THE HIT PROBABILITY OF A SINGLE AA ROUND AS A FUNCTION OF THE SYSTEM ERRORS

by

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ABSTRACT

Two methods are employed for estimating the hit-probability of a single A.A. round fired at a moving spherical target. First, a simulation-program for a digital-computer is constructed then an analytic formula is derived, which estimates the hit-probability as an explicit function of the accuracy of the values of some parameters, determining the trajectories of target and shell. Comparison of the estimates obtained by the two methods reveals that for a specific category of encounters the derived formula is a reliable and convenient instrument for hit-probability estimation.
1. INTRODUCTION

Whether or not a hit will be scored upon firing a single A.A. round at a moving target is dependent on many parameters which determine the trajectories of the target and the shell, among them the initial and instantaneous velocities of target and shell, atmospheric conditions at the time of fire, wind velocity, etc. Due to the difficulty of predicting beforehand the values of the various parameters at the time of fire, there is no assurance that aiming is indeed accurate, therefore it is customary to refer to the \"hit-probability of the A.A. round\".

In order to estimate the hit-probability two methods are used. First a simulation-program (SP) which simulates the aiming and firing process by \"Monte-Carlo\" methods is constructed. We then derive an analytic formula, which estimates the hit-probability as an explicit function of the accuracy of the values of some of the parameters which determine the trajectories of target and shell.

The SP serves as a \"laboratory\" for checking various shell-target encounters. Because of the myriad situations which can be derived from various combinations of target configurations, target trajectories, muzzle-velocities, etc., we restrict the investigations to spherical targets which move at constant speed along a straight line during the time of fire.

Comparison of the derived formula's estimates of the hit-probability with those of the SP, assuming the SP results to be accurate standards.
for comparison, reveals that in encounters similar to those which we investigate, the analytic formula serves as a reliable and convenient instrument.
2. THE SIMULATION-PROGRAM

In order to reach an accurate estimation of the hit-probability, the SP (based on the aiming principles stated in Wrigley [8], which are the same for most conventional A.A. systems) must correctly simulate both the gun-aiming process and the errors which are inherent to it.

The object of the gun-aiming system is to point the gun barrel toward the Theoretical Impact Point (TIP) defining the TIP to be the point in space at which a properly directed projectile fired at the right moment will strike the target, provided the estimates of target and shell trajectories are correct.

We divide the main errors appearing during this process roughly into two categories: errors which arise from erroneous prediction of the trajectories of target and shell and therefore lead to the wrong TIP - the prediction-errors; and errors which arise from deviations in the aiming process due to mechanical limitations of the gun system and ballistic dispersion of the shell - the dispersion-errors.

In order to implement the SP, we make the following assumptions:

2.1 The target is spherical with radius $r$ and moves at a constant speed $u$ along a straight line at the time of fire. This trajectory is determined by two independent angles $\theta$ and $\phi$, the dive and course angles respectively of the target in relation to the gun (see Fig. 1).

We derive the above restriction from the evidence that aircraft which attack
Figure 1: The coordinate system and symbols used in this work.
point ground targets with conventional armament, such as bombs and rockets, attempt to fly at a constant speed and in a straight line during the attack stage, which is also the stage at which the A.A. batteries usually fire at the aircraft (see Schreier [6]).

2.2 Shell trajectory is determined by the muzzle-velocity $v$ and by two independent angles $\alpha$ and $\beta$, the elevation and prediction angles respectively of the gun (see Fig. 1). The instantaneous position vector of the projectile is calculated by equations (1) and (2).

Usually, a system of differential equations must be solved in order to accurately calculate the ballistic trajectory of a shell (see Macshane [5]). For our purposes, equations (1) and (2) are sufficiently accurate, especially since we deal with short range flat trajectories.

2.3 The gun system estimates of target speed, dive and course angles are simulated as normally distributed random variables $N(u_o, \sigma_u^2)$, $N(\theta_o, \sigma_\theta^2)$, $N(\phi_o, \sigma_\phi^2)$ respectively, with the actual values of these parameters $u_o$, $\theta_o$, $\phi_o$ as means and $\sigma_u$, $\sigma_\theta$, $\sigma_\phi$, the standard deviations, as input variables.

2.4 The actual muzzle-velocity is simulated as a normally distributed random variable $N(v_o, \sigma_v^2)$ with the nominal muzzle-velocity as a mean, and the standard deviation as an input variable.

2.5 The elevation and prediction angles at which the shell is actually fired are simulated as two independent normally distributed random variables $N(\alpha_i, \sigma_\alpha^2)$, $N(\beta_i, \sigma_\beta^2)$ with $\alpha_i$, $\beta_i$ (i.e. the angles at which the shell
must be fired at the correct moment in order to hit a target moving at $u$, $\hat{\theta}$ and $\hat{\phi}$ as the means, and the standard deviations $\sigma_\alpha$, $\sigma_\beta$ as input variables.

Assumptions 2.3 and 2.4 allow us to simulate the prediction-errors and Assumptions 2.4 and 2.5, to simulate the dispersion-errors.

2.6 The target position at the moment of fire ($t = 0$) is known accurately. We assume that actual errors in estimating this position can be represented by the errors in estimating the trajectory and speed of the target.

2.7 A hit is scored when and if at any moment the centers of target and shell are at a distance of less than $r$ from each other.

The trajectories of target and shell are displayed in a three-dimensional rectangular coordinate system $(X,Y,Z)$, where the origin $O$ is at the center of the gun position; coordinate $Y$ is along the instantaneous position vector of target at time $t = 0$ (point $B$); the $X$ coordinate is perpendicular to the $Y$ coordinate, such that the $XY$ plane is defined by gun position $O$, target position at moment of fire $B$ and the TIP $C$; the $Z$ coordinate is perpendicular to the $XY$ plane (see Fig. 1).

Let $\hat{\gamma}(t; v, \hat{\alpha}, \hat{\beta})$ be the instantaneous position vector of the shell at time $t$, $0 \leq t \leq T$, where $T$ is a given time limit (e.g., self-destruction time of shell); let $w(t)$ be the average speed of the projectile along
where \( k \) is an input variable, a drag constant specific to each gun type, and \( v \) is the muzzle velocity.

Using (1), the formula for \( \dot{y}(t;v,\alpha,\beta) \) in the \((X,Y,Z)\) system is:

\[
\dot{y}(t;v,\alpha,\beta) = (w(t)t\cos\alpha\sin\beta, w(t)t\cos\alpha\cos\beta, w(t)t\sin\alpha)
\]  

(2)

The instantaneous position vector of the target at time \( t \) is denoted by \( \dot{x}(t;u,\theta,\phi) \) and \( d \) denotes the distance of target from gun position at \( t=0 \) (i.e., \( d = \dot{x}(0;u,\theta,\phi) \)). The formula for \( \dot{x}(t;u,\theta,\phi) \) is:

\[
\dot{x}(t;u,\theta,\phi) = (ut\cos\theta\sin\phi, d-ut\cos\theta\cos\phi, utsin\theta)
\]  

(3)

A concrete case for investigation by the SP is defined by the basic input data which includes values for the variables \( r, d, u_o, \theta_o, \phi_o, v_o, \sigma_u, \sigma_\theta, \sigma_\phi, \sigma_v, \sigma_\alpha, \sigma_\beta, k \).

The estimate of the hit-probability is derived by repeatedly simulating different encounters of the same case by the following method:

**Step 1:** The gun-system estimates of the three parameters \( u, \theta, \phi \), which determine the target's trajectory are sampled from the given normal distributions \( N(u_o,\sigma_u^2) \), \( N(\theta_o,\sigma_\theta^2) \), \( N(\phi_o,\sigma_\phi^2) \) respectively.

**Step 2:** With the above estimates, utilizing formulas (1), (2) and (3), the theoretical hit time \( t_o \), and \( \hat{\alpha}_1 \) and \( \hat{\beta}_1 \), are calculated by solving the equation

\[
|\dot{y}(t;v_o,\hat{\alpha},\hat{\beta}) - \dot{x}(t;u,\theta,\phi)|^2 = 0, \quad (4)
\]
or explicitly:

\[(u \cos \theta \sin \phi - w_o(t) t \cos \phi \sin \beta)^2 + (u \cos \theta \cos \phi - w_o(t) t \cos \phi \cos \beta)^2 + (u \sin \theta - w_o(t) t \sin \alpha)^2 = 0 \]

where

\[w_o(t) = v_o/(1 + kt)\]

Step 3: \(v, \alpha, \beta\) which serves as the actual values of the three parameters determining the projectile's trajectory are sampled from the normal distributions \(N(v_o, \sigma_v^2), N(\alpha_1, \sigma_\alpha^2), N(\beta_1, \sigma_\beta^2)\) respectively.

Step 4: The minimal distance (miss distance) \(R\) between target and shell during the time interval \([0,T]\) \((R = \min_{0 \leq t \leq T} |\dot{x}(t; u_o, \hat{\theta}_o, \hat{\phi}_o) - y(t; v, \hat{\alpha}, \hat{\beta})|)\) is found. If \(R \leq r\), a hit is recorded.

Let \(n\) denote the total number of trials for a specific case, \(m\) the total number of hits scored, and \(p_n\) the SP estimate of the hit-probability after \(n\) trials; then

\[p_n = m/n\]

Another variable calculated by the computer model is \(R_m\), the average of the square miss-distance

\[R_m = \left( \sum_{i=1}^{n} R_i^2 \right)^{1/2}/n\]
where $R_i$ is the minimal distance between target and shell in the $i$-th trial.

Systematic examination of the dependency of hit-probability on the values of the various parameters determining the trajectories and the accuracy in their estimation, was accomplished by repeated runs of the SP, each run differing in the value of a single input variable.

Represented in Fig. 2 is the change in hit-probability as a function of the distance of target from gun at the moment of fire, and in Fig. 3, the change as a function of the target course angle. In each figure are represented the results obtained for several different values of the standard deviations of the prediction and elevation angles and of the dive and course angles.
Figure 2: Dependence of hit-probability on distance of target from gun at t=0.
Figure 3: Dependence of hit-probability on target course angle.
3. A FORMULA FOR ESTIMATING $\lambda_m^2$

In order to develop a convenient formula for estimating the hit-probability we devise a method of estimating the expectation of the square of the miss-distance between shell and target $\lambda_m^2$ from the values of the parameters defined in Section 2. (The relation between the two concepts is discussed in Section 4.)

We let $\zeta$ denote these six parameters, that is to say,

$$\zeta = (v, \alpha, \beta, u, \theta, \phi). \quad (9)$$

$\zeta$ is a vector of independent normally distributed variables $N(\zeta_{i0}, \sigma_i^2)$, $i = 1, \ldots, 6$; let $\hat{z}(t; \zeta)$ and $\lambda(t; \zeta)$ be the instantaneous distance vector and instantaneous distance respectively between target and shell

$$\hat{z}(t; \zeta) = \hat{y}(t; v, \alpha, \beta) - \hat{x}(t; u, \theta, \phi) \quad (10)$$

$$\lambda(t; \zeta) = \|\hat{z}(t; \zeta)\| = \left[ \sum_{i=1}^{3} z_i(t; \zeta)^2 \right]^{1/2}. \quad (11)$$

We assume:

3.1 $T$ is a positive number such that for any value of $\zeta$, $(t; \zeta)$ has only a single local minimum in the time interval $[0, T]$. That is to say, if $R(\zeta)$ denotes that minimum and $t_1$ the time it occurs, then

$$R(\zeta) = \begin{case} 
\lambda(t_1; \zeta) < \lambda(t; \zeta) \quad & 0 \leq t_1 < T \quad \forall t, t \in [0, T], t \neq t_1 
\end{case} \quad (12)$$
3.2 If $\xi = \xi_0 = (v_o, \alpha_0, \beta_0, u_o, \theta_0, \phi_0)$ then the minimal distance vanishes

$$R(\xi_0) = 0.$$  \hspace{1cm} (13)

$R(\xi)$ is a continuous function of the random vector $\xi$, therefore

**Theorem:**

According to Assumptions 3.1 and 3.2 the expectation of $R^2$ is

$$\ell_m^2 = A_1 + A_2 + A_3 + E_1,$$  \hspace{1cm} (14)

where

$$A_1 = \left\{ \sigma_\theta^2 + [u_o + [w_o(t_o)/(l+kt_o)]\cos(\beta_o + \phi_o)]^2 \sigma_\phi^2/s^2 \right\} u_o^2 t_o^2,$$

$$A_2 = \left\{ \sigma_\alpha^2 + [u_o \cos(\beta_o + \phi_o) + w_o(t_o)/(l+kt_o)]^2 \sigma_\beta^2/s^2 \right\} w_o(t_o)^2 t_o^2,$$  \hspace{1cm} (15)

$$A_3 = \left\{ w_o(t_o)^2 u_o^2/(l+kt_o)^2 s^2 \right\} (\sigma_u^2/u_o^2 + \sigma_v^2/w_o(t_o)^2) t_o^2 \sin^2(\beta_o + \phi_o),$$

$$S^2 = u_o^2 + [w_o(t_o)/(l+kt_o)]^2 + 2u_o[w_o(t_o)/(l+kt_o)]\cos(\beta_o + \phi_o).$$  \hspace{1cm} (16)

and $t_o$, $\beta_o$ are calculated by solving equation (5).

$E_1$ is the error.

For proof of the theorem see Appendix A.

Examination of formula (15) reveals that $A_1$ and $A_2$ represent the contribution to the miss-distance of the angular-deviation factor of the prediction-error and dispersion-error respectively, while $A_3$ represents the contribution of the inaccuracy in estimating target speed, and the
deviation of the shell's speed.

In some 30 different cases, comparison between the estimates of $\ell_m^2$ derived by formula (a14) (see Remark in Appendix A) and those derived by the SP (taking $R_m^2$ to be the SP estimate), yields an average relative error ($|\ell_m - R_m| / R_m$) of 1.5%, and 0.35m as the standard deviation of the difference between the two types of estimates.
4. UTILIZING $\chi^2$ FOR HIT-PROBABILITY ESTIMATION

We define the target-plane (TP) as passing through the target center and as constantly perpendicular to gun-target line (GTL), and take the target center to be the origin of the $(\eta_1, \eta_2)$ rectangular coordinate system of TP, where $\eta_1$ is constantly horizontal.

We assume, as did Cunningham [1], Frazer [2], and Lind [4], that the most significant information about the hit-probability is derived from the dispersion of the intersection between shell trajectory and the TP. The dispersion is normally distributed in the two independent coordinates $(\eta_1, \eta_2)$, with the center of target as the mean and $\sigma_1$, $\sigma_2$ as the respective standard deviations.

As in Assumption 2.7, we consider a hit to be scored when

$$(\eta_1^2 + \eta_2^2)^{\frac{1}{2}} \leq r;$$

therefore, the hit probability $p$ is

$$p = \int \int_{\eta_1^2 + \eta_2^2 \leq r^2} N(0, \sigma^2_1) N(0, \sigma^2_2) d\eta_1 d\eta_2. \tag{18}$$

If $\sigma_1 = \sigma_2 = \sigma$ the hit-probability is given by

$$p = 1 - \exp(-r^2/2\sigma^2). \tag{19}$$

Alternatively, if...
\[
\rho = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}, \tag{20}
\]

then \(\rho\) is a Rayleigh distributed variable, and therefore

\[
\mathcal{L}_R^2 = \mathbb{E}(\rho^2) = 2\sigma^2 \tag{21}
\]

and

\[
p = 1 - \exp\{-r^2/\mathcal{L}_R^2\} \tag{22}
\]

We find that taking \(\sigma_\theta \sim \sigma_\phi\), \(\sigma_\alpha \sim \sigma_\beta\) and \(\phi = 0\) is a sufficient condition to cause \(\sigma_1 = \sigma_2\). This condition defines a category of cases where formula (22) can be used for hit-probability estimation.

Comparison of the hit-probability estimates derived by the SP and by formula (22), where \(\mathcal{L}_m^2\) (computed by formula (a14)) is taken to be the estimation of \(\mathcal{L}_R^2\), yields a relative difference of 4.5%, where the maximal relative difference recorded is 11.6%.
5. CONCLUSIONS

Sections 3 and 4 indicate that formula (22) in conjunction with (a14) is a reliable instrument for estimating the hit-probability in a specific category of encounters. It is also a convenient instrument for analyzing the dependence of the hit-probability on the various parameters defined throughout the paper. Some interesting relationships utilizing formulas (22) and (a14) are represented in figures 2, 3 and 4.

In figure 4, for instance, each curve represents the different combinations of $\sigma_\alpha$ and $\sigma_\theta$ which yield the same hit-probability, information which can be utilized for decision making in an attempt to improve a given AA system. Once the costs of changes in $\sigma_\alpha$ and $\sigma_\theta$ are defined, we can calculate the quality of the reductions in $\sigma_\alpha$ and $\sigma_\theta$ which will most efficiently shift the AA system from one indifference curve to a better one.

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Figure 4: Hit-probability indifference-curves
APPENDIX A

Proof of the Theorem

R is a continuous function of $\zeta$; therefore:

$$\mathbb{E}_m^2 = \mathbb{E}\{R^2\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R^2(\zeta) \prod_{i=1}^{L} N(\zeta_{i0}, \sigma_{i}^2) d\zeta_i \quad (a1)$$

since all partial derivatives of at least order 4 of $R^2(\zeta)$ exist, let us expand $R^2(\zeta)$ to a Taylor series around $\zeta_0$:

$$R^2(\zeta) = T(\zeta) + E_0(\zeta) \quad (a2)$$

where

$$T(\zeta) = R^2(\zeta_0) + \sum_{i=1}^{6} [\partial R^2(\zeta_0)/\partial \zeta_i] (\zeta_i - \zeta_{i0}) +$$

$$(1/2!) \sum_{j=1}^{6} [\partial^2 R^2(\zeta_0)/\partial \zeta_i \partial \zeta_j] (\zeta_i - \zeta_{i0})(\zeta_j - \zeta_{j0}) +$$

$$(1/3!) \sum_{i,j,k=1}^{6} [\partial^3 R^2(\zeta_0)/\partial \zeta_i \partial \zeta_j \partial \zeta_k] (\zeta_i - \zeta_{i0})(\zeta_j - \zeta_{j0})(\zeta_k - \zeta_{k0}); \quad (a3)$$

$$E_0(\zeta) = (1/4!) \sum_{i,j,k,l=1}^{6} [\partial^4 R^2(\zeta_0)/\partial \zeta_i \partial \zeta_j \partial \zeta_k \partial \zeta_l] \prod_{n=i,j,k,l} N(\zeta_n - \zeta_{n0}). \quad (a4)$$

Substituting (a2) in (a1), we obtain

$$\mathbb{E}\{R^2\} = \mathbb{E}\{T\} + \mathbb{E}\{E_0\}. \quad (a5)$$
As is well known, if \( x \) is a random variable with mean \( x_0 \) and standard deviation \( \sigma_x \), then

\[
E \{ x - x_0 \} = 0
\]

\[
E \{ (x - x_0)^2 \} = \sigma_x^2
\]  \hspace{1cm} (a6)

Formulas (13), (a3), (a6) and (a5) imply:

\[
E\{T\} = \sum_{i=1}^{6} \left[ \frac{\partial^2 R^2(\xi_0)}{\partial \xi_i^2} \right] \sigma_i^2
\]  \hspace{1cm} (a7)

and

\[
E_1 = E\{E_0\} = (9 \cdot 6/4!) \sum_{i,j=1}^{6} \left[ \frac{\partial^4 R^2(\xi_0)}{\partial \xi_i^2 \partial \xi_j^2} \right] \sigma_i^2 \sigma_j^2
\]  \hspace{1cm} (a8)

By formula (12) we obtain:

\[
\frac{\partial R^2(\xi_0)}{\partial \xi_i} = \frac{\partial^2 (t_1, \xi_0)}{\partial \xi_i} + \frac{\partial^2 (t_1, \xi_0)}{\partial t} \frac{\partial t}{\partial \xi_i}, \quad i = 1, \ldots, 6 \]  \hspace{1cm} (a9)

Since \( t_1 \) is the time instant where \( |\dot{z}(t)| \) is minimal:

\[
G(t; \xi) = \ddot{z}(t; \xi) \cdot (d\dot{z}(t; \xi)/dt) = 0
\]  \hspace{1cm} (a10)

consequently,

\[
\frac{\partial t}{\partial \xi_i} = - \left[ \frac{\partial G(t; \xi_0)}{\partial \xi_i} \right] / \left[ \frac{\partial G(t; \xi_0)}{\partial t} \right], \quad i = 1, \ldots, 6
\]  \hspace{1cm} (a11)
Substitution of (all) in (a9) and differentiation yield

\[ \frac{\partial^2 R^2(t_o)}{\partial \xi^2} = \frac{\partial^2 \xi^2(t_i; t_o)}{\partial \xi^2} - \left[ \frac{\partial^2 \xi^2(t_i; t_o)}{\partial \xi \partial t} \right]^2 + \frac{\partial^2 \xi^2(t_i; t_o)}{\partial t^2}, \quad i = 1, \ldots, 6. \]  
\[ (a12) \]

Solving (a12) explicitly for every parameter and substituting the result in (a7) we obtain:

\[ E(T) = A_1 + A_2 + A_3, \]  
\[ (a13) \]

where

\[ A_1 = \{ \sigma^2 \left[ u_o + [w_o/(1+kt_o)] \cos(\beta_o + \phi_o) \right]^2 \sigma^2 / S^2 \} u_o t_o^2, \]

\[ A_2 = \{ \sigma^2 \left[ w_o \cos(\beta_o + \phi_o) + w_o (t_o) / (1+kt_o) \right]^2 \sigma^2 / S^2 \} w_o^2 (t_o) t_o^2, \]

\[ A_3 = \{ w_o^2 (t_o) u_o^2 / [(1+kt_o)^2 S^2] \} \{ \sigma^2 / u_o^2 + \sigma^2 / w_o^2 (t_o) \} t_o^2 \sin^2(\beta_o + \phi_o), \]

and

\[ S^2 = u_o^2 + [w_o (t_o) / (1+kt_o)]^2 + 2u_o [w_o (t_o) / (1+kt_o)] \cos(\beta_o + \phi_o). \]

Q.E.D.
Remark:

Since

\[ \delta^4 R^2(\xi) / \delta u^4 = \delta^4 R^2(\xi) / \delta v^4 = \delta^4 R^2(\xi) / \delta u^2 \delta v^2 = 0 \]

and

\[ \delta^4 R^2(\xi) / \delta u^2 \delta \alpha^2 = \delta^4 R^2(\xi) / \delta u^2 \delta \beta^2 = \delta^4 R^2(\xi) / \delta v^2 \delta \theta^2 = \delta^4 R^2(\xi) / \delta v^2 \delta \phi^2 = 0. \]

and since, in effect, the standard-deviations of the angles \( \hat{\alpha}, \hat{\beta}, \hat{\theta} \) and \( \hat{\phi} \) are quite small, there are cases where \( E_2 \) can be neglected, and \( \lambda_m^2 \) can be estimated by

\[ \lambda_m^2 = E\{T\}. \]  

(a14)


