A NEW UNIVERSAL METHOD OF CODING
FOR THE CLASS OF ALL BERNOULLIAN SOURCES
(EQUAL-LENGTH-AT-OUTPUT CODING)

by

Boris Fitingof

Technical Report No.48
May 1974

Computer Science Department
TECHNION - Israel Institute of Technology
A NEW UNIVERSAL METHOD OF CODING FOR THE CLASS OF ALL BERNOULLIAN SOURCES (EQUAL-LENGTH-AT-OUTPUT CODING)

by

Boris Fitingof
NOTATION

1. \( A = \{a_1, \ldots, a_m\}, \ (2 \leq |A| = m < \infty) \) - a source alphabet.
2. \( B = \{b_1, \ldots, b_n\}, \ (2 \leq |B| = n < \infty) \) - a coding alphabet.
3. \( \Lambda \) - the empty word.
4. \( w_1 \preceq w_2 \) - the word \( w_1 \) is prefix of the word \( w_2 \).
5. \( \ell(w) \) - the length of a word \( w \) (number of symbols in the word).
6. \( C^k \) - the set of all the words of the length \( \ell(w) = k \) in an alphabet \( C \).
7. If \( z_0 \in Z \), then \( Z - z_0 \) - the set of all elements \( z \in Z \) such that \( z \neq z_0 \).
8. \( C = \bigcup_{k=1}^{\infty} C^k \) - the set of all words in the alphabet \( C \).
9. If \( w \) is a word in alphabet \( C \) and \( c \in C \), then \( \overline{wc} \) is the word we suffixed by the letter \( c \).
10. \( \overline{wc} \) - the set of all words \( \overline{wc} \), where \( c \in C \).
1. INTRODUCTION

The problem of information compression methods which do not depend on the probabilistic characteristics of the sources, yet ensure asymptotically and uniformly perfect elimination of redundancies was formulated and initially solved in 1964 [1]. Methods of this type were later called "universal coding methods" [2,3]. Incomplete but mutually-complementary reviews of the works on universal codes are given in [4,5]. Universal codes which use subdivision of messages into blocks of different lengths and yield blocks of equal length as a result of code mapping are considered in [5,6,7,8]. Such codes (not only universal) were called "codes of class B: [9,10] or "variable-length-to block codes" [11,12]. Both in [6,7,8] and here we use the term: "equal-length-at-output codes".

The proof of universality of the equal-length-at-output coding method in [6] is not quite correct. In the present note a modified method of this type is suggested and its universality strictly proved.
2. DEFINITIONS

Definition 1. A coding $\phi(X \rightarrow Y)$ is one-to-one mapping $X$ on $Y$, $y = \phi(x)$, (where $x \in X$, $y \in Y$, $X \subseteq A^*$, $Y \subseteq B^*$) such that:

$$
\begin{align*}
\text{if } x_1, x_2 &\in X, \; x_1 \neq x_2, \text{ then } x_1 \lla x_2 \\
\text{if } x &\in A^*, \text{ then } (\exists x_1 \in X) [x_1 \lla x \text{ or } x \lla x_1]
\end{align*}
$$

if $y_1, y_2 \in Y, \; y_1 \neq y_2, \text{ then } y_1 \nlla y_2$ (2)

A set of words obeying (2) is called a prefix set, and a set of words obeying (1) is called a complete prefix set. It is known that (1) is a sufficient condition for unique representability of any infinite sequence of words $x \in X$, and (2) is a sufficient condition of unique decipherability of any output coding sequence.

Definition 2. A method of coding is a sequence $(\phi_i) \ (1 \leq i < \infty)$ where all $\phi_i(X_i \rightarrow Y_i)$ are codings.

Definition 3. Source $p$ is a non-negative function $p(x)$, where $x \in A^*$, such that $p(A) = 1$ and $\sum_{x \in A} p(\overline{x}) = p(x)$.

Definition 4. The average length per source letter of source $p$ under coding $\phi_i(X_i \rightarrow Y_i)$ is the quantity

$$
\tau_{i, \text{ave}}(p) = \frac{\sum_{x \in X_i} p(x) \mathcal{L}(\phi_i(x))}{\sum_{x \in X_i} p(x) \mathcal{L}(x)}
$$
Definition 5. The average entropy per source letter of source letter of source $p$ under coding $\phi_i(X_i + Y_i)$ is the quantity

$$H_{i, ave}(p) = \frac{-\sum_{x \in X_i} p(x) \log p(x)}{\sum_{x \in X_i} p(x) \ell(x)}$$

($n = |B|$)

Definition 6. A universal method of coding (UMC) for class $P$ of sources $p \in P$ is any coding method $(\phi_i)$ for which:

$$(\forall \epsilon > 0)(\exists i_0 > 0)(\forall i > i_0)(\forall p \in P)$$

$$\tau_{i, ave}(p) < H_{i, ave}(p) + \epsilon.$$  \hspace{1cm} (3)

Definition 7. An asymptotically optimal method of coding (AOMC) is such a method of coding, that for the source $p$

$$(\forall \epsilon > 0)(\exists i_0 > 0)(\forall i > i_0)$$

formula (3) is valid.

Definition 8. An equal-length-at-output coding is a coding with equal lengths of all output code words $y \in Y(\ell(y) = \text{const})$. 

Technion - Computer Science Department - Technical Report CS0048 - 1974
3. DESCRIPTION OF CODING METHOD

Now we shall construct a new universal method of coding for the class \( P_0 \) of all Bernoullian sources. We define the sequence of codings \( (\phi_i)(X_i + Y_i) \) recursively:

1. \( X_1^* = A \).

2. If \( X_i^* \) is known, then we find \( X_{i+1}^* \) as following.

Consider the source \( p^* \) defined by (4):

\[
\frac{1}{p^*(x)} = K(x) = \frac{\lambda(x) + m-1}{m-1!} \prod_{a \in A} \lambda_a(x)!
\]

(4)

where \( m = |A|, \lambda_a(x) \) is the number of occurrences of a letter \( a \in A \) in the word \( x \). Denote by \( \overline{w_i^0} \) the word in \( X_1^* \) such that:

\[
M_i = \max_{a \in A} K(\overline{w_i^0}a) = \min_{x \in X_1} \max_{a \in A} K(xa)
\]

(5)

Then

\[
X_{i+1}^* = (X_i^* - \overline{w_i^0}) \cup \overline{w_i^0 A}
\]

(6)

Let \( N_i(x) \) be the alphabetical order number of the word \( x \in X_i^* \) given by the function \( y \in \phi_i^*(x) \) where \( \phi_i^*(x) \) is the number \( N_i(x) - 1 \), expressed by letters \( b_1, \ldots, b_n \in B \) as digits \( 0, \ldots, n-1 \), each word
by being of minimum possible constant length ($\ell(y)$ is the least integer not less than $\log_n |X'_1|$).

**Theorem:** The method of coding ($\phi_1^*$) described above is a universal method of equal-length-at-output coding for the class $P_0$ of all Bernoullian sources.

**Proof.** Consider the quasi-entropy [1] of a word $x$:

$$J(x) = \sum_{a \in A} \frac{\ell_a(x)}{\ell(x)} \log_2 \frac{\ell_a(x)}{\ell(x)}$$  \hspace{1cm} (7)

A sufficient condition for universality of a method of coding is [6] :

$$(\forall \varepsilon > 0)(\exists i_0')(\forall i > i_0)(\forall x \in X'_1)[\frac{\ell(\phi(x))}{\ell(x)} < J(x) + \varepsilon]$$  \hspace{1cm} (8)

It is known (e.g. [3]) that:

$$\frac{\ell(x)!}{\prod_{a \in A} \ell_a(x)!} \leq \frac{\ell(x)\ell(x)}{\prod_{a \in A} \ell_a(x)\ell_a(x)}$$  \hspace{1cm} (9)

From (4), (7), (9) we obtain:

$$(\forall \varepsilon > 0)(\exists i_0')(\forall i > i_0)(\forall x \in A'_1)[\frac{\log_2 K(x)}{\ell(x)} < J(x) + \varepsilon]$$  \hspace{1cm} (10)
It is readily shown that:

\[
\lim_{k \to \infty} \min_{x \in \mathcal{A}^k} K(x) = \infty \quad (11)
\]

By use of (11), and the description of the coding method, we obtain:

\[
\lim_{i \to \infty} \min_{x \in \mathcal{X}^*_i} (x) = \infty \quad (12)
\]

(The simplest way to prove (12) is by reductio ad absurdum)

From (10) and (12):

\[
(\forall \varepsilon > 0)(\exists i_0)(\forall i > i_0)(\forall x \in \mathcal{X}_i^*) \left[ \frac{\log_n K(x)}{\ell(x)} < J(x) + \varepsilon \right] . \quad (13)
\]

From (4) for any \(a \in A\)

\[
K(xa) = \frac{K(x)(\ell(x)+m)}{\ell_a(x) + 1}
\]

Therefore:

\[
K(x) \geq \frac{K(xa)}{\ell(x)+m} . \quad (14)
\]

Now consider the number \(M_i\) defined by using (14), with (12) taken into account. We obtain:

\[
(\forall \varepsilon > 0)(\exists i_0)(\forall i > i_0)(\forall x \in \mathcal{X}_i^*)
\]

\[
\frac{\log M_i}{\ell(x)} < \frac{\log_n K(x)}{\ell(x)} + \varepsilon . \quad (15)
\]
Using the explicit expression for \(K(x)\) (4), the definition of \(M_i\) (5) and the construction rule (6), it can be shown by induction over \(i\) that \(M_i\) increases monotonically with \(i\), and that

\[
(\forall x \in X_i^*) \{K(x) \leq M_i\} \tag{16}
\]

It is readily shown that for any source \(p(x)\) and any complete prefix set \(X_i\)

\[
\sum_{x \in X_i} p(x) = 1 \tag{17}
\]

Using (16) and (17) for the source \(p^*(x) = \frac{1}{K(x)}\) and the set \(X_i^*\), we can see that:

\[
|X_i^*| < M_i
\]

and then, by the description of the coding method, we obtain:

\[
\ell(\phi_i(x)) \leq 1 + \log M_i \tag{18}
\]

Finally, from (15) and from (18) we obtain (8).

Q.E.D.

**ACKNOWLEDGEMENT**

The author wishes to acknowledge the invaluable encouragement and advice of Professor Lev Levitin (Tel Aviv University).
REFERENCES


REFERENCES (cont'd)


