^1, |x| \geq \varepsilon : \\
\quad s(\nu)(x) = 0 \)

3) \( \forall \nu \in \mathbb{N}: \exists \nu \in \mathbb{N}, \nu \geq \nu_p : \\
\quad s(\nu)(0) = D_s(\nu)(0), \ldots D_s(\nu)(0) \\
\quad \vdots \\
\quad s(\nu)(0) = D_s(\nu)(0), \ldots D_s(\nu)(0) \neq 0 \)

The purpose of the present paper is to show that the above set \( Z_o^\delta \) can be enlarged in a way offering the possibility of increasing the amount of diagrams (A), by increasing the choices of the vector spaces \( S \).
An other consequence of enlarging the set $\mathbb{Z}^0$ is the possibility of stating certain pure algebraic conditions (see the conjecture, partially generalizing a well known property of the Vandermonde determinants in §5), which could permit the generalization of the distribution multiplication to the case of the distributions in $\mathcal{D}'(\mathbb{R}^n)$, with $n \geq 2$.

2. PROPERTIES OF INFINITE MATRICES

Denote by $M$ the set of all infinite vectors of complex numbers $X = (x_\mu | \mu \in \mathbb{N})$, $x_\mu \in \mathbb{C}$, $\forall \mu \in \mathbb{N}$, with at most a finite number of non-zero components $x_\mu$.

An infinite matrix of complex numbers $A = (a_{\nu\mu} | \nu, \mu \in \mathbb{N})$, $a_{\nu\mu} \in \mathbb{C}$, $\forall \nu, \mu \in \mathbb{N}$, is called line wise nonsingular only if:

$$\forall X \in M: A \cdot X \in M \implies X = 0$$

Proposition 1. The infinite matrix of complex number $A = (a_{\nu\mu} | \nu, \mu \in M)$ is line wise nonsingular only if:

\begin{align*}
&\forall \nu, \mu \in \mathbb{N}: \downarrow \mu \in \mathbb{N}, \nu_0, \ldots, \nu_\mu \in \mathbb{N}: \\
&1) \mu \geq \mu_\star \\
&2) \nu_\star \leq \nu_0 < \ldots < \nu_\mu \\
&3) \begin{vmatrix}
  a_{\nu_0 \mu} & \cdots & a_{\nu_0 \mu} \\
  \vdots & \ddots & \vdots \\
  a_{\nu_\mu} & \cdots & a_{\nu_\mu}
\end{vmatrix} \neq 0
\end{align*}
Proof. First, the "if" part. Suppose $X = (x_{\mu} \mid \mu \in \mathbb{N}) \in M$, then

$\exists \mu_0 \in \mathbb{N}: \forall \mu \in \mathbb{N}, \mu \geq \mu_0: x_{\mu} = 0$. Suppose now $A \cdot X \in M$, then

$\exists \nu_0 \in \mathbb{N}: \forall \nu \in \mathbb{N}, \nu \geq \nu_0:
\sum_{0 \leq i < \nu_0} a_{\nu i} x_i = 0$.

Assume $\mu, \nu_0, \ldots, \nu_\mu \in \mathbb{N}$ with the properties 1), 2) and 3) from Proposition 1, then, obviously, the first two properties imply

$a_{\nu_0} x_0 + \ldots + a_{\nu_\mu} x_\mu = 0$

$a_{\nu_0} x_0 + \ldots + a_{\nu_\mu} x_\mu = 0$

which, due to 3), will result in $x_0 = \ldots = x_\mu = 0$, hence $X = 0$.

Now, the "only if" part. Suppose $A$ is line wise nonsingular and the properties 1), 2) and 3) in Proposition 1, do not hold. Then

$\exists \nu_0, \nu \in \mathbb{N}: \forall \mu \in \mathbb{N}, \nu_0, \ldots, \nu_\mu \in \mathbb{N}$:

$\begin{pmatrix}
\mu \geq \mu_0 \\
\nu_0 \leq \nu < \ldots < \nu_\mu
\end{pmatrix} \quad \Rightarrow \quad \begin{vmatrix}
a_{\nu_0} & \ldots & a_{\nu_\mu} \\
\vdots & \ddots & \vdots \\
a_{\nu_0} & \ldots & a_{\nu_\mu}
\end{vmatrix} = 0$

Denote by $A_\nu$ the infinite matrix of complex numbers

$\begin{pmatrix}
a_{\nu_0} & \ldots & a_{\nu_\mu} \\
\vdots & \ddots & \vdots \\
a_{\nu_0} & \ldots & a_{\nu_\mu}
\end{pmatrix}$.
It results that any $\mu_{x^*} + 1$ lines in $A_{x^*}$ are linear dependent.

Therefore

$$3X_{x^*} = (x_0, ..., x_{\mu_{x^*}}, 0, 0, ..., 0, ...), X_{x^*} \neq 0: A_{x^*}X_{x^*} = 0.$$ 

Define, now $X = (x_0, ..., x_{\mu_{x^*}}, 0, 0, ..., 0, ...)$ then, obviously, $X \in M$ and $X \neq 0$. Denoting $Y = A^*X$ and supposing $Y = (y_U | \forall U \in N)$, it results $y_U = 0, \forall U \in N, U \geq U_{x^*}$, since $A_{x^*}X_{x^*} = 0$. Therefore, $A^*X = Y \in M$ and that contradicts the assumption of $A$ being line-wise nonsingular.

3. THE ENLARGEMENT OF $Z^0\delta$

Given $s \in W$ we denote by $W(s)(x), x \in R^1$, the following infinite matrix of functions

$$\begin{pmatrix}
  s(o)(x) & Ds(o)(x) & \ldots & D^ps(o)(x) & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  s(v)(x) & Ds(v)(x) & \ldots & D^ps(v)(x) & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}$$

We can now define the enlarged $Z^0\delta$, by

$$Z^0\delta = \left\{ s \in S_0 \mid \begin{array}{l}
1) \langle s, \cdot, \cdot \rangle = \delta_0 \\
2) \forall \varepsilon > 0: \exists \delta \in N: \forall U \in N, U \geq V_{\varepsilon}, x \in R^1, |x| \geq \varepsilon: \quad s(v)(x) = 0 \\
3) W(s)(o) \text{ is line wise nonsingular}
\end{array} \right\}$$
It is important to determine in the case of the usual way of constructing "\( \delta \) sequences" ([9], [10]), the necessary and sufficient conditions for such a sequence in order to belong to the enlarged \( Z^0_\delta \).

Suppose \( \in \mathcal{D}(\mathbb{R}^1) \) such that \( \int_{\mathbb{R}^1} f(x)dx = 1 \) and define \( s \in S_o \) with

\[
s_p(v)(x) = (v+1):((v+1)x), \forall v \in \mathbb{N}, x \in \mathbb{R}^1.
\]

Then, obviously, \( s \) satisfies the first two conditions in the above definition of \( Z^0_\delta \).

**Proposition 2.** \( s \in Z^p_\delta \) only if

\[
\forall p \in \mathbb{N}: D^p_0(0) \neq 0.
\]

**Proof:** We have only to find out the necessary and sufficient condition in order that \( s \) satisfies the condition 3) in the definition of the enlarged \( Z^0_\delta \).

Obviously,

\[
W(s)(0) = \\
\begin{pmatrix}
(0) & D(0) & \ldots & D^p(0) & \ldots \\
2(0) & 2^2D(0) & \ldots & 2^{p+1}D^p(0) & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
(v+1)(0) & (v+1)^2D(0) & \ldots & (v+1)^{p+1}D^p(0) & \ldots
\end{pmatrix}
\]
therefore, according to Proposition 1, 

\[ W(s^\circ)(o) \] 

is line wise nonsingular, only if

\[
\forall \nu_k, p_k \in N: \exists p \in N, \nu_0, \ldots, \nu_p \in N:
\]

1) \( p \geq p_k \)

2) \( \nu_k \leq \nu_0 < \cdots < \nu_p \)

3) \[
\begin{vmatrix}
(\nu_0+1)(o) & (\nu_0+1)^2D(o) & \ldots & (\nu_0+1)^{p+1}D^p(o) \\
(\nu_p+1)(o) & (\nu_p+1)^2D(o) & \ldots & (\nu_p+1)^{p+1}D^p(o)
\end{vmatrix} \neq 0
\]

Taking into account a well known property of the Vandermonde determinants, it results that 3) is equivalent with

3') \( (o) \cdots D^p(o) \cdots D^p(o) \neq 0 \).

Therefore, the above necessary and sufficient conditions for \( W(s_b^\circ)(o) \) being line wise nonsingular, can be written under the simpler form

\[
\forall p \in N: (o) \cdots D(o) \cdots D^p(o) \neq 0
\]

which completes the proof.
4. CONSEQUENCES IN THE DISTRIBUTION MULTIPLICATION THEORY

One can easily check, step by step, that all the results presented in [13], [14], [15], [16], [17], [18], [19] and [20] remain still valid if we use in the construction of the associative and commutative algebras with unit element and containing the distributions in $D'(R^1)$, the enlarged definition of $Z^0_\delta$.

5. REMARK ON POSSIBLE GENERALIZATION TO HIGHER DIMENSIONS

As was pointed out in [13], §12, the generalization to higher dimensions of the associative and commutative multiplication theory for the distributions in $D'(R^1)$ can be trivially accomplished - except Proposition 8 and its consequence, Theorem 4, [13], §7 - if a suitable generalization for $n \geq 2$ is found for the set $Z^0_\delta$.

As was mentioned at the end of §1, the enlarged definition of $Z^0_\delta$ presented in this paper, makes the generalization of $Z^0_\delta$ dependent on a conjecture, partially generalizing a well known property of the Vandermonde determinants. In the followings, we present this conjecture.

Suppose $n, m \in N$, $n \geq 1$ and denote

$$P(n,m) = \{ p = (p_1, \ldots, p_n) \in N^n \mid |p| = p_1 + \ldots + p_n \leq m \}$$

$$\ell(n,m) = \text{card} P(n,m).$$

One can see that $\ell(1,m) = m+1$, $\forall m \in N$ and $\ell(n+1,m) = \sum_{0 \leq k \leq m} \ell(n,k)$,
\( \forall n, m \in \mathbb{N}, n \geq 1. \)

One can also see that there exists a total order \( \leq \) on \( \mathbb{N}^n \), such that

1. \( \mathbb{N}^n = \{p(1), p(2), \ldots\} \)
2. \( p(1) \rightarrow p(2) \rightarrow \ldots \)
3. \( \forall m \in \mathbb{N} : P(n,m) = \{p(1), \ldots, p(l(n,m))\} \)

Let us conjecture a partial generalization of the well-known property of the Vandermonde determinants:

\[
\forall \ell \in \mathbb{N}, x_0, \ldots, x_\ell \in \mathbb{C}^1:
\begin{vmatrix}
1 & x_0^2 & \ldots & x_\ell^2 \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_0^2 & \ldots & x_\ell^2
\end{vmatrix} = 0 \iff \left( \begin{array}{c}
30 \leq l < j \leq \ell: \\
x_1 = x_j
\end{array} \right)
\]

as follows:

**Theorem VAN** Suppose \( n \in \mathbb{N}, n \geq 1 \) given. Then, for each \( a^* \in \mathbb{N}^n \) and \( m^* \in \mathbb{N} \), there exists \( a \in \mathbb{N}^n, a \geq a^* 2 \), and \( m \in \mathbb{N}, m \geq m^* \), such that

\[
\begin{vmatrix}
(a+p(1))p(1) & \ldots & (a+p(1))p(\ell(n,m)) \\
\vdots & \ddots & \vdots \\
(a+p(\ell(n,m)))p(1) & \ldots & (a+p(\ell(n,m)))p(\ell(n,m))
\end{vmatrix} \neq 0
\]

2) for \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n \), we denote \( a \geq b \), only if \( a_i \geq b_i, \forall 1 \leq i \leq n \).
3) for \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{N}^n \), \( a_1, \ldots, a_n \geq 1 \), we denote \( a = a_1 \ldots a_n \).
Now, $Z^\delta_\delta$ can be generalized to the dimensions $n \geq 2$, by the definition

$$Z^\delta_\delta = \begin{cases} \{ s \in S_\delta \} \\
1) \langle s, \ast \rangle = \delta_\delta \\
2) \forall \xi > 0: \forall \varepsilon_\varepsilon \in \mathbb{N}: \forall \varepsilon \in \mathbb{N}, \forall \xi \geq \varepsilon, x \in \mathbb{R}^n, ||x|| \geq \varepsilon: \ s(\varepsilon)(x) = 0 \\
3) W(s)(\varepsilon) \text{ is line wise nonsingular} \end{cases}$$

where, for $s: \mathbb{N} \rightarrow C^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote

$$W(s)(x) = \begin{pmatrix} D^p(1)s(o)(x) & \ldots & D^p(\mu)s(o)(x) & \ldots \\
\vdots & \ddots & \vdots & \ddots \\
D^p(1)s(\varepsilon)(x) & \ldots & D^p(\mu)s(\varepsilon)(x) & \ldots \\
\vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

Indeed, supposing that Theorem VAN holds, we can prove the following proposition, generalizing Proposition 3, §5, [13], whose validity was the precondition for the construction of the multiplication theory in the case of the distributions in $\mathcal{D}'(R^1)$.

**Proposition 3.** $Z^\delta_\delta \neq \emptyset, \forall n \in \mathbb{N}, n \geq 2$.

**Proof.** Suppose $\alpha \in C^\infty(R^1), \beta \in \mathcal{D}(R^1)$ such that $\alpha(x) = e^x, \beta(x) \geq 0, \forall x \in R^1$ and $\beta(x) = 1, \forall x \in [-1,1]$.

Denote $\gamma = \alpha \ast \beta$, then $\gamma \in \mathcal{D}(R^1), D^p\gamma(o) = 1, \forall p \in \mathbb{N},$ and

$$K = \int_{R^1} \gamma(x) dx > 0.$$
We define $s: \mathbb{N} \rightarrow C_\infty(\mathbb{R}^n)$ such that $s(v)(x) = \mu_1 \cdots \mu_n \gamma(\mu_1 x_1) \cdots \gamma(\mu_n x_n)/K^n$ where $v \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \ldots, \mu_n) = \nu(n+1) + e$, with $e = (1, \ldots, 1) \in \mathbb{N}^n$.

Obviously, $<s, \nu> = \delta_0$ and

$$D^q s(v)(o) = (\nu(n+1) + e)^q / K^n, \forall \nu \in \mathbb{N}, q \in \mathbb{N}^n.$$ 

Therefore, the matrix $W(s)(o)$ becomes

$$
\begin{pmatrix}
(p(1)+e)p(1)+e/K^n & \ldots & (p(1)+e)p(h)+e/K^n \\
\vdots & \ddots & \vdots \\
(p(k)+e)p(1)+e/K^n & \ldots & (p(k)+e)p(h)+e/K^n
\end{pmatrix}
$$

And, $W(s)(o)$ will be line wise nonsingular, only if

$$
\begin{vmatrix}
(p(1)+e)p(1) & \ldots & (p(1)+e)p(h) \\
\vdots & \ddots & \vdots \\
(p(k)+e)p(1) & \ldots & (p(k)+e)p(h)
\end{vmatrix}
$$

is line wise nonsingular.

Now, supposing Theorem VAN true, the above requirement will be granted. Indeed, suppose $k_*, h_* \in \mathbb{N}, k_*, h_* \geq 1$ given. Assume $a^* \in \mathbb{N}^n$ and $m^* \in \mathbb{N}$ such that

1) $a^* \geq p(i)+e$, $a^* \neq p(i)+e$, $\forall 1 \leq i \leq k_*$

2) $\ell(n,m^*) \geq h_*$
and consider $a \in \mathbb{N}^n$, $a \geq a^*$ and $m \in \mathbb{N}$, $m \geq m^*$ resulting according to Theorem VAN.

Due to 1) above, it results

$$a + p(j) \geq p(i)+e, \ a + p(j) \neq p(i)+e, \ \forall l \leq i \leq k^*_n, \ 1 \leq j \leq \ell(n,m).$$

Therefore,

1. $k_1, \ldots, k_{2}(n,m) \in \mathbb{N}$:
2. $k^*_n \leq k_1, \ldots, k_{2}(n,m)$ and pair wise different
3. $a + p(j) = p(k_j)+e, \ \forall l \leq j \leq \ell(n,m)$.

Since 2) above, it results $\ell(n,m) \geq h^*_n$.

Finally, Theorem VAN will result in

$$
\begin{vmatrix}
(p(k_1)+e)p(1) & \cdots & (p(k_1)+e)p(\ell(n,m)) \\
\vdots & \ddots & \vdots \\
(p(k_{2}(n,m))+e)p(1) & \cdots & (p(k_{2}(n,m))+e)p(\ell(n,m))
\end{vmatrix} \neq 0
$$

If we take into account the above relations 4).


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