NP-COMPLETENESS OF SEVERAL ARRANGEMENT PROBLEMS

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ABSTRACT

Through polynomial reductions, starting with the satisfiability problem for conjunctive normal forms with three literals per clause, we show that the following problems are NP-complete: The maximum cut of an undirected graph with unit weights of the edges; the minimum linear arrangement of the vertices of an undirected graph; the minimum linear arrangement of the vertices of a directed graph; the minimum linear arrangement of the vertices of a bipartite graph. The last two results are new.
I. INTRODUCTION

The satisfiability problem for conjunctive normal forms with three literals per clause can be defined as follows:

3-SAT: Given a set of literals \( X = \{x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m\} \) and a family of clauses \( C_1, C_2, \ldots, C_p \) such that for every \( 1 \leq j \leq p \), \( |C_j| = 3 \) and \( C_j \subseteq X \), the problem is to determine whether there is a "true" and "false" assignment for the literals such that \( C_j \) contains at least one literal with a "true" value. Clearly, for every \( 1 \leq i \leq m \), exactly one of \( x_i \) and \( \bar{x}_i \) must be "true" while the other is "false".

Cook showed [1] that 3-SAT is NP-complete. We shall now consider several combinatorial decision problems which are all trivially in the NP class. We shall show that they are NP-complete through a sequence of polynomially bounded reductions. The notation \( \text{PROB1} \preceq \text{PROB2} \) means that the problem PROB1 is polynomially reducible to the problem PROB2; if PROB1 is known to be NP-complete then the existence of such a reduction proves that PROB2 is NP-complete.

Karp [2] proved** that the maximum cut problem for undirected graphs, where the edges may have arbitrary weights is NP-complete. Cirey, Johnson and Stockmeyer [3] proved that even if all edges weights are 1, the

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* Also, we assume that no clause contains both \( x_i \) and \( \bar{x}_i \), or it may be dropped altogether. Furthermore, \( \bar{x}_i = x_i \).

** There is a typographical error in his reduction. One should have \( W = \frac{1}{4} (\Sigma C_i)^2 \) instead of \( W = \frac{1}{4} \Sigma C_i^2 \).
maximum cut problem is NP-complete. They proceed to show that the minimum
linear arrangement for undirected graphs is also NP-complete.

In this paper we give alternative proofs for the completeness of
the maximum cut with unit weights and the minimum linear arrangement
problems and proceed to prove the completeness of the minimum linear
arrangement for directed graphs and bipartite graphs. It is interesting
to note here, that the case of linear arrangement of trees with unit
weights has been solved by us through an algorithm whose complexity
is bounded by $O(n^{2.2})$, (the paper is in preparation) and that the case of
linear arrangement of directed (rooted) trees, was solved by Adolphson
and Hu [4] and the complexity of their algorithm is $O(n\log n)$.

II. THE REDUCTIONS

We shall use the following notation. $G(V,E)$ is a finite undirected
graph; $V$ is its set of vertices and $E$, its set of edges. Also, if
$S \subseteq V$, then the cut $C(S;\overline{S})$ consists of all edges with one of their end
vertices in $S$, and the other in $\overline{S}$.

**MAXBC** (Maximum cut of a graph whose edges have a bounded weight.)

Given a graph $G(V,E)$ where $|V| = n$ with a bounded weight function
$\omega$ for the edges, that is, for every $e \in E$, $\omega(e)$ is a positive integer and
$\omega(e) \leq n^3$. Also, an integer $W$ is given. The problem is to determine
whether there exists a cut $C(S;\overline{S})$ such that

$$\sum_{e \in C(S;\overline{S})} \omega(e) \geq W.$$
Theorem 1: 3-SAT $\leq$ MAXBC.

Proof: Let us define a weighted graph and a MAXBC problem on it, using the data of the 3-SAT problem, as follows:

$$V = V \cup \{v_j \mid 1 \leq j \leq p\} \cup \{v_0\}$$

For every $1 \leq j \leq p$ let

$$A_j = \{v_0\} \cup \{v_j\} \cup C_j.$$  

Now

$$E = \{<u,v> \mid \exists j \exists u,v \in A_j \cup \{x_i, \overline{x_i} \mid 1 \leq i \leq m\}.\}$$

The weight function $\omega$ is defined by

$$\omega(<\xi, v_0>) = \sum_{j=1}^{p} |C_j \cap \{\overline{\xi}\}| \quad \text{if } \xi \text{ is a literal}$$

$$\omega(<\xi', \overline{\xi''}> ) = \sum_{j=1}^{p} |C_j \cap \{\overline{\xi'}\}| \cdot |C_j \cap \{\overline{\xi''}\}| \quad \text{if both } \xi' \text{ and } \xi'' \text{ are literals and } \overline{\xi'} \neq \overline{\xi''},$$

$$\omega(<x_i, \overline{x_i}> ) = |E| \cdot p \quad \text{for every } 1 \leq i \leq m \text{ and}$$

$$\omega(<v_j, u>) = 1 \quad \text{for every } 1 \leq j \leq p \text{ and every } u \in A_j.$$  

Finally, $W = (m \cdot |E| + 6) \cdot p$.

This is a construction of a graph with $n = 2m + p + 1$ vertices and its number of edges, $|E|$, is bounded by $3m + 7p$. Every clause $C_j$ is represented by the clique $A_j$. In addition we have an edge between $x_i$ and $\overline{x_i}$ whose weight is $|E| \cdot p$. (This weight is designed to overweigh the sum of weights of all edges of the other type, and yet it is bounded by $n^3$.}
In fact it is bounded by \( O(n^2) \). The weight of the edge connecting a literal \( \xi \) and \( v_0 \) is equal to the number of times \( \xi \) appears in the clauses. The weight of the edge connecting the literals \( \xi' \) and \( \xi'' \) is equal to the number of clauses in which both \( \xi' \) and \( \xi'' \) appear.

The weight of all edges adjacent to a \( v_j \) is 1.

We now claim that the answer to the given 3-SAT problem is the same as the answer to the MAXBC problem defined above.

First, assume the answer to the 3-SAT problem is affirmative, and let \( T \) be the set of literals which have the "true" value in an assignment which satisfies the requirements. Let \( T \subseteq S \) while \( \overline{T} \subseteq \overline{S} \). Clearly, one of \( x_i \) and \( \overline{x_i} \) is in \( S \) while the other is in \( \overline{S} \). Thus, the contribution of the \( \{x_i, \overline{x_i}\} \) edges to the cut is \( m \cdot |E| \cdot p \). We put \( v_0 \in \overline{S} \).

It is convenient to interpret the rest of the edges and their weights as follows: Each \( A_j \) is a clique, and each edge appears as many times as it belongs to such cliques. In each of these \( p \) cliques there is at least one vertex in \( \overline{S} \), i.e. \( v_0 \), and at least one vertex in \( S \), since at least one literal in \( C_j \) is "true". Thus, we can put \( v_j \) in a proper side (either \( S \) or \( \overline{S} \)) so that there will be 2 vertices of the clique on one side and 3 on the other. Thus, each clique will contribute 6 to the cut, and the value of the cut will be \( (m \cdot |E| + 6) \cdot p \).

The considerations used above prove that the cut cannot exceed the value \( W \). Thus, if the answer to the MAXBC problem is affirmative, then there exists a cut \( C(S; \overline{S}) \) whose weight is exactly \( W \). Let us show that its existence assures an affirmative answer to the given 3-SAT problem.
Assume that \( v_0 \in \bar{S} \), or change the names of \( S \) and \( \bar{S} \) to assure it. Since \( |E| \cdot p \) is greater than the sum of all weights of edges which are not of the \( <x_i, \bar{x}_i> \) type, there is no way to reach \( m \cdot |E| \cdot p \) except by a cut which satisfies

\[
|\{x_i\} \cap S| + |\{\bar{x}_i\} \cap S| = 1
\]

for every \( 1 \leq i \leq m \). Thus, the assignment of "true" to \( \xi \) if \( \xi \in S \), and "false" if \( \xi \in \bar{S} \), is consistent. Also, the only way to gain the additional \( 6 \cdot p \) is to have in each clique a 2-3 split, and if all literals in \( C_j \) are "false" then there are 4 vertices of \( A_j \) on the \( \bar{S} \) side — a contradiction.

Q.E.D.

**MAXC:** Given a graph \( G'(V', E') \) and an integer \( W' \).

Determine whether there exists a cut \( C(S'; \bar{S}') \) such that

\[
|C(S'; \bar{S}')| \geq W' .
\]

**Theorem 2:** \( \text{MAXBC} \propto \text{MAXC} \).

**Proof:** Given a graph \( G(V, E) \) with a bounded weight function \( \omega(e) \), we construct a new graph \( G' \) as follows. Each edge \( e = <u, v> \) is replaced by \( \omega(e) \) parallel paths of length 3 connecting \( u \) and \( v \). See Fig.1. (Clearly, the weights of all edges are now 1; the vertices \( a_1, a_2, \ldots, a_{\omega(e)} \) and \( b_1, b_2, \ldots, b_{\omega(e)} \) are new in the graph and uniquely correspond to \( e \).)

The weight \( W' \) for the corresponding MAXC problem is defined by
\[ W' = 2 \sum_{e \in E} \omega(e) + W. \]

\[ \text{Figure 1.} \]

First, assume \( S \) satisfies the condition

\[ \sum_{e \in C(S'; \overline{S}')} \omega(e) \geq W'. \]

Let us display a cut \( C(S'; \overline{S}') \) of \( G' \) such that \( |C(S'; \overline{S}')| \geq W' \).

Put \( S \subset S' \) and \( \overline{S} \subset \overline{S}' \). Now, if both \( u \) and \( v \) are on the same side of the cut, i.e., either both belong to \( S' \) or both belong to \( \overline{S}' \), then assign all the \( a_i \)'s to \( S' \) and all the \( b_i \)'s to \( \overline{S}' \). Thus, the edges of the \( <u,v> \) structure contribute \( 2\omega(e) \) to the cut. However, if \( u \) and \( v \) are on different sides of the cut, then assign all the \( b_i \)'s to \( u \)'s side and all the \( a_i \)'s to \( v \)'s side; the \( <u,v> \) structure contributes, now \( 3\omega(e) \). Thus, the assertion holds.
Conversely, assume there is a cut \( C(S'; \bar{S}') \) such that \( |C(S'; \bar{S}')| \geq W' \).

Consider now two vertices \( u, v \in V \) such that \( \langle u, v \rangle \in E \). If both are on the same side of the cut then the contribution of the \( \langle u, v \rangle \) structure to the cut is bounded by \( 2 \cdot \omega(e) \), and if they are not, it is bounded by \( 3 \cdot \omega(e) \). Thus,

\[
|C(S'; \bar{S}')| \leq 2 \cdot \sum_{e \in E} \omega(e) + \sum_{e \in C(S; \bar{S})} \omega(e)
\]

where \( S = S' \cap V \) and \( C(S; \bar{S}) \) is the cut in \( G \) defined by this \( S \).

Since \( |C(S'; \bar{S}')| \geq W' \), we have

\[
\sum_{e \in C(S; \bar{S})} \omega(e) \geq W.
\]

This proves the assertion. Note that since \( \omega(e) \leq n^3 \), the increase in the size of the problem in this reduction is polynomially bounded, as required.

Q.E.D.

**MAXLR:** (Maximum linear arrangement.) Given an undirected graph \( G(V,E) \) and an integer \( C \). The problem is to determine whether there exists a one-one mapping \( f \) from \( V \) onto \( \{1, 2, \ldots, |V|\} \) such that

\[
\sum_{\langle u, v \rangle \in E} |f(u) - f(v)| \geq C.
\]

This is a very primitive placement problem, where we want to place the vertices of the graph in a linear array as to maximize the total length of the edges. In practice, we try to minimize the length, but as we shall see, the two problems are almost identical.
Theorem 3: MAXC \neq MAXLAR.

Proof: Given a MAXC problem in which \( G'(V',E') \) and \( W' \) are specified, define a MAXLAR as follows:

\[
V = V' \cup \{u_1, u_2, \ldots, u_n\},
\]
\[
E = E',
\]
\[
C = W' \cdot n^3.
\]

Here \( n = |V'|. \) Also, \( u_1, u_2, \ldots, u_n \) are isolated vertices, but if we interpose an isolated vertex between \( u \) and \( v \) then \( |f(u) - f(v)| \) is increased by 1. Now, first assume that the answer to the MAXC is affirmative and \( |C(S';\overline{S}')| \geq W' \). Define an arrangement mapping \( f \) which satisfies the following constraints:

1. If \( v \in S' \) then \( 1 \leq f(v) \leq |S'| \)
2. If \( v \in \{u_1, u_2, \ldots, u_n\} \) then \( |S'| < f(v) \leq |S'| + n^3. \)
3. If \( v \in \overline{S}' \) then \( |S'| + n^3 < f(v) \leq n + n^3. \)

The number of edges which span over the \( n^3 \) vertices interposed in between \( S' \) and \( \overline{S}' \) is \( |C(S';\overline{S}')| \), and each has a length which exceeds \( n^3 \). Thus,

\[
\sum_{<u,v> \in E} |f(u) - f(v)| > |C(S';\overline{S}')| \cdot n^3 \geq W' \cdot n^3,
\]

and the answer to the corresponding MAXLAR problem is affirmative too.

Next, assume \( f \) is an arrangement such that

\[
\sum_{<u,v> \in E} |f(u) - f(v)| \geq C.
\]
For each \( 1 \leq i < |V| \) the set

\[ S_i = \{ v \mid v \in V \text{ and } f(v) \leq i \} \]
defines a cut \( C(S_i; \overline{S}_i) \). Now, let \( j \) satisfy

\[ C(S_j; \overline{S}_j) = \max_i C(S_i; \overline{S}_i). \]

We now rearrange the vertices as follows. All the vertices in \( V' \) remain in their relative position but the vertices \( u_1, u_2, \ldots, u_3 \) are all moved to interpose between \( S_j \cap V' \) and \( \overline{S}_j \cap V' \). Clearly, each \( u_k \) contributes now the maximum value to the total length of the edges.

Thus, the new arrangement, \( f' \), satisfies

\[ \sum_{(u,v) \in E} |f'(u) - f'(v)| \geq \sum_{(u,v) \in E} |f(u) - f(v)| \geq W' \cdot n^3. \]

Now, the total length of the edges in the arrangement \( f' \) can be divided into two parts:

1. \( n^3 \cdot C(S_j; \overline{S}_j) \), which is the length caused by the interposition of \( u_1, u_2, \ldots, u_3 \).

2. The total length of the edges if \( u_1, u_2, \ldots, u_3 \) were dropped and the gap between \( S_j \cap V' \) and \( \overline{S}_j \cap V' \) was closed; this length is clearly bounded by

\[ (n-1) \cdot 1 + (n-2) \cdot 2 + \ldots + 1 \cdot (n-1) = \frac{n(n^2-1)}{6}. \]

Thus,

\[ n^3 \cdot C(S_j; \overline{S}_j) + \frac{n(n^2-1)}{6} \geq W' \cdot n^3. \]
or
\[ |C(S_j; \overline{S}_j)| + \frac{n^2 - 1}{6n^2} \geq W'. \]

Since \(|C(S_j; \overline{S}_j)|\) and \(W'\) are integers, this implies that
\[ |C(S_j; \overline{S}_j)| \geq W'. \]

Finally, define \(S = S_j \cap V'\). Clearly
\[ C(S_j; \overline{S}_j) = C(S; \overline{S}) \]
where \(C(S; \overline{S})\) is a cut of \(G'\). Thus, the original MAXC problem is answered affirmatively too.

Q.E.D.

**MINLAR:** (Minimum linear arrangement.) Given an undirected graph \(G'(V', E')\) and an integer \(C'\). The problem is to determine whether there exists a one-one mapping \(f'\) from \(V'\) onto \(\{1, 2, \ldots, |V'|\}\) such that
\[ \sum_{<u, v> \in E'} |f'(u) - f'(v)| \leq C'. \]

**Theorem 4:** MAXLAR \(\leq\) MINLAR.

**Proof:** The reduction in this case is trivial. \(G'\) is simply the complement of \(G\). That is,
\[ V' = V \]
\[ E' = \{<u, v> | u, v \in V \text{ and } <u, v> \notin E\}. \]
Also,
\[ C' = \frac{n(n^2-1)}{6} - C, \text{ where } n = |V|. \]

Now, let \( f' = f \). Clearly
\[ \sum_{<u,v> \in E} |f(u) - f(v)| + \sum_{<u,v> \in E'} |f(u) - f(v)| = \frac{n(n^2-1)}{6} \]

Thus one can reach the total length \( C \) in the MAXLAR problem if and only if the total length of the corresponding MINLAR problem can be kept at less than or equal to \( C' \).

Q.E.D.

MINLARD: (Minimum linear arrangement of a directed graph.) Given a directed graph \( G(V,E) \) and an integer \( C \). The problem is to determine whether there exists a one-one mapping \( f \) of \( V \) onto \( \{1,2,...,|V|\} \) such that the following two conditions are satisfied:

(1) If \( <u,v> \in E \) then \( f(u) < f(v) \);

(2) \[ \sum_{<u,v> \in E} [f(v) - f(u)] \leq C. \]

Clearly, a necessary condition for an arrangement to exist is that \( G(V,E) \) contains no directed circuits.

Theorem 5: MINLAR \( \preceq \) MINLARD

Proof: Given a MINLAR problem defined by \( G'(V',E') \) and \( C' \), we define a MINLARD problem as follows (\( n = |V'| \)):
\[ V = V' \cup \{ v_{ij} | \langle v_i, v_j \rangle \in E' \} \cup \{ u_{ik} | 1 \leq i \leq n, 1 \leq k \leq n^4 \}, \]

\[ E = \{ \langle v_i, v_j \rangle, \langle v_j, v_{ij} \rangle | \langle v_i, v_j \rangle \in E' \} \cup \{ \langle u_{ik}, v_i \rangle | 1 \leq i \leq n, 1 \leq k \leq n^4 \}, \]

\[ C = C'(n^4 + 1) + 2 \cdot |E'|^2 + \frac{n^5(n^4 + 1)}{2}. \]

First assume that \( f' \) is a one-one function from \( V' \) onto \( \{1, 2, \ldots, n\} \) which satisfies

\[ \sum_{\langle u, v \rangle \in E'} |f'(u) - f'(v)| \leq C'. \]

Define an arrangement function \( f \) for \( G(V, E) \) which satisfies the following 3 conditions:

(a) If \( f'(v_i) < f'(v_j) \) then \( f(v_i) < f(v_j) \);

(b) \( f(u_{ik}) = f(v_i) - k \) for every \( 1 \leq i \leq n \) and \( 1 \leq k \leq n^4 \).

(c) If \( f'(v_i) < f'(v_j) \), \( \langle v_i, v_j \rangle \in E' \) and \( f'(v_k) = f'(v_j) + 1 \)

then \( f(v_{ij}) < \min_{k} f(u_{ik}) \).

Clearly, there exists an \( f \) which satisfies all these conditions and condition (1). In fact there are many such \( f \)'s. It remains to show that condition (2) is satisfied too.

The length of edge \( \langle u_{ik}, v_i \rangle \) is \( k \). Thus, the total length of these \( n^4 \) edges is

\[ \frac{n^4(n^4 + 1)}{2}, \]

and the total length for all \( i \) is
For each edge \(<v_i, v_j> \in E'\) whose length in the \(f'\) arrangement is \(\lambda\) we have now two edges: \(<v_i, v_{ij}>\) and \(<v_j, v_{ij}>\). If \(f'(v_i) < f'(v_j)\) then the length of the first is bounded by \(\lambda \cdot (n^4 + 1) + |E|\) while the length of the second is bounded by \(|E'|\).

Thus, the contribution of edges of this type to the total length is bounded by:

\[
\frac{n^5(n^4 + 1)}{2} \cdot \sum_{<v_i, v_j> \in E'} |f'(v_i) - f'(v_j)| + 2 \cdot |E'|^2
\]

Thus,

\[
\sum_{<u, v> \in E} |f((v) - f(u)| \leq C' \cdot (n^4 + 1) + 2 \cdot |E'|^2 + \frac{n^5(n^4 + 1)}{2}, \text{ as required.}
\]

Next, assume \(f\) is an arrangement function of \(G(V,E)\) which satisfies conditions (1) and (2). Let us show that there exists an arrangement function \(f_m\) of \(G(V,E)\) which satisfies (1) and (2) and in addition has the following property: If \(f_m(v_i) < f_m(v_j)\) then for all \(1 \leq k \leq n^4\), \(f_m(v_i) < f_m(u_{jk})\).

We derive \(f_m\) from \(f\) by successive circular shifts of consecutive sections of vertices, without violating conditions (1) and (2). Assume \(v_i\) does not satisfy the property above and let \(u_{jp}\) be the right most vertex on \(v_i\)'s left hand side, for which \(f(v_i) < f(v_j)\). We shift all vertices

\[\text{We shall use the name } f \text{ for all the intermediate arrangements, as well as the initial one.}\]
on $u_{jp}$'s right hand side, up to an including $v_i$, one place to the left, and place $u_{jp}$ in $v_i$'s previous position. Clearly, condition (1) is preserved. Let us show that the total length of the edges decreases, thus, preserving condition (2). Assume there are originally $q$ vertices of type $u_{ik}$ between $u_{jp}$ and $v_i$. The shift shortens the edge $<u_{jp}, v_j>$ by at least $q + 1$. In addition, each of the $n^4 - q$ edges $<u_{ik}, v_i>$, where $u_{ik}$ has been on $u_{jp}$'s left hand side, is shortened by one — a saving of $n^4 - q$. So far, we have saved at least $n^4 + 1$. Edges of the type $<v_a, v_{ab}>$ may have lengthened by the shift, each by one, if $v_a$ has been between $u_{jp}$ and $v_i$ and $v_{ab}$ has been on the right hand side of $v_i$. However, the total increase of these edges, is bounded by $2|E'|$. Since $|E'| \leq n(n-1)/2$, the total length of the edges has decreased.

Let us change $f_m$ further, to $f_\mu$, by assuring that if $<v_i, v_j> \in E'$ and $f_\mu(v_i) < f_\mu(v_j) < f_\mu(v_k)$ then $f_\mu(v_{ij}) < \min_{k} f_\mu(u_{ik})$. This may be achieved through successive circular shifts which preserve conditions (1) and (2). If $v_{ij}$ violates the above, we shift all the vertices between $v_j$ and $v_{ij}$ one place to the right and place $v_{ij}$ on $v_j$'s right hand side, in the place adjacent to it. It is easy to see that the saving in the length of $<v_i, v_{ij}>$ and $<v_j, v_{ij}>$ compensates for the length of all edges which enter this section from the left, while the rest of the edges may only decrease. Thus, $f_\mu$ is achieved. We have,

$$
\sum_{<u,v> \in E} [f_\mu(v) - f_\mu(u)] \leq C'(n^4 + 1) + 2|E'|^2 + \frac{n^5(n^4+1)}{2}
$$
Let $f'$ be the arrangement of $V'$ induced by $f$, where we drop the rest of $V$'s vertices and close all the gaps. By an argument similar to that of the first part of the proof we have

$$\sum_{<u,v> \in E'} |f'(u)-f'(v)| + \frac{\binom{n}{2}(n+1)}{2} \leq \sum_{<u,v> \in E} |f'(v)-f'(u)|.$$ 

Thus,

$$\sum_{<u,v> \in E'} |f'(u)-f'(v)| \leq C' \cdot (n+1) + 2 \cdot |E'|^2$$

Since $n+1 > 2 \cdot |E'|^2$ the former inequality implies that

$$\sum_{<u,v> \in E'} |f'(u)-f'(v)| \leq C'.$$

Q.E.D.

**MINLARB**: (Minimum linear arrangement of a bipartite graph.)

Given a bipartite graph $G(X,Y,E)$ and an integer $C$. The problem is to determine whether there exists a one-one mapping $f$ of $X \cup Y$ onto $\{1,2,\ldots,|X\cup Y|\}$ such that

$$\sum_{<x,y> \in E} |f(x)-f(y)| \leq C.$$ 

**Theorem 6**: $\text{MINLAR} \preceq \text{MINLARB}$.

**Proof**: This proof is a little harder than the previous one, but is similar to it. We shall only discuss it briefly.

Given a MINLAR problem, define the corresponding MINLARB problem, where $n = |V'|$:
\[ X = V', \]
\[ Y = \{ v_{ij} | \langle v_i, v_j \rangle \in E' \} \cup \{ u_{ik} | 1 \leq i \leq n \text{ and } 1 \leq k \leq 2n^2 \}, \]
\[ E = \{ \langle v_i, v_{ij} \rangle, \langle v_j, v_{ij} \rangle | \langle v_i, v_j \rangle \in E' \} \]
\[ \cup \{ \langle v_i, u_{ik} \rangle | 1 \leq i \leq n \text{ and } 1 \leq k \leq 2n^4 \}, \]
\[ C = C' \cdot (2 \cdot n^4 + 1) + |E'|^2 + n^5 (n^4 + 1). \]

The significant difference between the construction of the previous proof and the present one is that in the previous construction the graph is directed, while here it is not. Thus, the \( u_{ik} \) vertices may be located on both sides of \( v_i \).

It is easy to see that if an \( f' \) exists which satisfies

\[ \sum_{u, v \in E'} |f'(u) - f'(v)| \leq C'. \]

then an \( f \) exists which satisfies

\[ \sum_{x, y \in E} |f(x) - f(y)| \leq C. \]

Next, assume \( f \) satisfies the latter inequality. One can show that an arrangement \( f_m \) of \( G \) exists which satisfies all the following conditions:

1. \[ \sum_{x, y \in E} |f_m(x) - f_m(y)| \leq \sum_{x, y \in E} |f(x) - f(y)|; \]

2. If \( f_m(v_i) < f_m(v_j) \) then for all \( 1 \leq k \leq 2n^2 \)

\[
\begin{align*}
&f_m(u_{ik}) < f_m(v_i), \\
&f_m(v_i) < f_m(u_{jk}) \text{ and for all } 1 \leq k \leq 2n^2
\end{align*}
\]

\[ f_m(u_{ik}) < f_m(u_{jk}). \]
\[ g \cdot F.\]

\[ \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \Rightarrow (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]

and since \[ |\mathcal{E}| > 2n^2 + 1, \] it follows that

\[ \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \Rightarrow \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]

This implies that

\[ \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \Rightarrow \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]

It is then easy to see that

\[ (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \cdot \]

\[ \forall \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \Rightarrow \exists \lambda, f \in \mathcal{F}, \exists \mathcal{E}, \exists x \in \mathcal{E}, \forall n \in \mathbb{N}, \exists i, j \in \mathbb{Z}, (\lambda, f) \in \mathcal{E}, f(x) \in \mathcal{E}, i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]

Now, we use \( w \) to denote an arrangement \( \mathcal{E} \) for \( G \). For all \( i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \)

\[ \frac{z}{u} + \frac{1}{u} \frac{u}{n} \leq \| (\lambda, f) \|_\mathcal{E} \leq \| (\lambda, f) \|_\mathcal{E} \leq \| \lambda, f \|_\mathcal{E} \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]

\[ \text{for all } i \leq j \leq \frac{z}{u} + \frac{1}{u} \frac{u}{n} \]
REFERENCES


