ON THE SINGULARITIES OF MATRICES ARISING IN
THE ANALYSIS OF STOCHASTIC BEHAVIOUR OF
MASS-STORAGE DEVICES

by

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On the Singularities of Matrices Arising in the Analysis of the stochastic Behaviour of Mass-Storage Devices.

1. The purpose of this note is to present a phenomenon which is apparently rather common, and yet not quite understood. In our particular instance it arose when we considered operating policies for mass-storage devices in computer systems, such as disks and drums [1].

2. The simplest situation that apparently still possesses all the salient points can be presented in queueing-theoretical terminology as follows:
A single server facility. Customers come with i.i.d. interarrival times following an exponential distribution with parameter $\lambda$. There are $N$ types of customers; the probability that a customer is of type $i$ is $p_i (\geq 0)$, independently of the past or the state of the system. A customer of type $i$, immediately preceded by a customer of type $j$ requires a service $S_{ji}$ with duration governed by the distribution $F_{ji}(\cdot)$ — whether an idle period intervened or not. The selection for service is FCFS.

3. We are only interested in steady state behaviour of the processes involved, particularly the waiting times and the occupancy levels.

4. Obviously, $(X_n, I_n)$ — where

- $X_n$ — the number of customers in the system following the departure epoch of the $n$-th customer, and
- $I_n$ — the type of the $n$-th customer

— is an imbedded aperiodic, irreducible — and for a small enough $\lambda$ — ergodic M.C.*

*Clearly, ergodicity requires that $\lambda E(S) < 1$, where $E(S) = \sum_j \sum_i p_i p_j E(S_{ij})$. 

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5. The dynamics of the system are completely determined by the relation
\[ x_{n+1} = x_n - u_n + y_{n+1} \]  
where \( u_n = 1 \) when \( x_n > 0 \), and 0 otherwise; \( y_{n+1} \) is the number of customers that arrive during the service of the \( n+1 \)-st customer.

6. Defining \( p_i(x) = \lim_{n \to \infty} P(X_n = x | I_n = i) \)
\[ G_i(z) = \sum_{x=0}^{\infty} p_i(x) z^x \]
we obtain from (1) the linear equations
\[ A(z) \vec{G}(z) = B(z) \vec{\eta} \]
where \( A \) and \( B \) are matrices, \( \vec{G} \) and \( \vec{\eta} \) are vectors:
\[ A_{ij}(z) = z \sum_{j=a}^{\infty} p_{ij} L_{ij}(a) \]
\[ B_{ij}(z) = p_{ij}(z-1) L_{ij}(a) \]
\[ G_i(z) = G_i(z) \]
\[ \vec{\eta}_i = p_i(0) \]
\[ L_{ij}(s) = \text{Laplace-Stieltjes transform of } P_{ij}(s) \text{ at } s. \]
\[ a = \chi(1-z) \]

We note in passing that \( A(z) \) is characterized by the selection for service method, whereas \( B(z) \) by the idle time behaviour of the server.

The obvious solution of (2) is
\[ \vec{G}(z) = A^{-1}(z) B(z) \vec{\eta}, \]
with \( \vec{\eta} \) determined through normalization and analyticity requirements.

7. It is to this last issue that this note is addressed. First, we note that \( A(z) \) is singular at \( z=1 \), and at least at \( N-1 \) more values of \( z \) in the unit disk. That \( A(1) \) is singular is shown by substitution and noting that the sum of all the \( p_i \) is one. The second claim results from observing that when \( \chi = 0 \), \( \det(A) = (z-1)z^{N-1} \). When \( \chi \) increases continuously from 0 to its
operational value, the roots of the determinant break up and depart from
the origin continuously and separately. (If we write the \( L_{ij} \) as a power
series in \( \lambda: L_{ij}(\lambda) = \sum_{k=0}^{\infty} b_{ijk} \lambda^k + o(\lambda^r) \), we obtain that \( \det(A) \) is a
polynomial in \( z \), of degree \( N \), with \( z=0 \) a root of multiplicity \( N-r-1 \), gene-
really. We used here the fact that these roots are \( o(\lambda) \) for small \( \lambda \).
The Levy-Desplanques theorem assures us that for \( |z|=1 \), only \( z=1 \) is a
root of \( \det(A)=0 \). Hence, although the roots depart from the origin they
remain in the unit disk.

8. Let \( z_1 \) be the roots in the \( z \) plane of \( \det(A(z))=0 \). Define the matrix
\( C(z) = \det(A(z))A^{-1}(z) \). The \( \Pi \) are then to be determined by the totality
of the equations
\[
C(z_i)B(z_i)\Pi_i = 0 \quad i=1(1)N-1 \tag{4}
\]
\[
\sum_i p_i \Pi_i = 1 - \lambda \mathbb{E}(S) \tag{5}
\]
This gives rise to \( N^2-N+1 \) equations, for \( N \) unknowns. The question of how
the process of selecting \( N \) independent equations from this collection can
be automated has not been resolved. We tend to think that checking the
dependence numerically is not reliable, since practically nothing is
guaranteed about the condition numbers of the matrices \( C \) and \( B \). In the
few experiments we conducted the rank of \( CB \) was found to be \( N-1 \), for all
\( z_1 \), except 1. Thus just (5) and one set of (4) were necessary to obtain
all the \( \Pi_i \). We conjecture that this is generally true. Continuity argu-
ments point that way too; however, experiments in a somewhat related situation

*The theorem states, in one of its forms, that if matrix \( C \) satisfies
\( \left| c_{ii} \right| > \sum_{j \neq i} \left| c_{ij} \right| \) then \( \det(C) \neq 0 \). In our case this simply reduces
to \( |L_{ij}(\lambda(1-z))| < 1 \), which is true when \( \lambda > 0 \) and \( \text{Re}(1-z) > 0 \). [3]
References

